Journal of Algebra and Its Applications
(2023) 2350205 (19 pages)

DOI: 10.1142/S0219498823502055

On power-associative modules

J. C. Gutierrez Fernandez* and A. Grishkov†
Departamento de Matemática e Estatística
Universidade de São Paulo, São Paulo, Brazil
**jcgf@ime.usp.br*
†*grishkov@ime.usp.br*

E. O. Quintero Vanegas
Departamento de Matemática
Universidade Federal do Amazonas, Manaus
Amazonas, Brazil
eoquinterov@ufam.edu.br

Received 14 May 2021
Revised 25 October 2021
Accepted 25 October 2021
Published 23 June 2022

Communicated by V. Futorny

The aim of this paper is to study the structure of irreducible modules in the variety $\mathcal{M}$ of commutative power-associative nilalgebras of nilindex ≤ 4. If $A \in \mathcal{M}$ with dimension at most 5, then we prove that A^2 is contained in the annihilator of every irreducible A-module in the variety $\mathcal{M}$. Also, we consider the enveloping algebra of an algebra A in the variety $\mathcal{M}$ and we obtain a new example of a commutative power-associative non-nilpotent nilalgebra of dimension 9.

Keywords: Commutative; nilalgebra; power-associative algebra.

Mathematics Subject Classification 2020: 17A05, 17A30

1. Introduction

The theory of bimodules in a class of algebras defined by multilinear identities has been introduced by Eilenberg [7] and it is a powerful tool in the study of the structure of this class of algebras. For every algebra A in $\mathcal{M}$ we have its associative enveloping algebra $\mathcal{U}(A)$. This construction is useful in order to pass from a non-associative structure to a more familiar (associative) algebra over the same field while preserving the representation theory. Commutative power-associative algebras are a natural generalization of associative, alternative, and Jordan algebras. Every

*Corresponding author.

Jordan algebra is a commutative power-associative algebra. Reciprocally, Albert [2, 3] and Kokoris [25] showed that commutative power-associative algebras with certain additional conditions are Jordan. Every simple finite-dimensional commutative power-associative algebra over a field of characteristic zero is either Jordan or nil [25]. For Jordan algebras it is true that every finite-dimensional nilalgebra must be nilpotent. It is natural to ask if the same occurs in the variety of commutative power-associative algebras. This question was answered by Suttles [31], who constructed a 5-dimensional algebra which is commutative power-associative, nil of index 4, but it is not nilpotent. However, this algebra is solvable. If every finite-dimensional commutative power-associative nilalgebra over a field of characteristic different from 2 is solvable remains so far unknown, [13, Problem 1]. This problem is usually called Albert's problem [1] and has different formulations, in relation to radicals and simplicity [18]. Albert's problem has connections with classical problems in affine algebraic geometry [32] and it is still open. In some particular cases, positive answer to Albert's question was obtained [5, 6, 11, 14–16, 21, 28, 29, 33]. The algebra is nilpotent in the following cases: (i) dimension ≤ 4; (ii) is Jordan; (iii) the nilindex is greater than or equal to the dimension. For dimension ≤ 9 over an algebraically closed field of characteristic 0, the algebra is solvable. Also, we know that if the nilindex k of the algebra satisfies $n-3 \leq k$, where n is the dimension of the algebra, then it is solvable (with suitable characteristic).

On the other hand, it is known that every commutative nilalgebra of nilindex 3 generated by r elements over a field of characteristic $\neq 2, 3$ is nilpotent of index less than or equal to $r+5$ [19, 33]. However, for finite-dimensional commutative power-associative nilalgebra of nilindex 4 it is unknown if the algebra is solvable.

An approach to Albert's question can be achieved using modules (Shestakov's generalized theorem, Lemma 1). The goal of this paper is to study the structure of irreducible $\mathcal{M}$-modules, where $\mathcal{M}$ is the variety of commutative power-associative nilalgebras of nilindex ≤ 4.

This paper is organized as follows. Section 2 provides the enveloping algebra of an algebra in a variety, along with some of its properties, which has its own independent interest. The study of commutative power-associative representations for a finite-dimensional algebra A in $\mathcal{M}$ is equivalent to studying the (associative) representations of the associative enveloping algebra of A. In Sec. 3, we give basic definitions and results about modules in the variety $\mathcal{M}$, and we show the connection between Albert's problem and power-associative irreducible modules. Section 4 is devoted to irreducible modules in the variety $\mathcal{M}$. We prove that: If A is a commutative power-associative nilalgebra of nilindex ≤ 4 and dimension ≤ 5, and M is a finite-dimensional irreducible A-module in the variety $\mathcal{M}$, then $A^2M = 0$. Also, we give a new example of a commutative finite-dimensional power-associative non-nilpotent nilalgebra as the null split extension of an algebra of dimension 4 and an A-module of dimension 5 resulting a nilalgebra of nilindex 5 where $A^2M \neq 0$. This paper is a continuation of [18] where the power-associative modules over a trivial algebra of dimension 2 were described.

2. Bimodules and Enveloping Algebras

Throughout this paper, F will be a field of characteristic $\neq 2, 3$ and 5 and all the algebras will be consider over the field F. Let $\mathcal{V}$ be a variety of algebras over F, and take an algebra A in the variety $\mathcal{V}$. An A-bimodule in the variety $\mathcal{V}$, or a $\mathcal{V}$-bimodule over A, is a vector space M over the field F and two bilinear mapping $\lambda : A \times M \rightarrow M$ and $\rho : M \times A \rightarrow M$ sending (a, m) to am and (m, a) to ma such that the algebra $E = A \oplus M$ with multiplication given by $(a + m)(b + n) = ab + (an + mb)$ for all $a, b \in A$ and $m, n \in M$, belongs to $\mathcal{V}$. For example, the algebra $S = A \oplus M$ is associative if A is associative and $(m, a_1, a_2) = 0$, $(a_1, m, a_2) = 0$ and $(a_2, a_2, m) = 0$ for all $a_1, a_2 \in A$ and $m \in M$. For each element $a \in A$ we define the linear transformations of the vector space M to itself, denoted by $\mathbf{L}_a$ and $\mathbf{R}_a$, by $\mathbf{L}_a(m) = am$ and $\mathbf{R}_a(m) = ma$ for all $m \in M$. The pair of mappings $(\mathbf{L}_a, \mathbf{R}_a)$ is called a $\mathcal{V}$-bipresentation of A. Every bimodule M over A is an associative left module over the algebra generated by the elements $\mathbf{L}_a$ and $\mathbf{R}_a$, where $a \in A$. Moreover, for every algebra A from the variety $\mathcal{V}$ there exists a universal associative algebra denoted by $\mathcal{U}_{\mathcal{V}}(A)$, (for simplicity $\mathcal{U}(A)$) such that every $\mathcal{V}$-bimodule over A is an associative left module over $\mathcal{U}(A)$ and vice versa, every associative left module over $\mathcal{U}(A)$ is an $\mathcal{V}$-bimodule over A. The algebra $\mathcal{U}(A)$ is called the $\mathcal{V}$-multiplicative enveloping algebra for A (or enveloping algebra of A in the variety $\mathcal{M}$) [24, Chap. II]. A natural question is about the description of the multiplicative enveloping algebra for an algebra A in an arbitrary variety $\mathcal{V}$ and this description has considerable difficulties. If A is a finite-dimensional alternative or Jordan algebra, then the multiplicative enveloping algebra is also finite-dimensional [23, 30]. If A is the 2-dimensional trivial algebra, in the sense that $A^2 = 0$, and $\mathcal{M}$ is the variety of commutative power-associative nilalgebras of nilindex ≤ 4, then L'vov [27] obtained a basis of $\mathcal{U}_{\mathcal{M}}(A)$ and a classification of its irreducible modules.

Because the algebras considered in this paper are commutative, we can use only left multiplications. Note that in the case of commutative algebras the notion of module and bimodule is the same.

Let A be a commutative algebra. For each element $x \in A$ we define inductively the *powers* of x by $x^1 = x$ and $x^k = xx^{k-1}$ for all $k \geq 2$. The algebra A is called power-associative if $x^i x^j = x^{i+j}$ for all positive integers i, j. Basic properties of these algebras were given by Albert in [1]. In particular, a commutative algebra over the field F is a power-associative algebra if and only if it satisfies the fourth power-associative identity $x^4 = x^2x^2$. A power-associative algebra A is called nil, or nilalgebra, if for every $x \in A$, there exists a positive integer k such that $x^k = 0$. If there exists a positive integer k where $x^k = 0$ for all $x \in A$, then we say that A is of bounded nilindex. For a power-associative nilalgebra A of bounded nilindex, the smallest positive integer k such that $x^k = 0$ for all $x \in A$, is called nilpotent index (or nilindex) of A.

