The polynomial-exponential distribution: A continuous probability model allowing for occurrence of zero values

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ABSTRACT
This paper deals with a new two-parameter lifetime distribution with increasing, decreasing and constant hazard rate. This distribution allows the occurrence of zero values and involves the exponential, linear exponential and other combinations of Weibull distributions as submodels. Many statistical properties of the distribution are derived. Maximum likelihood estimation of the parameters and a bias corrective approach is investigated with a simulation study for performance of the estimators. Four real data sets are analyzed for illustrative purposes and it is noted that the distribution is a highly alternative to the gamma, Weibull, Lognormal and exponentiated exponential distributions.

KEYWORDS
Hazard rate; Two-parameter distributions; Reliability and statistical measures; Maximum Likelihood Estimation; Data applications.

MSC 62E10, 62F10

1. Introduction with motivations

In analysis of the lifetime data, monotone hazard rates are common. Such data can be modeled using the log-normal, Weibull and gamma distributions. The Weibull distribution is more popular than log-normal and gamma because the survival and hazard rate functions of the last two distributions have not a closed form and hence numerical integrations are required. Gupta and Kundu (2001) introduced the exponentiated exponential (EE) distribution as an extension to the exponential distribution and also as an alternative to the gamma distribution. Further developments on the exponentiated exponential distribution can be seen in Gupta and Kundu (2007).

In many practical applications, continuous probability models that allow occurrence of zero values have vast importance, for example in forecast models when we observe

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the monthly rainfall precipitation, it is common in dry periods the non occurrence of precipitation, therefore the occurrence of zero values can be observed in different measures, such as the average, maximum and minimum. In survival analysis, we may observe data with instantaneous failure due to construction problem. Another example of zero occurrence is the hydrologic data in arid and semiarid regions, like annual peak flow discharges. Moreover, zero occurrence can be met in many areas, e.g. manufacturing defects, medical consultations, hydrology, ecology and econometrics.

Unfortunately, the above four distributions (log-normal, Weibull, gamma and exponentiated exponential) do not provide the characteristic of zero occurrence. In order to reach this aim, new distributions was proposed, including the Kuş distribution by Kuş (2007) and the Nadarajah-Haghighi distribution by Nadarajah and Haghighi (2011), both well-established in statistical modelling. A complete discussion on such distributions can be found in Louzada et al. (2018). In this paper, a new continuous two-parameter distribution that allows occurrence of zero values is introduced. Following the spirit of the Nadarajah-Haghighi distribution, the corresponding cumulative distribution function (cdf) is based on an original combination of polynomial and exponential functions. It is given by

$$F(x) = \begin{cases} 1 - e^{-\lambda x \frac{x^{\alpha}-1}{x-1}}, & x \in (0, +\infty)\backslash\{1\}, \\ 1 - e^{-\lambda\alpha}, & x = 1, \end{cases} \tag{1}$$

where $\alpha > 0$ and $\lambda > 0$ (and $F(x) = 0$ if $x < 0$) (we have considered a natural continuous extension for $x = 1$ to avoid a discontinuity at this point). For the purposes of this paper, we call it the Polynomial-exponential (PE) distribution and we denoted it by PE(α, λ), when the parameters need to be specified. In order to relate the PE distribution to the existing literature, let us notice that the corresponding cdf can be rewritten under the form $F(x) = G[H(x)]$, where $G(x)$ is the cdf associated to the exponential distribution with parameter $\lambda > 0$ and

$$H(x) = x\frac{x^{\alpha} - 1}{x - 1}$$

(with $H(1) = \alpha$) is a positive increasing function with $H(0) = 0$ and $\lim_{x\to+\infty} H(x) = +\infty$. Hence, it belongs to the so-called transformed-transformer (T-X) family by Alzaatreh et al. (2013) with the exponential distribution as main generator and, to the best of our knowledge, a new function $H(x)$. Thus defined, we claim that the PE represents a powerful alternative to the mentioned distributions above for the following reasons. First of all, it has closed form expressions for its cdf and hazard rate function (which is not the case, for the log-normal and gamma distributions, for instance). Also, it provides an extension to exponential, linear exponential and other combinations of Weibull distributions. Indeed, for $\alpha = 1$, the PE distribution gives the exponential and when $\alpha = 2$, it reduces to the one parameter linear exponential distribution. Furthermore, when α is an integer, we can express $F(x)$ as

$$F(x) = 1 - e^{-\lambda \sum_{k=1}^{\alpha} x^{k}}, \quad x \geq 0.$$

Hence, the survival function $S(x) = 1 - F(x)$ of the PE distribution is the product of the survival functions of Weibull distributions with parameters $(\lambda, 1)$, $(\lambda, 2), \ldots,$ and (λ, α) with respect to value of k, which means the distribution has an exponential general polynomial. Furthermore, the corresponding hazard rate function is very flexible; it has constant, increasing or decreasing shape, which remains an important quality in terms of statistical modelling. Last but not least, by the way of numerical studies, the analyzes of some practical data sets are favorable to the PE model in comparison to the well-established Kuş and Nadarajah-Haghighi models. All these attractive properties are discussed and illustrated in the paper.

Owing to its practical importance, an emphasis is put on the inferential properties of the PE distribution. To be more specific, the inferential procedure for the parameters of PE distribution is presented using the maximum likelihood estimation. Then, it is shown that one of the estimators can be obtained in closed-form and this allows us to obtain the estimates solving a simple one non-linear equation. The obtained estimators are biased for small samples, therefore, we discuss a bias corrective approach based on bootstrap resampling method proposed by Efron (1979). The performance of the MLEs and the bias corrected MLEs are compared using extensive numerical simulations.

The paper is organized as follows. In Section 2 we introduce the PE distribution. Section 3 is devoted to some of its mathematical properties. Estimations of the parameters via the maximum likelihood method are investigated in Section 4. A simulation analysis is given in Section 5. In Section 6 we apply our proposed model in four real data sets. Finally, in Section 7, we conclude the paper.

2. Polynomial-exponential distribution

Upon differentiation of $F(x)$ given by (1), almost surely, the probability density function (pdf) of the PE distribution is given by

$$f(x) = \lambda \frac{\alpha x^{\alpha+1} - (\alpha+1)\, x^{\alpha} + 1}{(x-1)^2} e^{-\lambda x \frac{x^{\alpha}-1}{x-1}}, \quad x \in (0, +\infty)\backslash\{1\}, \tag{2}$$

with the continuous extension for $x = 1$: $f(1) = \frac{\lambda\alpha(\alpha+1)}{2} e^{-\lambda\alpha}$. Figure 1 presents some plot of the PE distribution for different values of α and λ, and showing various shapes of the density function with left skewness.

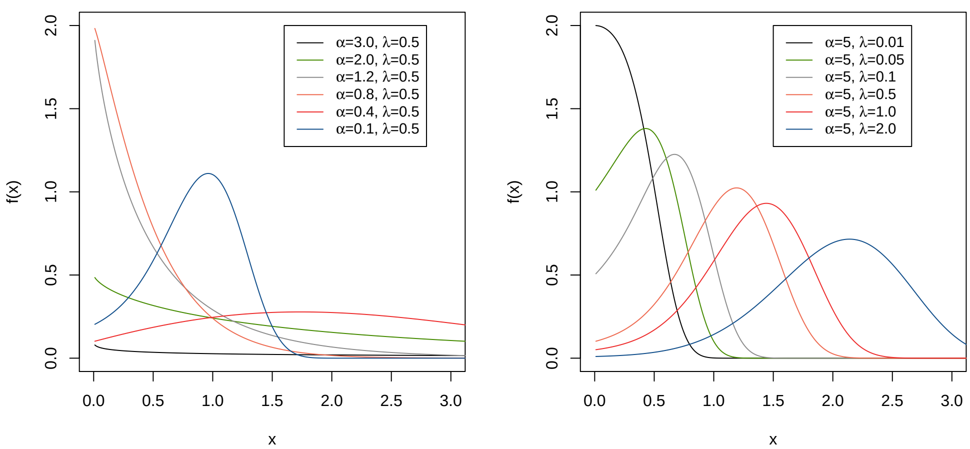

Figure 1. Density function shapes for PE distribution considering different values of α and λ.

As we know, many distributions such as log-normal, Weibull, gamma and exponentiated exponential, to list a few, do not allow occurrence of zero values. In this regard, the following remark shows that the PE distribution can be used as a model with occurrence of zero values.