In the following, $\mathcal{M}$ will be the variety of commutative power-associative nilalgebras of nilindex ≤ 4. A commutative algebra belongs to the variety $\mathcal{M}$ if and only if this algebra satisfies the following two identities:

$$x^4 = 0, \quad x^2x^2 = 0. \tag{1}$$

Using the process of linearization of identities [34], we obtain the following useful identities for the variety $\mathcal{M}$ [12]:

$$x^3y + x(x^2y) + 2x(x(xy)) = 0, \quad x^2(xy) = 0, \tag{2}$$

$$x(xy^2) + y(yx^2) + 2x(y(xy)) + 2y(x(xy)) = 0, \quad 2(xy)^2 + x^2y^2 = 0, \tag{3}$$

$$2[x(x(yz)) + x(y(xz)) + x(z(xy)) + y(x(xz)) + z(x(xy))] + y(zx^2) + z(yx^2) = 0, \tag{4}$$

$$x^2(yz) + 2(xy)(xz) = 0, \tag{5}$$

$$s(x_1, x_2, x_3, x_4) = 0, \quad (xy)(zt) + (xz)(yt) + (xt)(yz) = 0, \tag{6}$$

where $s(x_1, x_2, x_3, x_4) := \frac{1}{2}\sum_{\sigma\in S_4} x_{\sigma(1)}(x_{\sigma(2)}(x_{\sigma(3)}x_{\sigma(4)}))$ and S_4 is the set of all permutations on the set $\{1, 2, 3, 4\}$.

We shall denote by span$\{X\}$, the vector subspace of A spanned by a subset X of A.

Proposition 1. *Let $\mathfrak{A}$ be the free algebra generated by $\{a, b\}$ in the variety defined by the identities $x_1x_2 = x_2x_1$ and $x_1(x_2x_3) = 0$. If $\mathcal{U}$ is the enveloping algebra of $\mathfrak{A}$ in the variety $\mathcal{M}$ and $\overline{\mathfrak{A}} = \mathfrak{A}/\mathfrak{A}^2$ is the trivial 2-dimensional commutative algebra, then*

$$\mathcal{U}/J(\mathcal{U}) \cong \mathcal{U}(\overline{\mathfrak{A}}),$$

where $J(\mathcal{U})$ is the Jacobson radical of $\mathcal{U}$.

Proof. Let M be an arbitrary irreducible $\mathfrak{A}$-module in the variety $\mathcal{M}$. We need to prove that $\mathbf{L}_{c^2} = 0$ for all $c \in \mathfrak{A}$. Since the action of the group End$(\mathfrak{A})$ on $\overline{\mathfrak{A}}$ is transitive, it is enough to prove that $\mathbf{L}_{a^2} = 0$. Because $cd^2 = 0$ for all $c, d \in \mathfrak{A}$, the identity (5) implies

$$\mathbf{L}_{c^2}\mathbf{L}_{d^2} = -2\mathbf{L}_{cd^2}\mathbf{L}_c = 0. \tag{7}$$

We will now prove by induction on n that

$$\mathbf{L} := \mathbf{L}_{a^2}\mathbf{L}_{c_1}\cdots\mathbf{L}_{c_n}\mathbf{L}_{a^2} = 0, \tag{8}$$

for all $c_1, \dots, c_n \in \mathfrak{A}$. The case $n = 0$ is given in (7). Assume now that $n > 0$. By linearity, we can assume that $c_1, c_2, \dots, c_n \in \{a, b, a^2, b^2, ab\}$. If $c_1 \in \{a, a^2, ab, b^2\}$, then $\mathbf{L} = 0$ by (2) and (7). In the remaining cases, $c_1 = b$. If $n = 1$, then $\mathbf{L}_{a^2}\mathbf{L}_b\mathbf{L}_{a^2} = -\mathbf{L}_{a^2}\mathbf{L}_{a^2}\mathbf{L}_b - \mathbf{L}_b\mathbf{L}_{a^2}\mathbf{L}_{a^2} = 0$ by (4) and (7). If $c_1 = c_2 = b$, then $\mathbf{L}_{a^2}\mathbf{L}_b\mathbf{L}_b = -\mathbf{L}_b\mathbf{L}_{a^2}\mathbf{L}_b - \mathbf{L}_b\mathbf{L}_b\mathbf{L}_{a^2}$ by (4) and (7) and now $\mathbf{L} = 0$, by induction hypothesis. If $c_2 = a$, then $\mathbf{L}_{a^2}\mathbf{L}_b\mathbf{L}_a = -\mathbf{L}_{a^2}\mathbf{L}_a\mathbf{L}_b - \mathbf{L}_b\mathbf{L}_{a^2}\mathbf{L}_a - \mathbf{L}_b\mathbf{L}_a\mathbf{L}_{a^2} - \mathbf{L}_a\mathbf{L}_{a^2}\mathbf{L}_b - \mathbf{L}_a\mathbf{L}_b\mathbf{L}_{a^2} = 0$ by (2), (6), (7) and induction hypothesis. If $c_2 \in \{a^2, ab, b^2\}$, then

$\mathbf{L}_{a^2}\mathbf{L}_b\mathbf{L}_{c_2} = -\mathbf{L}_{c_2}\mathbf{L}_b\mathbf{L}_{a^2} = 0$ by (6), (7) and induction hypothesis, since ab belongs to the vector space spanned by $\{d^2 : d \in \mathfrak{A}\}$. Thus, we have proved that $\mathbf{L} = 0$ in all the cases.

Finally, we will prove by contradiction that $\mathbf{L}_{a^2} = 0$. Assume that $\mathbf{L}_{a^2} \neq 0$. We can take $v \in M$ such that $a^2v \neq 0$. Because M is irreducible, M is spanned by all the elements of the form $a_1(\cdots(a_m(a^2v))\cdots)$ where $a_1,\ldots,a_m \in \mathfrak{A}$. But, Eq. (8) implies that $a^2(a_1(\cdots(a_m(a^2v))\cdots)) = 0$ so that $a^2M = 0$, which gives a contradiction. □

3. Power-Associative Modules

The interested reader can find equivalences to Albert's problem and a brief overview about the structure of finite-dimensional commutative nilalgebras and the Albert's problem in [17, 18]. The study of finite-dimensional irreducible modules over the variety $\mathcal{M}$ is motivated by the following generalization of a Shestakov's theorem [18, Lemma 1].

Lemma 1. *Let A be a finite-dimensional commutative power-associative nilalgebra over a field of characteristic zero and V a finite-dimensional irreducible A-bimodule in the variety of commutative power-associative algebras. Assume that we can define a product on V with values in A, $(v,w) \mapsto v{\cdot}w \in A$, such that $V{\cdot}V = A$. If the vector space $Q = A \oplus V$ with the multiplication $(a+v)(b+w) = a\cdot b + v\cdot w + (aw+bv)$ is a commutative power-associative algebra, then Q is nil, $Q^2 = Q$, and gives a counterexample to Albert's problem.*

Previous lemma can be extended in the following sense. Let A be a finite-dimensional commutative power-associative nilalgebra over a field of characteristic zero and V and W two finite-dimensional irreducible A-bimodules in the variety of commutative power-associative algebras. Let $f : V \times W \to A$ a surjective bilinear map. If the vector space $Q = A \oplus V \oplus W$ with the multiplication $(a+v+w)(b+v'+w') = a\cdot b + f(v,w') + f(v',w) + (av' + aw' + bv + bw)$ is a commutative power-associative algebra, then Q is nil and $Q^2 = Q$. In this case, for each maximal ideal I of Q, we have that Q/I is a simple finite-dimensional power-associative nilalgebra.

The first step to study the irreducible modules in the variety of commutative power-associative nilalgebras was given in [18] where the irreducible A-modules when A is the 2-dimensional algebra with zero multiplication are described in [18, Theorem 1].