Let us observe that $f(0) = \lambda > 0$ for all $\alpha > 0$ and $\lambda > 0$. Further, it follows from equation (2) that $f(x) \sim \lambda\alpha x^{\alpha-1}e^{-\lambda x^{\alpha}}$ as $x \to \infty$. Therefore, the upper tail behavior of the pdf is a product of a polynomial power and an exponential polynomial power decay, both of them depend only on α. Obviously, larger values of α lead to faster decay of the upper tail, which interprets α as a shape parameter.

The study of the behavior of the hazard function is not an easy task. Glaser (1980) lemma is difficult to be implemented since $\{\log(f(x))\}'$ does not has a simple form, i.e.,

$$
\begin{aligned}
\{\log(f(x))\}' = &-\frac{2}{x-1} + \alpha(\alpha+1)x^{\alpha-1}\frac{x-1}{\alpha x^{\alpha+1} - (\alpha+1)\,x^{\alpha} + 1} \\
&- \lambda\frac{\alpha x^{\alpha+1} - (\alpha+1)\,x^{\alpha} + 1}{(x-1)^2}.
\end{aligned}
$$

Thus, a critical point x_* of $f(x)$ satisfies the non-linear equation $\{\log(f(x))\}' \,|_{x=x_*} = 0$ and its nature depends on the sign of $\{\log(f(x))\}'' \,|_{x=x_*}$.

The hazard rate function (hrf) is given by

$$
h(x) = \lambda\frac{\alpha x^{\alpha+1} - (\alpha+1)\,x^{\alpha} + 1}{(x-1)^2}, \qquad x \in (0,+\infty)\backslash\{1\},
$$

with the continuous extension for $x = 1$ given by $h(1) = \frac{\lambda\alpha(\alpha+1)}{2}$. Now, a critical point x_o for $h(x)$ is given as the solution by the non-linear equation: $\{\log(h(x))\}' \,|_{x=x_o} = 0$, where

$$
\{\log(h(x))\}' = -\frac{2}{x-1} + \alpha(\alpha+1)x^{\alpha-1}\frac{x-1}{\alpha x^{\alpha+1} - (\alpha+1)\,x^{\alpha} + 1}.
$$

So the non-linear equation becomes

$$
(x-1)^2\alpha(\alpha+1)x^{\alpha-1} = 2[\alpha x^{\alpha+1} - (\alpha+1)\,x^{\alpha} + 1]
$$

We see that x_o does not depend on λ; only α plays an important role. Also, the fact that α is greater or not to 1 seems to have an effect due to the term $x^{\alpha-1}$. Indeed, from graphical analysis we observed that the hazard function presents an decreasing hazard rate for $\alpha > 1$ and $\lambda > 0$, increasing hazard rate for $\alpha < 1$ and $\lambda > 0$ and constant rate for $\alpha = 1$, and for this purpose some plots of the hazard function with various values for the parameters α and λ are presented in Figure 2.

Moreover, note that, $h(x) \sim \lambda$ as $x \to 0$, and $h(x) \sim \lambda\alpha x^{\alpha-1}$ as $x \to \infty$. Hence, we conclude that, the lower tail of the hazard rate function is a constant, while its upper tail is a polynomial which allows for increasing, decreasing and constant hazard rate shapes.

Remark 1. Using the indicator function: $\mathbf{1}_A(x) = 1$ if $x \in A$ and 0 elsewhere, we

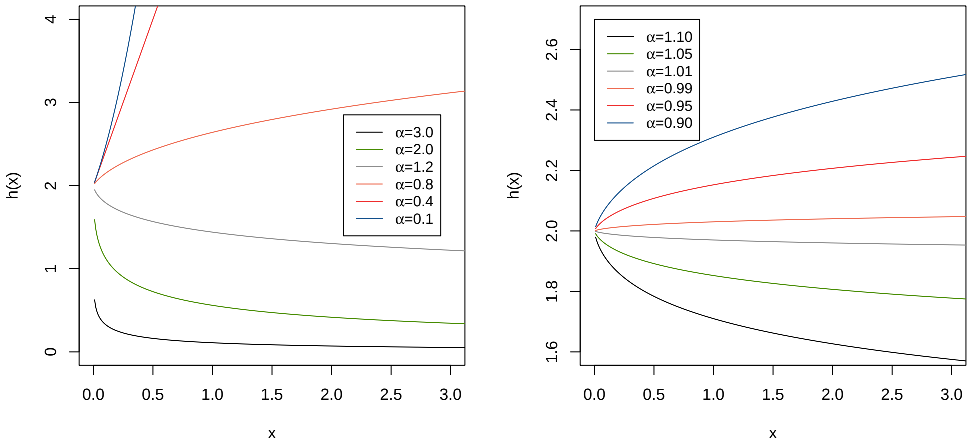

Figure 2. Hazard function shapes for PE distribution for $\lambda = 2$ and considering different values of α.