Theorem 1. *Let $A = \operatorname{span}\{a,b\}$ be a 2-dimensional algebra with zero multiplication. Then every irreducible power-associative A-module M has dimension 1 or 3. If M has dimension 1, then $AM = 0$. If M has dimension 3, then we can choice a suitable basis $\{u,v,w\}$ of M and a scalar $0 \neq \lambda \in F$ such that*

$$au = v, \quad av = w, \quad aw = 0, \quad bu = 0, \quad bv = \lambda u, \quad bw = -\lambda v. \tag{9}$$

The goal of this paper is to study the structure of irreducible $\mathcal{M}$-modules, where $\mathcal{M}$ is the variety of commutative power-associative nilalgebras of nilindex ≤ 4. We know that every algebra of this variety satisfies the following Engel identity [12]:

$$x(x(x(x(xy)))) = 0. \tag{10}$$

The following lemma is an immediate consequence of the above identity.

Lemma 2. *Let A be an algebra in the variety $\mathcal{M}$ and M a nonzero finite-dimensional A-module in the variety $\mathcal{M}$. Then $aM \subsetneq M$ for all $a \in A$. In particular, if* $\dim M = 1$, *then* $AM = 0$.

Let A be an algebra in the variety $\mathcal{M}$ and M an irreducible finite-dimensional A-module in the variety $\mathcal{M}$. For each $a \in A$, we define the following set:

$$V_a = \ker(\mathbf{L}_a) \cap \ker(\mathbf{L}_{a^2}).$$

The next lemma will be used throughout the paper without explicit mention.

Lemma 3. *Let M be an A-module in the variety $\mathcal{M}$. Take $a \in A$ and $m \in M$.*

(i) *If W is a nonzero vector subspace of M invariant under the action of a, that is $aW \subseteq W$, then* $\ker(\mathbf{L}_a) \cap W \neq 0$;
(ii) *If $a \in V_a$, then $a^3 m = 0$;*
(iii) *If $M \neq 0$, then $V_a \neq 0$.*

Proof. We get (i) and (ii) from (10) and first identity of (2), respectively. Second identity of (2) implies that $\ker(\mathbf{L}_{a^2})$ is invariant under the action of a. Now (i) and (ii) force (iii). □

Theorem 2. *Let A be an algebra in* $\operatorname{span}\{\mathbf{L}_a, \mathbf{L}_{a^2}\}$, *the subalgebra generated by $\mathbf{L}_a = L$ and $\mathbf{L}_{a^2} = U$, is spanned, as a vector space, by the following operators: $L, L^2, U, L^3, LU, L^4, L^2U$. Furthermore, we have the following identities:*

$$\mathbf{L}_{a^3} = -LU - 2L^3, \quad UL = 0, \quad UU = -2\mathbf{L}_{a^3}L = 4L^4 \tag{11}$$

and $p(a, b) = 0$ for every non-associative monomial $p(x, y)$ with x-degree ≥ 5 and y-degree 1.

Proof. First and second multiplication identities of (11) are an immediate consequence of (2). Next, $UU(b) = a^2(a^2b) = -2a^3(ab) = -2\mathbf{L}_{a^3}L(b) = 2LUL(b) + 4L^4(b) = 4L^4(b)$ by (5) and previous identities. Now, using (5) we obtain that $U\mathbf{L}_{a^3}(b) = a^2(a^3b) = -2a^4(ab) = 0$ and $\mathbf{L}_{a^3}U(b) = a^3(a^2b) = (a^2a)(a^2b) = -(1/2)(a^2a^2)(ab) = 0$ since $a^4 = a^2a^2 = 0$. Then by first identity of (11) we have that $L^3U = -(1/2)LUU = -2L^5 = 0$. We can now turn to prove that $L^2\mathbf{L}_{a^3} = -L^3U - 2L^5 = 0$ and $L\mathbf{L}_{a^3}L = -2L^5 = 0$. In order to prove the last statement of the theorem, it is sufficient to show that $UL^2U = 0$ and $UL^4 = 0$. Both relations are obviously true from the multiplication identity $UL = 0$. □

Previous theorem can be reformulated as follows.

Theorem 3. *Let $\mathfrak{A}$ be the free algebra in the variety $\mathcal{M}$ with free generating system $\{a\}$. Then $\mathcal{U}(\mathfrak{A})$ in the variety $\mathcal{M}$ is a graded 7-dimensional associative algebra*

$$\mathcal{U}(\mathfrak{A}) = \bigoplus_{i\geq 1} \mathcal{U}_i,$$

where $\mathcal{U}_1 = \mathrm{span}\{\mathfrak{a}\}$, $\mathcal{U}_2 = \mathrm{span}\{\mathfrak{a}^2, \mathfrak{b}\}$, $\mathcal{U}_3 = \mathrm{span}\{\mathfrak{a}^3, \mathfrak{a}\mathfrak{b}\}$, $\mathcal{U}_4 = \mathrm{span}\{\mathfrak{a}^4, \mathfrak{a}^2\mathfrak{b}\}$, $\mathcal{U}_n = 0$ *for all* $n \geq 5$ *and*

$$\mathfrak{b}\mathfrak{a} = 0, \quad \mathfrak{b}^2 = 4\mathfrak{a}^4.$$

The above theorem suggests the following conjecture.

Conjecture 1. *Let $p(x, y)$ be a non-associative monomial with x-degree n and y-degree m. If $|m - n| \geq 4$ then $p(x, y) = 0$ is an identity in the variety $\mathcal{M}$.*

By Theorem 2, we know that above conjecture is true for $m = 1$.

4. Irreducible $\mathcal{M}$-Modules

In the following A will be an algebra in $\mathcal{M}$ and M a finite-dimensional irreducible A-module in the variety $\mathcal{M}$. An interesting open question is the following conjecture.

Conjecture 2. *Let A be a commutative finite-dimensional power-associative nilalgebra of nilindex ≤ 4. If M is a finite-dimensional irreducible A-module in the variety $\mathcal{M}$, then $A^2M = 0$.*

We confirm this conjecture in a positive way if $\dim A \leq 5$. The condition, nilindex ≤ 4, cannot be eliminated as the following example shows.

Example 1. Let A be the 4-dimensional algebra spanned by a, b, a^2 and ab such that $b^2 = 0$ and $A^3 = 0$. As an F-vector space, M is spanned by v_1, v_2, v_3, v_4 and v_5 subject to the relations

$$\begin{aligned}
&av_1 = v_2, \quad a^2v_3 = -2v_5, \quad bv_1 = -5v_4, \quad (ab)v_1 = v_5,\\
&av_3 = v_4, \quad a^2v_5 = \frac{1}{2}v_2, \quad bv_2 = 4v_5, \quad (ab)v_5 = -v_4,\\
&av_4 = v_5, \quad bv_5 = v_3,\\
&av_5 = v_1,
\end{aligned}$$

and $av_2 = a^2v_1 = a^2v_2 = a^2v_4 = bv_3 = bv_4 = (ab)v_2 = (ab)v_3 = (ab)v_4 = 0$. Then, M is a commutative power-associative irreducible A-module. If $u = \lambda_1 a + \lambda_2 a^2 + \lambda_3 b + \lambda_4 ab + \sum_{i=1}^5 \mu_i v_i$, with $\lambda_i, \mu_i \in F$, is a generic element of $A \oplus M$, then

$$\begin{aligned}
u^2 &= \lambda_1^2 a^2 + 2\lambda_1\lambda_3 ab + 2\lambda_1\mu_5 v_1 + (2\lambda_1\mu_1 + \lambda_2\mu_5)v_2 + 2\lambda_3\mu_5 v_3 + (2\lambda_1\mu_3\\
&\quad - 10\lambda_3\mu_1 - 2\lambda_4\mu_5)v_4 + (2\lambda_1\mu_4 - 4\lambda_2\mu_3 + 8\lambda_3\mu_2 + 2\lambda_4\mu_1)v_5,
\end{aligned}$$

$$\begin{aligned}
u^3 &= 2\lambda_1(\lambda_1\mu_4 - 2\lambda_2\mu_3 + 4\lambda_3\mu_2 + \lambda_4\mu_1)v_1 \\
&\quad + ((5/2)\lambda_1^2\mu_5 + \lambda_1\lambda_2\mu_4 - 2\lambda_2^2\mu_3 + 4\lambda_2\lambda_3\mu_2 + \lambda_2\lambda_4\mu_1)v_2 \\
&\quad + 12\lambda_3(\lambda_1\mu_4 - 2\lambda_2\mu_3 + 4\lambda_3\mu_2 + \lambda_4\mu_1)v_3 \\
&\quad + (-10\lambda_1\lambda_3\mu_5 - 2\lambda_1\lambda_4\mu_4 + 4\lambda_2\lambda_4\mu_3 - 8\lambda_3\lambda_4\mu_2 - 2\lambda_4^2\mu_1)v_4, \\
u^2u^2 = u^4 &= (\lambda_1\mu_4 - 2\lambda_2\mu_3 + 4\lambda_3\mu_2 + \lambda_4\mu_1)(2\lambda_1^2 v_2 - 8\lambda_1\lambda_3 v_4), \\
u^5 &= 0.
\end{aligned}$$

Thus, the commutative algebra $B = A \oplus M$ is power-associative and nil with nilindex 5. In order to prove that M is irreducible, we first observe that M, as A-module, is generated by v_5. Furthermore, for each $x \in M$, $x \neq 0$, we have that v_5 belongs to the submodule generated by x. Let $x = \sum_{i=1}^{5} \mu_i v_i$, with $\mu_i \in F$, a nonzero element of M. Then

- If $\mu_3 \neq 0$, then $a((ab)(a^2x)) = 2\mu_3 v_5$;
- If $\mu_3 = 0$ and $\mu_5 \neq 0$, then $b(a^2x) = 2\mu_5 v_5$;
- If $\mu_3 = \mu_5 = 0$ and $\mu_1 \neq 0$, then $(ab)x = \mu_1 v_5$;
- If $\mu_1 = \mu_3 = \mu_5 = 0$, then $ax = \mu_4 v_5$ and $bx = 4\mu_2 v_5$.

Until now, the Suttles algebra S was essentially the only known commutative power-associative finite-dimensional and non-nilpotent nilalgebra, where $S = \text{span}\{a, a^2, a^3, b, a^2b\}$ and the non-obvious nonzero products are

$$a(a^2b) = a^2, \quad a^3b = -a^2. \tag{12}$$

The algebra $B = A \oplus M$ is a new example of commutative finite-dimensional power-associative nilalgebra that is not nilpotent. We observe that the Suttles algebra S can also be given through irreducible modules, because $S = A_0 \oplus M_0$ where A_0 is the trivial 2-dimensional algebra and M_0 is the 3-dimensional irreducible A_0-module defined in Theorem 1.

Many different classification results of finite-dimensional commutative nilalgebras have appeared in the literature. Gerstenhaber and Myung [15] showed that commutative, power-associative, nilalgebras of dimension 4 are nilpotent and determined their isomorphic classes. Thus, Suttles's example is the commutative, power-associative nilalgebra not nilpotent of least possible dimension. Elgueta and Suazo [9] described the commutative power-associative nilalgebras of dimension 5 and the commutative Jordan nilalgebras of dimension 6 in [9, 10], respectively. The commutative right-nilalgebras of right-nilindex 4 and dimension at most 4 were classified by Elduque and Labra in [8], thus extending the classification by Gerstenhaber and Myung.

4.1. *The dimension of A is less than or equal to* 4

In this section, we confirm Conjecture 2 for all algebras with dimension at most 4.

Lemma 4. *Let* $\dim A = 2$. *Then the dimension of* M *is either* 1 *or* 3. *If* $\dim M = 3$, *then* $A^2 = 0$ *and the action of* A *on the vector space* M *is given as in Theorem* 1.

Proof. Every (irreducible) A-module in the variety $\mathcal{M}$ is in fact an (irreducible) power-associative A-module. Thus, the case $A^2 = 0$ is clear from Theorem 1 since the algebra $A \oplus M$ is trivial if $\dim M = 1$ and it is isomorphic to the Suttles algebra when $\dim M = 3$. Let us assume now $A^2 \neq 0$. Take an element $a \in A$ with $a^2 \neq 0$. Then $A = \text{span}\{a, a^2\}$. First, we observe that $U = \{u \in M : au = 0\}$ is a non-zero submodule of M since by (2) we get that $a(a^2u) = -a^3u - 2a(a(au)) = 0$ for all $u \in U$. Therefore, $AU = a^2U \subseteq U$. Because M is irreducible, we have that $M = U$ and hence $aM = 0$. Let now V be the nonzero vector subspace of M given by the set $\{u \in M : a^2u = 0\}$. Since $aV = 0$ and also $a^2V = 0$, we have that $AV = 0 \subseteq V$ and hence $M = V$. Therefore, $AM = 0$ so that $\dim M = 1$. □

As a more general result, we have the following.

Proposition 2. *Let* B *be an algebra with zero multiplication where* $\dim B = \dim A - \dim A^2$. *Take a vector subspace* V *of* A *such that* $A = V \oplus A^2$ *and a bijective linear mapping* $f : V \to B$. *If* M *is a power-associative* (*irreducible*) B-*module, then* M *is an* (*irreducible*) A-*module in the variety* $\mathcal{M}$ *via* $A^2M = 0$ *and* $vx = f(v)x$ *for all* $v \in V$ *and* $x \in M$. *Reciprocally, if* N *is an* (*irreducible*) A-*module in the variety* $\mathcal{M}$ *and* $A^2N = 0$, *then* N *is a* (*irreducible*) *power-associative* B-*module via* $by = f^{-1}(b)y$ *for all* $b \in B$ *and* $y \in N$.

Proof. For simplicity, we can assume that $B = V$ and f is the identity mapping. Let M be a power-associative B-module and $u = (a, x)$ an arbitrary element of the algebra $A \oplus M$. We have a unique decomposition $a = v + a_0$ where $v \in V$ and $a_0 \in A^2$. Then $u^2 = (a^2, 2ax) = (a^2, 2vx)$, $u^2u^2 = (a^2a^2, 4a^2(vx)) = (0, 0)$, $u^3 = (a, x)(a^2, 2vx) = (a^3, 2a(vx) + a^2x) = (a^3, 2v(vx))$ and $u^4 = (a, x)(a^3, 2v(vx)) = (a^4, 2a(v(vx)) = (0, 2v(v(vx))) = (0, -v^3x - v(v^2x) + 4v^2(vx)) = (0, 0)$. This proves the first part of the proposition. The reciprocal case can be obtained in a similar way. □

In [15], the authors classified all the power-associative nilalgebras of dimension ≤ 4. In particular, for dimension 3 and with non-zero multiplication, each one of those algebras is isomorphic to one of the following:

- $A_1 := \text{span}\{a, a^2, a^3\}$, where $A_1^4 = 0$;
- $A_2(\alpha) := \text{span}\{a, a^2, b\}$, where $ab = 0$, $b^2 = \alpha a^2$ and $(A_2(\alpha))^3 = 0$.

Lemma 5. *If* $\dim A = 3$, *then* $A^2M = 0$. *Furthermore*

(i) *If* $\dim A^2 = 2$, *then* $\dim M = 1$ *and* $AM = 0$;
(ii) *If* $\dim A^2 = 1$ *and* $AM \neq 0$, *then* $\dim M = 3$ *and there exist* $a, b \in A$ *and a basis* $\{v_1, v_2, v_3\}$ *of* M *such that* $av_1 = v_2$, $av_2 = v_3$, $av_3 = 0$, $bv_1 = 0$, $bv_2 = v_1$, $bv_3 = -v_2$.

Proof. Case i, ($A^3 \neq 0$). Take an element $a \in A$ such that $a^3 \neq 0$. Obviously, $A = \operatorname{span}\{a, a^2, a^3\}$ and $A^4 = 0$. Because $a^3x + a(a^2x) + 2a(a(ax)) = 0$ and $a^2(ax) = 0$ for all $x \in M$, we have that $a^3(a(ax)) = -a(a^2(a(ax))) - 2a(a(a(a(ax)))) = 0$. This implies that $a(aM)$ is a submodule of M and since M is irreducible and $a(aM) \subseteq aM \subsetneq M$, we have that $a(aM) = 0$. Furthermore, $a^2(aM) = 0$ and for Lemma 3 we get that aM is a proper submodule of M, so $aM = 0$. Since we have proved that $aM = 0$, where a is an arbitrary element of A such that $a^3 \neq 0$, it follows that $AM = 0$ and hence $\dim M = 1$.