have following analytic expressions for $F(x)$, $f(x)$ and $h(x)$:

$$F(x) = \left(1 - e^{-\lambda x \frac{x^{\alpha}-1}{x-1}}\right)^{\mathbf{1}_{(0,+\infty)\setminus\{1\}}(x)} \left(1 - e^{-\lambda\alpha}\right)^{\mathbf{1}_{\{1\}}(x)} \mathbf{1}_{(0,+\infty)}(x),$$

$$\begin{aligned} f(x) &= \left(\lambda \frac{\alpha x^{\alpha+1} - (\alpha+1)\, x^{\alpha} + 1}{(x-1)^2} e^{-\lambda x \frac{x^{\alpha}-1}{x-1}}\right)^{\mathbf{1}_{(0,+\infty)\setminus\{1\}}(x)} \times \\ &\quad \times \left(\frac{\lambda\alpha(\alpha+1)}{2} e^{-\lambda\alpha}\right)^{\mathbf{1}_{\{1\}}(x)} \mathbf{1}_{(0,+\infty)}(x), \end{aligned} \tag{3}$$

and

$$h(x) = \left(\lambda \frac{\alpha x^{\alpha+1} - (\alpha+1)\, x^{\alpha} + 1}{(x-1)^2}\right)^{\mathbf{1}_{(0,+\infty)\setminus\{1\}}(x)} \left(\frac{\lambda\alpha(\alpha+1)}{2}\right)^{\mathbf{1}_{\{1\}}(x)} \mathbf{1}_{(0,+\infty)}(x).$$

These analytic expressions will be useful in the next.

3. Mathematical properties

3.1. *Some useful expansions*

The result below presents a polynomial expansion of the cdf $F(x)$ given by (1).

Proposition 3.1. *We have the following expansion, for* $x \in (0, +\infty)\setminus\{1\}$,

$$F(x) = 1 - \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \sum_{j=0}^{\infty} A_{k,\ell,j} \left[(-1)^k x^{k+\alpha\ell+j} \mathbf{1}_{(0,1)}(x) + x^{\alpha\ell-j} \mathbf{1}_{(1,+\infty)}(x)\right],$$

where

$$A_{k,\ell,j} = \binom{k}{\ell}\binom{-k}{j}\frac{1}{k!}(-1)^{\ell+j}\lambda^k.$$

Proof of Proposition 3.1. First of all, let us now investigate an expansion for $e^{-\lambda x \frac{x^\alpha - 1}{x-1}}$ by distinguishing the case $x \in [0,1)$ and the case $x > 1$.

- If $x \in [0,1)$, note that $\frac{x^\alpha-1}{x-1} = (1-x^\alpha)\frac{1}{1-x}$. The exponential and binomial series give

$$\begin{aligned} e^{-\lambda x \frac{x^\alpha - 1}{x-1}} &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\lambda x \frac{x^\alpha - 1}{x-1}\right)^k = \sum_{k=0}^{\infty} \frac{1}{k!}(-\lambda)^k x^k (1-x^\alpha)^k \frac{1}{(1-x)^k} \\ &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{j=0}^{\infty}\binom{k}{\ell}\binom{-k}{j}\frac{1}{k!}(-1)^{\ell+j}(-\lambda)^k x^{k+\alpha\ell+j}. \end{aligned}$$

- If $x > 1$, note that $x\frac{x^\alpha-1}{x-1} = -(1-x^\alpha)\frac{1}{1-\frac{1}{x}}$. It follows from the exponential and binomial series that

$$\begin{aligned} e^{-\lambda x \frac{x^\alpha - 1}{x-1}} &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\lambda x \frac{x^\alpha - 1}{x-1}\right)^k = \sum_{k=0}^{\infty} \frac{1}{k!}\lambda^k (1-x^\alpha)^k \frac{1}{\left(1-\frac{1}{x}\right)^k} \\ &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{j=0}^{\infty}\binom{k}{\ell}\binom{-k}{j}\frac{1}{k!}(-1)^{\ell+j}\lambda^k x^{\alpha\ell-j}. \end{aligned}$$

Hence

$$e^{-\lambda x \frac{x^\alpha - 1}{x-1}} = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{j=0}^{\infty} A_{k,\ell,j}\left((-1)^k x^{k+\alpha\ell+j}\mathbf{1}_{(0,1)}(x) + x^{\alpha\ell-j}\mathbf{1}_{(1,+\infty)}(x)\right).$$

Therefore

$$F(x) = 1 - e^{-\lambda x \frac{x^\alpha - 1}{x-1}} = 1 - \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{j=0}^{\infty} A_{k,\ell,j}\left((-1)^k x^{k+\alpha\ell+j}\mathbf{1}_{(0,1)}(x) + x^{\alpha\ell-j}\mathbf{1}_{(1,+\infty)}(x)\right).$$

This complete the proof of Proposition 3.1. □

The result below presents an expansion of the pdf $f(x)$ given by (2) via polynomial and the exponential function $e^{-\lambda x}$, which will be important to ensure the permutation of sum and integral in several probabilistic quantities.

Proposition 3.2. *We have the following expansion, for* $x \in (0,+\infty)\backslash\{1\}$,

$$f(x) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{j=0}^{\infty} B_{k,\ell,j}\left[(-1)^k R_{k,\ell,j}(x)\mathbf{1}_{(0,1)}(x) + S_{k,\ell,j}(x)\mathbf{1}_{(1,+\infty)}(x)\right]e^{-\lambda x},$$

where

$$B_{k,\ell,j} = \binom{k}{\ell}\binom{-k-2}{j}\frac{1}{k!}(-1)^{\ell+j}\lambda^{k+1}, \tag{4}$$

$$R_{k,\ell,j}(x) = \alpha x^{2k+(\alpha-1)\ell+j+\alpha+1} - (\alpha+1)\,x^{2k+(\alpha-1)\ell+j+\alpha} + x^{2k+(\alpha-1)\ell+j}$$

and

$$S_{k,\ell,j}(x) = \alpha x^{k+(\alpha-1)\ell-j+\alpha-1} - (\alpha+1)\,x^{k+(\alpha-1)\ell-j-2+\alpha} + x^{k+(\alpha-1)\ell-j-2}.$$

Proof of Proposition 3.2. Let us observe that

$$\lambda x\frac{x^\alpha-1}{x-1} = \lambda x\left(\frac{x^\alpha-1}{x-1}-1\right)+\lambda x = \lambda x^2\frac{x^{\alpha-1}-1}{x-1}+\lambda x.$$

Therefore, we can express the pdf $f(x)$ as

$$f(x) = \lambda(\alpha x^{\alpha+1} - (\alpha+1)\,x^\alpha + 1)e^{-\lambda x} \times \frac{1}{(x-1)^2}e^{-\lambda x^2\frac{x^{\alpha-1}-1}{x-1}}.$$

Let us now investigate an expansion for $\frac{1}{(x-1)^2}e^{-\lambda x^2\frac{x^{\alpha-1}-1}{x-1}}$ by distinguishing $x\in[0,1)$ and $x>1$.

- If $x\in[0,1)$, it follows from the exponential and binomial series that

$$\begin{aligned}\frac{1}{(x-1)^2}e^{-\lambda x^2\frac{x^{\alpha-1}-1}{x-1}} &= \sum_{k=0}^{\infty}\frac{1}{k!}\frac{1}{(x-1)^2}\left(-\lambda x^2\frac{x^{\alpha-1}-1}{x-1}\right)^k\\ &= \sum_{k=0}^{\infty}\frac{1}{k!}(-\lambda)^k x^{2k}(1-x^{\alpha-1})^k\frac{1}{(1-x)^{k+2}}\\ &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{j=0}^{\infty}\binom{k}{\ell}\binom{-k-2}{j}\frac{1}{k!}(-1)^{\ell+j}(-\lambda)^k x^{2k+(\alpha-1)\ell+j}.\end{aligned}$$