Case ii, ($A^3 = 0$). There exist two elements $a, b \in A$ and a scalar $\alpha \in F$ such that $A = \operatorname{span}\{a, a^2, b\}$, $ab = 0$ and $b^2 = \alpha a^2$. Then $a^2(ax) = 0$ and by (5) we have $a^2(bx) = -2(ab)(ax) = 0$ and $a^2(a^2x) = -2a^3(ax) = 0$. Therefore $A^2(AM) = 0$. Obviously, AM is a submodule of M. Since M is an irreducible module, we have that either $AM = 0$ or $AM = M$. In both cases, $A^2M = 0$. The last statement of the lemma follows from Lemma 4 and Proposition 2. □

A classification of commutative power-associative nilalgebras of dimension 4 was obtained by Gerstenhaber and Myung [15] based on a classification of Kruse and Price [26] for nilalgebras of nilindex 3. Also, [20] gives a classification of commutative nilalgebras with nilindex and dimension 4.

Lemma 6. *If* $\dim A = 4$, *then* $A^2M = 0$.

Proof. By [15, 20] we know that A is isomorphic to one of the algebras listed below. Each case will be analyzed independently.

Case 1. $A = \operatorname{span}\{a, a^2, a^3, b\}$ where $ab = \lambda a^2$, $a^4 = 0$, $bA^2 = 0$, $A^2A^2 = 0$ and $b^2 = \alpha a^2 + \beta a^3$ with $\alpha, \beta, \lambda \in F$. Observe that $A^4 = 0$. Let $U = \{x \in M : a^2x = 0\}$. Since $aU \subseteq U$ and $U \neq 0$ by (10), we can take $x \in U$, $x \neq 0$ such that $ax = 0$. This element satisfies $ax = a^2x = 0$. We will prove inductively that $a^2(a_1(a_2(\cdots(a_kx)\,))) = 0$ and $a^3(a_1(a_2(\cdots(a_kx)\,))) = 0$ for all non-negative integers k and $a_1, a_2, \ldots, a_k \in A$. It is sufficient to consider $a_i \in \{a, a^2, a^3, b\}$. The case $k = 0$ is obvious. If $k = 1$, then $a^2(a_1x) = -2(aa_1)(ax) = 0$ and $a^3(a_1x) = -(aa_1)(a^2x) - (a^2a_1)(ax) = 0$. Let now $k > 1$. By induction hypothesis and properties of a, we can assume that $a_i \in \{a, b\}$ for all i. Take $y = a_3(a_4(\cdots(a_kx)))$. Then $a^t(a_1(a_2(\cdots(a_kx)))) = a^t(a_1(a_2y)) = -a^t(a_2(a_1y))$ for $t = 2, 3$, by first identity of (6) and induction hypothesis. Thus, the product vanishes if $a_1 = a_2$. Finally, $a^2(a(by)) = 0$ and $a^3(a(by)) = -(1/2)a^2(a^2(by)) = 0$. Because M is an irreducible A-module, this one is spanned as vector space by all the element of the form $a_1(a_2(\cdots(a_kx)))$ where k is a non-negative integer and $a_i \in A$. Therefore, we have proved that $A^2M = \operatorname{span}\{a^2, a^3\}M = 0$.

Case 2. $A = \operatorname{span}\{a, a^2, b, c\}$ where $b^2 = \alpha a^2$, $c^2 = \beta a^2$ and all other products of basic elements vanish. Then $a^2(ax) = 0$, $a^2(bx) = -2(ab)(ax) = 0$, $a^2(cx) = -2(ac)(ax) = 0$ and $a^2(a^2x) = -2a^3(ax) = 0$ for all $x \in M$. Therefore, $N = aM + bM + cM$ is a submodule of M. If $N = M$, then $A^2M = a^2M = a^2N = 0$.

Otherwise, $N = 0$ and we can take and element $0 \neq x \in M$ where $a^2x = 0$. Then $Ax = 0$ so that $M = Fx$.

Case 3. $A = \text{span}\{a, a^2, b, ab\}$ where $b^2 = \alpha a^2$ and all other products of basic elements vanish. Using (5) and relation $A^3 = 0$, we get that

$$c(dx) = 0, \tag{13}$$

for all $x \in M$ and $c, d \in A^2$. Let $U = \{x \in M : a^2x = 0\}$. Since $aU \subseteq U$ and $U \neq 0$ by (10), we can take $0 \neq x \in U$ such that $ax = 0$. This element satisfies $ax = a^2x = 0$. We will prove inductively that $A^2(a_1(a_2(\cdots(a_kx)))) = 0$ for all positive integers k and $a_1, a_2, \ldots, a_k \in A$. By linearity, it is sufficient to consider $a_i \in \{a, a^2, b, ab\}$. If $k = 1$, then $a^2(a_1x) = -2(aa_1)(ax) = 0$, $(ab)(bx) = -(1/2)b^2(ax) = 0$ and by (13) we have that $(ab)(wx) = 0$ for all $w \in A^2$. Thus, $A^2(Ax) = 0$. Let now $k > 1$. By induction hypothesis and properties of A, we can assume that $a_i \in \{a, b\}$. Let $y = a_3(a_4(\cdots(a_kx)))$. If $w \in A^2$, then $w(a_1(a_2(\cdots(a_kx)))) = w(a_1(a_2y)) = -w(a_2(a_1y))$ by first identity of (6), relation $A^3 = 0$ and induction hypothesis. Thus, the product vanishes if $a_1 = a_2$. Furthermore, $a^2(a(by)) = 0$ and $(ab)(a(by)) = -(1/2)a^2(b(by)) = 0$. Therefore, the product vanishes in all the cases as we wanted to prove. Since the module is irreducible, previous fact implies that either $AM = 0$ or $A^2M = 0$. □

4.2. *The dimension of A is 5 and its nilindex 3*

For a nontrival algebra A in the variety $\mathcal{M}$ with dimension 5, the nilindex is either 3 or 4. We will now analyze the case when A has nilindex 3. Every commutative nilalgebra of nilindex 3 is Jordan [4, 34]. Elgueta and Suazo [9] gave a description of all commutative power-associative (Jordan) nilalgebras of dimension 5 and nilindex 3. We have the following (Hegazi and Abdelwahab [22]).

Proposition 3. *If A has dimension 5 and it is not an associative algebra, then its nilindex is 3 and there exist $a, b \in A$ such that $\{a, a^2, b, ab, a^2b\}$ is a basis of A, $a(ab) = -(1/2)a^2b$, $b^2 = b(ab) = 0$ and $A^4 = 0$. Furthermore, $B = \text{span}\{a, a^2, ab, a^2b\}$ is a subalgebra of A and every irreducible B-submodule N of M has dimension 1.*

Proof. The first part could be found in [9]. From Lemmas 4 and 6 and Proposition 2 we have that the dimension of N is either 3 or 1. Suppose that $\dim N = 3$. Then, we can take a basis $\{v_1, v_2, v_3\}$ of N such that $av_1 = v_2$, $av_2 = v_3$, $(ab)v_1 = 0$, $(ab)v_2 = v_1$, $(ab)v_3 = -v_2$ and $a(a(aN)) = a^2N = (a^2b)N = 0$. From $s(a, a, b, v_2) = 0$ and $s(a, a, b, v_3) = 0$, we get

$$v_2 = -a(a(bv_2)) - a(bv_3), \quad v_3 = a(a(bv_3)). \tag{14}$$

Multiplying first equation of (14) by $2a$, we obtain $2v_3 = -2a(a(a(bv_2))) -2a(a(bv_3)) = a(a^2(bv_2)) - 2a(a(bv_3)) = -2a((ab)(av_2)) - 2v_3 = 0$ a contradiction. Thus, $\dim N = 1$. □

Lemma 7. *If A has dimension 5 and it is not an associative algebra, then* $A^2M = 0$.