- If $x>1$, exponential and binomial series give

$$\begin{aligned}\frac{1}{(x-1)^2}e^{-\lambda x^2\frac{x^{\alpha-1}-1}{x-1}} &= \sum_{k=0}^{\infty}\frac{1}{k!}\frac{1}{(x-1)^2}\left(-\lambda x^2\frac{x^{\alpha-1}-1}{x-1}\right)^k\\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\lambda^k x^k(1-x^{\alpha-1})^k x^{-2}\frac{1}{\left(1-\frac{1}{x}\right)^{k+2}}\\ &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{j=0}^{\infty}\binom{k}{\ell}\binom{-k-2}{j}\frac{1}{k!}(-1)^{\ell+j}\lambda^k x^{k+(\alpha-1)\ell-j-2}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{(x-1)^2}e^{-\lambda x\frac{x^\alpha-1}{x-1}} &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{j=0}^{\infty}\binom{k}{\ell}\binom{-k-2}{j}\frac{1}{k!}(-1)^{\ell+j}\lambda^k \\ &\times \left[(-1)^k x^{2k+(\alpha-1)\ell+j}\mathbf{1}_{(0,1)}(x)+x^{k+(\alpha-1)\ell-j-2}\mathbf{1}_{(1,+\infty)}(x)\right].\end{aligned}$$

Owing to this equality, we obtain the desired expansion:

$$\begin{aligned}f(x) &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{j=0}^{\infty}B_{k,\ell,j}\left[(-1)^k x^{2k+(\alpha-1)\ell+j}\mathbf{1}_{(0,1)}(x)+x^{k+(\alpha-1)\ell-j-2}\mathbf{1}_{(1,+\infty)}(x)\right] \\ &\times \left(\alpha x^{\alpha+1}-(\alpha+1)\,x^\alpha+1\right)e^{-\lambda x} \\ &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{j=0}^{\infty}B_{k,\ell,j}\left((-1)^k R_{k,\ell,j}(x)\mathbf{1}_{(0,1)}(x)+S_{k,\ell,j}(x)\mathbf{1}_{(1,+\infty)}(x)\right)e^{-\lambda x}.\end{aligned}$$

This ends the proof of Proposition 3.2. □

3.2. *Moments and moment generating function*

Here and after, we consider a random variable X following the PE(α,λ) distribution with $\alpha>0$ and $\lambda>0$.

We define the upper incomplete gamma function as $\Gamma(s,x)=\int_x^{+\infty}t^{s-1}e^{-t}dt$ and the lower incomplete gamma function as $\gamma(s,x)=\int_0^x t^{s-1}e^{-t}dt$, $s>0$, $x\geq 0$.

Let $r\geq 0$. Using the notations and the result of Proposition 3.2, the r-moments of X is given by

$$E(X^r)=\int_{-\infty}^{\infty}x^r f(x)dx=\sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{j=0}^{\infty}B_{k,\ell,j}\left((-1)^k U_{k,\ell,j,r}+V_{k,\ell,j,r}\right),$$

where $U_{k,\ell,j,r}=\int_0^1 x^r R_{k,\ell,j}(x)e^{-\lambda x}dx$ and $V_{k,\ell,j,r}=\int_1^{+\infty}x^r S_{k,\ell,j}(x)e^{-\lambda x}dx$. We have

$$\begin{aligned}U_{k,\ell,j,r} &= \alpha\int_0^1 x^{r+2k+(\alpha-1)\ell+j+\alpha+1}e^{-\lambda x}dx-(\alpha+1)\int_0^1 x^{r+2k+(\alpha-1)\ell+j+\alpha}e^{-\lambda x}dx \\ &+ \int_0^1 x^{r+2k+(\alpha-1)\ell+j}e^{-\lambda x}dx \\ &= \alpha\frac{\gamma(r+2k+(\alpha-1)\ell+j+\alpha+2,\lambda)}{\lambda^{r+2k+(\alpha-1)\ell+j+\alpha+2}}-(\alpha+1)\,\frac{\gamma(r+2k+(\alpha-1)\ell+j+\alpha+1,\lambda)}{\lambda^{r+2k+(\alpha-1)\ell+j+\alpha+1}} \\ &+ \frac{\gamma(r+2k+(\alpha-1)\ell+j+1,\lambda)}{\lambda^{r+2k+(\alpha-1)\ell+j+1}}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
V_{k,\ell,j,r} &= \alpha \int_1^{+\infty} x^{r+k+(\alpha-1)\ell-j+\alpha-1} e^{-\lambda x} dx - (\alpha+1) \int_1^{+\infty} x^{r+k+(\alpha-1)\ell-j-2+\alpha} e^{-\lambda x} dx \\
&+ \int_1^{+\infty} x^{r+k+(\alpha-1)\ell-j-2} e^{-\lambda x} dx \\
&= \alpha \frac{\Gamma(r+k+(\alpha-1)\ell-j+\alpha,\lambda)}{\lambda^{r+k+(\alpha-1)\ell-j+\alpha}} - (\alpha+1) \frac{\Gamma(r+k+(\alpha-1)\ell-j-1+\alpha,\lambda)}{\lambda^{r+k+(\alpha-1)\ell-j-1+\alpha}} \\
&+ \frac{\Gamma(r+k+(\alpha-1)\ell-j-1,\lambda)}{\lambda^{r+k+(\alpha-1)\ell-j-1}}.
\end{aligned}$$

The moment generating function of X is given by, for $t < \lambda$,

$$M(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x) dx = \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \sum_{j=0}^{\infty} B_{k,\ell,j} \left((-1)^k U^*_{k,\ell,j}(t) + V^*_{k,\ell,j}(t) \right),$$

where $U^*_{k,\ell,j}(t) = \int_0^1 e^{tx} R_{k,\ell,j}(x) e^{-\lambda x} dx$ and $V^*_{k,\ell,j}(t) = \int_1^{+\infty} e^{tx} S_{k,\ell,j}(x) e^{-\lambda x} dx$. We have

$$\begin{aligned}
U^*_{k,\ell,j}(t) &= \alpha \int_0^1 x^{2k+(\alpha-1)\ell+j+\alpha+1} e^{-(\lambda-t)x} dx - (\alpha+1) \int_0^1 x^{2k+(\alpha-1)\ell+j+\alpha} e^{-(\lambda-t)x} dx \\
&+ \int_0^1 x^{2k+(\alpha-1)\ell+j} e^{-(\lambda-t)x} dx \\
&= \alpha \frac{\gamma(2k+(\alpha-1)\ell+j+\alpha+2,\lambda-t)}{(\lambda-t)^{2k+(\alpha-1)\ell+j+\alpha+2}} - (\alpha+1) \frac{\gamma(2k+(\alpha-1)\ell+j+\alpha+1,\lambda-t)}{(\lambda-t)^{2k+(\alpha-1)\ell+j+\alpha+1}} \\
&+ \frac{\gamma(2k+(\alpha-1)\ell+j+1,\lambda-t)}{(\lambda-t)^{2k+(\alpha-1)\ell+j+1}}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
V^*_{k,\ell,j}(t) &= \alpha \int_1^{+\infty} x^{k+(\alpha-1)\ell-j+\alpha-1} e^{-(\lambda-t)x} dx - (\alpha+1) \int_1^{+\infty} x^{k+(\alpha-1)\ell-j-2+\alpha} e^{-(t-\lambda)x} dx \\
&+ \int_1^{+\infty} x^{k+(\alpha-1)\ell-j-2} e^{-(\lambda-t)x} dx \\
&= \alpha \frac{\Gamma(k+(\alpha-1)\ell-j+\alpha,\lambda-t)}{(\lambda-t)^{k+(\alpha-1)\ell-j+\alpha}} - (\alpha+1) \frac{\Gamma(k+(\alpha-1)\ell-j-1+\alpha,\lambda-t)}{(\lambda-t)^{k+(\alpha-1)\ell-j-1+\alpha}} \\
&+ \frac{\Gamma(k+(\alpha-1)\ell-j-1,\lambda-t)}{(\lambda-t)^{k+(\alpha-1)\ell-j-1}}.
\end{aligned}$$

3.3. *On other means and moments*

The following result proposes an expansion of the primitive $\int_0^t x^r f(x) dx$, with $t > 0$. It will be useful in the next.