Proof. Let B be the subalgebra in the latter lemma and $N \subseteq M$ an irreducible B-module. Then, the latter lemma implies that there exists a nonzero element $x \in M$ such that $Bx = 0$. It will be proved by induction on k that $um = 0$ whenever $m = a_1(a_2(\cdots a_k x))$ with $a_1, \ldots a_k \in A$ and $u \in A^2$. It is sufficient to consider the cases $a_i \in \{a, b\}$ and $u \in \{a^2, ab, a^2b\}$. The case $k = 0$ is obvious. For $k = 1$ we have that $ax = 0$, $a^2(bx) = -2(ab)(ax) = 0$, $(ab)(bx) = -(1/2)b^2(ax) = 0$ and $(ba^2)(bx) = -(1/2)b^2(a^2x) = 0$. Let $k > 1$. If $y = a_3(\cdots(a_k x)\cdots)$, then $0 = s(u, a_1, a_2, y) = u(a_1(a_2y)) + u(a_2(a_1y))$ so that

$$u(a_1(a_2y)) = -u(a_2(a_1y)). \tag{15}$$

Thus, it is enough to consider the case $a_1 = a$ and $a_2 = b$. Now, $a^2(a(by)) = 0$, $(ab)(a(by)) = -(1/2)a^2(b(by)) = 0$ by (15), $(a^2b)(a(by)) = -(a^2b)(b(ay)) = (1/2)b^2(a^2(ay)) = 0$. Hence, $A^2M = 0$. □

Assume now that A is an associative algebra with dimension 5 and nilindex 3. By Elgueta and Suazo [9] we know that $A^3 = 0$ and $1 \le \dim A^2 \le 3$. We will study each case separately.

Lemma 8. *If A is an associative algebra of dimension* 5, *nilindex* 3 *and* $\dim A^2 = 3$, *then* $A^2M = 0$.

Proof. It is clear that A^2 is an ideal of A since $A^2A = A^3 = 0$. Take $a, b \in A$ where $A = \text{span}\{a, b\} \oplus A^2$. Then $\{a, a^2, b, ab, b^2\}$ is a basis of A. Therefore, A is the free algebra in the variety defined by the identities $x_1x_2 = x_2 = x_1$ and $x_1(x_2x_3) = 0$ with free generating system $\{a, b\}$. Proposition 1 gives $A^2M = 0$. □

Lemma 9. *If A is an associative algebra of dimension* 5, *nilindex* 3 *and* $\dim A^2 = 2$, *then* $A^2M = 0$.

Proof. We know that there exist $a, b, c \in A$ and scalars $\alpha, \beta, \gamma, \lambda, \mu \in F$ such that $\{a, a^2, b, ab, c\}$ is a basis of A and

$$ac = 0, \quad b^2 = \gamma a^2, \quad bc = \alpha a^2 + \beta ab, \quad c^2 = \lambda a^2 + \mu ab.$$

The set $B = \text{span}\{a, a^2, b, ab\}$ is a 4-dimensional subalgebra of A. Let us consider $N \subset M$ an irreducible B-module. We already know that $B^2N = 0$. Take a nonzero element $x \in N$ such that $ax = 0$. We will now prove that $A^2(a_1(a_2(\cdots(a_k x)))) = 0$ for all non-negative integers k and $a_1, a_2, \ldots, a_k \in \{a, b, c\}$. The case $k = 0$ is obvious. For $k = 1$ we have that $a^2(ux) = -2(au)(ax) = 0$ for all $u \in A$ and $(ab)(bx) = -(1/2)b^2(ax) = 0$, $(ab)(cx) = -(ac)(bx) - (bc)(ax) = 0$. Let us assume now that $k \geq 2$ and denote by y the element $a_3(\cdots(a_k x))$ for simplicity. Combining identity $s(v, a_1, a_2, y) = 0$ with induction hypothesis we get the relation

$$v(a_1(a_2 y)) = -v(a_2(a_1 y)), \tag{16}$$

for all $v \in A^2$. By (16) we can assume, without loss of generality, that $a_1 \in \{a, b\}$, $a_2 \in \{b, c\}$ and $a_1 \neq a_2$. Now

$$a^2(b(cy)) = -a^2(c(by)) = 2(ac)(a(by)) = 0,$$

$$(ab)(a(a_2 y)) = -(1/2)a^2(b(a_2 y)) = 0 \quad \text{if } a_2 \in \{b, c\},$$

$$(ab)(b(cy)) = -\frac{1}{2}b^2(a(cy)) = 0.$$

Therefore, we have proved that the vector subspace T of M spanned by all the elements of the form $a_1(a_2(\cdots(a_k x)))$ where $a_i \in \{a, b, c\}$ is in fact a non-zero A-submodule of M and also $A^2 T = 0$. Because M is an irreducible A-module, we have that $T = M$ and $A^2 M = 0$. □

Lemma 10. *If A is an associative algebra of dimension* 5, *nilindex* 3 *and* $\dim A^2 =$ 1. *Then* $A^2 M = 0$.

Proof. We know that there exist $a_1, a_2, a_3, a_4 \in A$ such that $\{a_1, a_1^2, a_2, a_3, a_4\}$ is a basis of A where $a_i^2 \in F a_1^2$ for $i = 2, 3, 4$ and the other products of basic elements are zero. First, we observe that $N = \sum_{i=1}^4 a_i M$ is an A-submodule of M since $a_1^2(a_1 x) = 0$ and $a_1^2(a_i x) = -2(a_1 a_i)(a_1 x) = 0$ for $i = 2, 3, 4$ and all $x \in M$. Thus, if $N \neq 0$, then $M = N$ and hence $A^2 M = A^2 N = a_1^2 N = 0$. Let us consider now the case $N = 0$. Then $a_j M = 0$ for $j = 1, 2, 3, 4$. Take a non-zero element $y \in M$ where $a_1^2 y = 0$. Then $Ay = 0$ so that $T = Fy$ is an A-submodule of M. Because M is irreducible, it follows that $M = T$ and $AM = 0$. □

4.3. *The dimension of A is* 5 *and its nilindex* 4

We will now analyze the case when A has nilindex 4. Elgueta and Suazo in [9] gave a description of all commutative power-associative nilalgebras of dimension 5 and nilindex 4.

Proposition 4. *Let A be a commutative power-associative nilalgebra of dimension* 5 *and nilindex* 4. *The following affirmations are equivalents*: (i) $\dim A^3 = 3$; (ii) A *is not Jordan*; (iii) A *is not nilpotent.*

If A satisfies the equivalent previous conditions, then A is isomorphic to the Suttles algebra.

Furthermore, each nilalgebra in $\mathcal{M}$ of dimension 5 and nilindex 4 satisfies the following property [9, Lemma 1.4]:

$$1 \leq \dim A^3 < \dim A^2 \leq 3$$

and it is isomorphic to one algebra of the following four types of commutative algebras, where we indicate only the non-obvious nonzero products of the elements of the basis:

- $S = \text{span}\{a, a^2, a^3, b, a^2b\}$, the Suttles algebra, where $a(a^2b)a^2$ and $a^3b = -a^2$.
- $C = \text{span}\{a, a^2, a^3, b, a^2b\}$, where $b^2 \in C^2 = \text{span}\{a^2, a^3, a^2b\}$.
- $D = \text{span}\{a, a^2, a^3, b, ab\}$, where $b^2 \in D^2 = \text{span}\{a^2, a^3, ab\}$ and $a(ab), b(ab) \in D^3 = \text{span}\{a^3\}$.
- $E = \text{span}\{a, a^2, a^3, b, c\}$, where $b^2, bc, c^2 \in E^2 = \text{span}\{a^2, a^3\}$ and $ab, ac \in \text{span}\{a^2\}$.

Let A be an algebra in the variety $\mathcal{M}$ and M be an irreducible finite-dimensional A-module in the variety $\mathcal{M}$. For each $a \in A$, we define the following set:

$$W = V_a \cap \ker \mathbf{L}_{a^2b} = \{x \in M : ax = a^2x = (a^2b)x = 0\}.$$

Lemma 11. *If A is isomorphic to the Suttles algebra S, then $A^2M = 0$.*

Proof. We can assume that $A = S$. By (ii) of Lemma 3 we know that $V_a \neq 0$. This vector subspace V_a is invariant under the action of a^2b since for each $x \in V_a$ we have that $0 = s(a, b, a^2, x) = a((ba^2)x) + a(a^2(bx)) + x(a(a^2b)) + x(a^3b) = a((ba^2)x) + a(a^2(bx)) = a((ba^2)x) - 2a((ab)(ax)) = a((ba^2)x)$ as well as $a^2((ba^2)x) = -2(a(a^2b))(ax) = 0$. Therefore, W is different from 0 since $\mathbf{L}_{a^2b}$ is nilpotent by (10). Let's show that $T = W + bW + b(bW)$ is a submodule of M. Obviously,

$$AW \subseteq bW.$$

Now, $a(a(bw)) = (1/2)s(a, a, b, w) = 0$, $a^2(a(bw)) = 0$ and

$$(a^2b)(a(bw)) = (a^2b)(a(bw)) - \frac{1}{2}a(b^2(a^2w)) = (a^2b)(a(bw)) + a((a^2b)(bw))$$
$$= s(a^2b, a, b, w) = 0.$$

Thus, we have proved that $a(bW) \subseteq W$. Also, we have that $A^2(bW) \subseteq (Ab)(AW) \subseteq (Ab)(bW) \subseteq b^2(AW) = 0$. Summarizing, we have proved the following relation:

$$A(bW) \subseteq W + b(bW).$$

On the other hand, we have that $0 = s(b, b, a, w) = 2b(a(bw)) + 2a(b(bw))$ and hence $a(b(bw)) = -b(a(bw)) \subseteq b(a(bW)) \subseteq bW$. Furthermore, we obtain

$$a^2(b(bw)) = -2(ab)(a(bw)) = 0,$$
$$a^3(b(bw)) = -a(a^2(b(bw))) - 2a(a(a(b(bw)))) = -2a(a(a(b(bw))))$$
$$\in a(a(bW)) \subseteq aW = 0,$$
$$(a^2b)(b(bw)) = -(1/2)b^2(a^2(bw)) = 0,$$
$$b(b(bw)) = -(1/2)b^3w - (1/2)b(b^2w) = 0.$$

This implies $A(b(bW)) \subseteq bW$. We have proved that T is a non-zero submodule of M. Because M is an irreducible module, we get that $M = T$. Therefore, $A^2M = A^2T = 0$. □

Using (5) and (6) we immediately obtain the following.

Lemma 12. *Let N be a C-module in the variety $\mathcal{M}$. Then*

$$a^2(bx) = 0, \quad a^3(bx) = -(a^2b)(ax)$$

for all $x \in N$.

Lemma 13. *If A is isomorphic to the algebra C, then $A^2M = 0$.*

Proof. Assume that $A = C$. First, we shall prove that $W \neq 0$. We already know that $V_a \neq 0$ by Lemma 3. For any $x \in V_a$, we have $0 = s(a, a^2, b, x) = a(a^2(bx)) + a((a^2b)x) = -2a((ab)(ax)) + a((a^2b)x) = a((a^2b)x)$ and $a^2((a^2b)x) = -2(a(a^2b))(ax) = 0$. Hence, $(a^2b)V_a \subseteq V_a$ and this implies that $W \neq 0$. Now, we will show that $T = W + bW + b(bW)$ is a non-zero submodule of M and $A^2T = 0$. Obviously $AW = bW$. For any $w \in W$, $a^2(bw) = -2(ab)(aw) = 0$, $a^3(bw) = -(ab)(a^2w) - (a^2b)(aw) = 0$ and $(a^2b)(bw) = -(1/2)b^2(a^2w) = 0$. Thus, $A^2(bW) = 0$ and

$$A(bW) = a(bW) + b(bW).$$

In order to prove that $a(bW) \subseteq W$, note that $s(a, a, b, w) = 0$ forces $a(a(bw)) = 0$. Furthermore, $a^2(a(bx)) = a^3(a(bx)) = 0$ and $0 = s(a^2b, a, b, x) = (a^2b)(a(bx))$. Hence,

$$A(bW) \subseteq W + b(bW).$$

Let $u \in \{a, a^2, a^3, a^2b\}$. Then, $0 = s(u, b, b, w) = 2b(b(uw)) + 2b(u(bw)) + 2b(w(bu)) + 2u(b(bw)) + 2w(b(bu)) + u(wb^2) + w(ub^2) = 2b(u(bw)) + 2u(b(bw)$. If $u \neq a$ we obtain $u(b(bw)) = 0$. On the other hand, $a(b(bw)) = -b(a(bw)) \in b(a(bW)) \subseteq bW$. Finally, $2b(b(bw)) = -b^3w - b(b^2w) = 0$ since $A^2W = 0$. Therefore, T is a submodule of M and using that M is irreducible we get that $M = T$. □

Lemma 14. *If A is isomorphic to the algebra D, then $A^2M = 0$.*

Proof. Assume that $A = D$. Let $B = \operatorname{span}\{a, a^2, a^3, ab\}$ and N be an irreducible B-submodule of M. We already prove that $B^2N = 0$ and $\dim N$ is either 1 or 3. We will analyze each case separately.

Case 1, ($\dim N = 1$). $N = Fx$. We will prove by induction on k that

$$A^2(a_1(a_2(\cdots(a_kx)))) = 0,$$

for all $a_1, a_2, \ldots, a_k \in \{a, b\}$. The case $k = 0$ is obvious. By Lemma 2, we have that $ax = 0$. Consequently, $a^2(bx) = -2(ab)(ax) = 0$, $a^3(bx) = -(ab)(a^2x) - (a^2b)(ax) =$

0 and $(ab)(bx) = -(1/2)b^2(ax) = 0$. Furthermore,

$$Ax = F(bx), \quad A(Ax) = \operatorname{span}\{a(bx), b(bx)\}.$$

Assume now $k \geq 2$ and let $y = a_3(\cdots(a_k x))$ and $z = a_1(a_2 y)$. Take $u \in A^2$. We get that $0 = s(u, a_1, a_2, y) = u(a_1(a_2 y)) + u(a_2(a_1 y))$ by induction hypothesis. Therefore,

$$u(a_1(a_2 y)) = -u(a_2(a_1 y)). \tag{17}$$

Thus, it is sufficient to prove the relation for $a_1 = a$ and $a_2 = b$. In this case, $a^2 z = 0$, $a^3 z = a^3(a(by)) = -(1/2)a^2(a^2(by)) = 0$ by (11) and induction hypothesis and $(ab)(a(by)) = -(1/2)a^2(b(by)) = 0$ by (17).

Therefore, we have proved that T, the vector subspace spanned by all the elements of the form $a_1(a_2(\cdots(a_k x)))$ where $a_1, \ldots, a_k \in \{a, b\}$ and k is a non-negative integer, is a non-zero A-submodule of M and $A^2 T = 0$. Because M is irreducible, we get $T = M$ and also $A^2 M = 0$.

Case 2, ($\dim N = 3$). There exists a non-zero element $x \in M$ such that $N = \operatorname{span}\{x, ax, a(ax)\}$, $a(a(ax)) = 0$, $(ab)x = 0$, $(ab)(ax) = x$ and $(ab)(a(ax)) = -ax$ and $B^2 N = 0$. Now,

$$a^2(bx) = -2(ab)(ax) = -2x$$

and hence

$$a^2(b(a^2(bx))) = 4x. \tag{18}$$

On the other hand,

$$a^2((bx)(a^2 b)) = 0 \quad \text{by } a^2 b = 0,$$

$$b(a^2(a^2(bx))) = -2b(a^2((ab)(ax))) = -2b(a^2 x) = 0,$$

$$(bx)(a^2(a^2 b)) = 0,$$

$$a^2(a^2(b(bx))) = -2a^2((ab)(a(bx))) = 4(a(ab))(a(a(bx))) \in Fa^3(a(a(bx))) = 0,$$

by Theorem 2. Combining previous identities with $s(a^2, a^2, bx, b) = 0$ we get that $a^2(b(a^2(bx))) = 0$. This contradicts relation (18) and hence Case 2 is impossible.

□

Lemma 15. *If A is isomorphic to the algebra E, then $A^2 M = 0$.*

Proof. Assume that $A = E$. Let $B = \operatorname{span}\{a, a^2, a^3, b\}$ and N an irreducible B-submodule of M. We already know that $B^2 N = 0$ and $\dim N$ is either 1 or 3.

Case 1, ($\dim N = 1$). $N = Fx$. We will prove by induction on k that

$$A^2(a_1(a_2(\cdots(a_k x)))) = 0 \tag{19}$$

for all $a_1, a_2, \ldots, a_k \in \{a, b, c\}$. The case $k = 0$ is obvious. Let $u \in \{a, b, c\}$. If $k = 1$, then $a^2(ux) = -2(au)(ax) = 0$ and $a^3(ux) = -(au)(a^2 x) - (a^2 u)(ax) = 0$.

Therefore, $A^2(Ax) = 0$. Assume now $k \geq 2$ and let $y = a_3(\cdots(a_kx))$. If $j \in \{2,3\}$, then $0 = s(a^j, a_1, a_2, y) = a^j(a_1(a_2y)) + a^j(a_2(a_1y))$ so that $a^j(a_1(a_2y)) = -a^j(a_2(a_1y))$. Thus, it is enough to consider $(a_1, a_2) \in \{(a,b),(a,c),(b,c)\}$. Now, by induction hypothesis, we get

$$a^2(b(cy)) = -2(ab)(a(cy)) = -2\lambda a^2(a(cy)) = 0,$$

$$a^3(b(cy)) = -(ab)(a^2(cy)) - (a^2b)(a(cy)) = 0,$$

$$a^3(a(uy)) = -(1/2)(a^2(a^2(uy))) = 0.$$

Because M is irreducible, (19) implies that $A^2M = 0$.

Case 2, $(\dim N = 3)$. We can take an element $x \in N$ such that N is spanned by $x, ax, a(ax)$. Furthermore, $a(a(ax)) = 0$, $bx = 0$, $b(ax) = x$, $b(a(ax)) = -ax$ and $B^2N = 0$. We will prove by induction on k that

$$A^2(a_1(a_2(\cdots(a_kx)))) = 0 \tag{20}$$

for all $a_1, a_2, \ldots, a_k \in \{a,b,c\}$. The case $k = 0$ is clear. Let $k \geq 1$ and $y = a_2(\cdots(a_kx))$. Then (for $u = a,b,c$)

$$a^2(uy) = -2(au)(ay) \in \text{span}\{a^2(ay)\} = 0,$$

$$a^3(uy) = -(au)(a^2y) - (a^2u)(ay) = -(a^2u)(ay) \in F(a^3(ay)) = F(a^2(a^2y)) = 0,$$

by induction hypothesis. This proves that $A^2M = 0$. □

Summarizing, we have proved the following.

Theorem 4. *Let A be a commutative power-associative nilalgebra in the variety $\mathcal{M}$ with dimension ≤ 5. If M is an irreducible A-module in the variety $\mathcal{M}$, then $A^2M = 0$.*

We can reformulate Theorem 4 in terms of the enveloping algebra.

Corollary 1. *Let A be an algebra of dimension ≤ 5 in the variety $\mathcal{M}$. If $\mathcal{U} = \mathcal{U}(A)$ is the enveloping algebra of $\mathfrak{A}$ in the variety $\mathcal{M}$, then*

$$\frac{\mathcal{U}}{J(\mathcal{U})} \cong \mathcal{U}\left(\frac{A}{A^2}\right),$$

where $J(\mathcal{U})$ is the Jacobson radical of $\mathcal{U}$.

Acknowledgments

The work is partially supported by the projects 2018/13705-6, 2018/23690-6, the São Paulo Research Foundation (FAPESP) and by the Coordenacão de Aperfeicoamento de Pessoal de Nível Superior – Brasil (CAPES) – Finance Code 001.

The authors would like to thank the anonymous reviewer for his/her valuable suggestions and comments that improved the presentation of this paper.

References

[1] A. A. Albert, Power-associative rings, *Trans. Amer. Math. Soc.* **64** (1948) 552–593.

[2] A. A. Albert, A theory of power-associative commutative algebras, *Trans. Amer. Math. Soc.* **69** (1950) 503–527.

[3] A. A. Albert, On commutative power-associative algebras of degree two, *Trans. Amer. Math. Soc.* **74** (1953) 323–343.

[4] D. Burde and A. Fialowski, Jacobi-Jordan algebras, *Linear Algebra Appl.* **459** (2014) 586–594.

[5] I. Correa, I. R. Hentzel and L. A. Peresi, On the solvability of the commutative power-associative nilalgebras of dimension 6, *Linear Algebra Appl.* **369** (2003) 185–192.

[6] I. Correa and L. A. Peresi, On the solvability of the five dimensional commutative power-associative nilalgebras, *Results Math.* **39** (2001) 23–27.

[7] S. Eilenberg, Extensions of general algebras, *Ann. Soc. Polon. Math.* **21** (1948) 125–134.

[8] A. Elduque and A. Labra, On the classification of commutative right-nilalgebras of dimension at most four, *Comm. Algebra* **35** (2007) 577–588.

[9] L. Elgueta and A. Suazo, Jordan nilalgebras of nilindex N and dimension $N+1$, *Comm. Algebra* **30** (2002) 5547–5561.

[10] L. Elgueta and A. Suazo, Jordan nilalgebras of dimension 6, *Proyecciones* **21** (2002) 277–289.

[11] L. Elgueta and A. Suazo, Solvability of commutative power-associative nilalgebras of nilindex 4 and dimension, *Proyecciones* **23** (2004) 123–129.

[12] L. Elgueta, A. Suazo and J. C. Gutiérrez Fernández, Nilpotence of a class of commutative power-associative nilalgebras, *J. Algebra* **291** (2005) 492–504.

[13] V. T. Filipov, V. K. Kharchenko and I. P. Shestakov, Dniester notebook: unsolved problems in the theory of rings and modules, in *Non-Associative Algebra and Its Applications*, Lecture Notes in Pure Applied Mathematics, Vol. 246 (Chapman & Hall/CRC, Boca Raton, FL, 2006), pp. 461–516.

[14] M. Gerstenhaber, On nilalgebras and linear varieties of nilpotent matrices II, *Duke Math. J.* **27** (1960) 21–31.

[15] M. Gerstenhaber and H. C. Myung, On commutative power-associative nilalgebras of low dimension, *Proc. Amer. Math. Soc.* **48** (1975) 29–32.

[16] J. C. Gutierrez Fernandez, On commutative power-associative nilalgebras, *Comm. Algebra* **32** (2004) 2243–2250.

[17] J. C. Gutierrez Fernandez and C. I. Garcia, On commutative finite-dimensional nilalgebras, *São Paulo J. Math. Sci.* **10** (2016) 104–121.

[18] J. C. Gutierrez Fernandez, A. Grishkov, M. L. R. Montoya and L. S. I. Murakami, Commutative power-associative algebras of nilindex four, *Comm. Algebra* **39** (2011) 3151–3165.

[19] J.C. Gutierrez Fernández and C. I. Garcia, On Jordan-nilalgebras of index 3, *Comm. Algebra* **44** (2016) 4277–4293.

[20] J. C. Gutierrez Fernández, C. I. Garcia and M. L. R. Montoya, On power-associative nilalgebras of nilindex and dimension n, *Rev. Colombiana Mat.* **47** (2013) 1–11.

[21] J. C. Gutierrez Fernández and A. Suazo, Commutative power-associative nilalgebras of nilindex 5, *Results Math.* **47** (2005) 296–304.

[22] A. S. Hegazi and H. Abdelwahab, Classification of five-dimensional nilpotent Jordan algebras, *Linear Algebra Appl.* **494** (2016) 165–218.

[23] N. Jacobson, Structure of alternative and Jordan bimodules, *Osaka J. Math.* **6** (1954) 1–71.

[24] N. Jacobson, *Structure and Representations of Jordan Algebras* (American Mathematical Society Colloquium Publications, Providence, RI, 1968).
[25] L. A. Kokoris, Simple power-associative algebras of degree two, *Ann. Math.* **64** (1956) 544–550.
[26] R. L. Kruse and D. T. Price, *Nilpotent Rings* (Gordon and Breach Science Publishers, New York, 1969).
[27] I. L'vov, About representations of generalized Clifford algebras in *Proc. All Union XVII Algebraic Conference* (USSR, Minsk, 1983), p. 118.
[28] E. O. Quintero Vanegas and J. C. Gutierrez Fernandez, Power associative nilalgebras of dimension 9, *J. Algebra* **495** (2018) 233–263.
[29] E. O. Quintero Vanegas and J. C. Gutierrez Fernandez, Nilpotent linear spaces and Albert's problem, *Linear Algebra Appl.* **518** (2017) 57–78.
[30] R. Schafer, Representations of alternative algebras, *Trans. Am. Math. Soc.* **72** (1952) 1–17.
[31] D. Suttles, A counterexample to a conjecture of Albert, *Notices Amer. Math. Soc.* **19** (1972) A-566.
[32] U. Umirbaev, Polarization algebras and their relations, *J. Commut. Algebra* **11** (2019) 433–451.
[33] E. I. Zelmanov, Jordan nil algebras of bounded index, *Dokl. Akad. Nauk SSSR* **249** (1979) 30–33.
[34] K. A. Zhevlakov, A. M. Slin'ko, I. P. Shestakov and A. I. Shirshov, *Rings that are Nearly Associative*, Pure and Applied Mathematics, Vol. 104 (Academic Press, Harcourt Brace Jovanovich, Publishers, New York, London, 1982).