Proposition 3.3. *For any $r \geq 0$ and $t > 0$, we have*

$$\int_0^t x^r f(x)dx = \sum_{k=0}^{\infty}\sum_{\ell=0}^{k}\sum_{j=0}^{\infty} B_{k,\ell,j}\left((-1)^k U^{\circ}_{k,\ell,j,r}(t) + V^{\circ}_{k,\ell,j,r}(t)\mathbf{1}_{[1,+\infty)}(t)\right), \tag{5}$$

where $B_{k,\ell,j}$ is defined by (4),

$$\begin{aligned} U^{\circ}_{k,\ell,j,r}(t) &= \alpha\frac{\gamma(r+2k+(\alpha-1)\ell+j+\alpha+2, \lambda\min(t,1))}{\lambda^{r+2k+(\alpha-1)\ell+j+\alpha+2}} \\ &- (\alpha+1)\frac{\gamma(r+2k+(\alpha-1)\ell+j+\alpha+1, \lambda\min(t,1))}{\lambda^{r+2k+(\alpha-1)\ell+j+\alpha+1}} \\ &+ \frac{\gamma(r+2k+(\alpha-1)\ell+j+1, \lambda\min(t,1))}{\lambda^{r+2k+(\alpha-1)\ell+j+1}} \end{aligned}$$

and

$$\begin{aligned} V^{\circ}_{k,\ell,j,r}(t) &= \alpha\frac{\Gamma(r+k+(\alpha-1)\ell-j+\alpha, \lambda t)}{\lambda^{r+k+(\alpha-1)\ell-j+\alpha}} - (\alpha+1)\frac{\Gamma(r+k+(\alpha-1)\ell-j-1+\alpha, \lambda t)}{\lambda^{r+k+(\alpha-1)\ell-j-1+\alpha}} \\ &+ \frac{\Gamma(r+k+(\alpha-1)\ell-j-1, \lambda t)}{\lambda^{r+k+(\alpha-1)\ell-j-1}}. \end{aligned}$$

The proof of Proposition 3.3 follows from Proposition 3.2 with $U^{\circ}_{k,\ell,j,r}(t) = \int_0^{\min(t,1)} x^r R_{k,\ell,j}(x)e^{-\lambda x}dx$ and $V^{\circ}_{k,\ell,j,r}(t) = \int_1^t x^r S_{k,\ell,j}(x)e^{-\lambda x}dx$. The expressions of these integrals in terms of upper incomplete gamma function and the lower incomplete gamma function is obtained proceeding as Subsection 3.2.

Several crucial conditional moments use the integral $\int_0^t x^r f(x)dx$ for various values of r. The most useful of them are presented below. For any $t > 0$,

- The r-th conditional moments of X is given by,

$$E(X^r \mid X > t) = \frac{1}{1-F(t)}\int_t^{+\infty} x^r f(x)dx = \frac{1}{1-F(t)}\left(E(X^r) - \int_0^t x^r f(x)dx\right).$$

- The r-th reversed moments of X is given by

$$E(X^r \mid X \leq t) = \frac{1}{F(t)}\int_0^t x^r f(x)dx.$$

Let $\mu = E(X)$.

- The mean deviations of X about μ is given by

$$\delta = E(|X-\mu|) = 2\mu F(\mu) - 2\int_0^{\mu} xf(x)dx$$

- The mean deviations of X about the median M is given by

$$\eta = E(|X-M|) = \mu - 2\int_0^{M} xf(x)dx.$$

Residual life parameters can be also determined using $E(X^r)$ and $\int_0^t x^r f(x)dx$ for several values of r. In particular,

- The mean residual life is defined as

$$K(t) = E(X - t \mid X > t) = \frac{1}{S(t)}\left(E(X) - \int_0^t xf(x)dx\right) - t$$

and the variance residual life is given by

$$V(t) = Var(X-t \mid X > t) = \frac{1}{S(t)}\left(E(X^2) - \int_0^t x^2 f(x)dx\right) - t^2 - 2tK(t) - [K(t)]^2.$$

- The mean reversed residual life is defined as

$$L(t) = E(t - X \mid X \leq t) = t - \frac{1}{F(t)}\int_0^t xf(x)dx$$

and the variance reversed residual life is given by

$$W(t) = Var(t - X \mid X \leq t) = 2tL(t) - [L(t)]^2 - t^2 + \frac{1}{F(t)}\int_0^t x^2 f(x)dx.$$

3.4. *Stress-strength reliability*

Let X be a random variable following the PE(α, λ_1) distribution with pdf denoted by $f_X(x)$ and Y be a random variable following the PE(α, λ_2) distribution with cdf denoted by $F_Y(x)$, with $\alpha > 0$, $\lambda_1 > 0$ and $\lambda_2 > 0$. Then the stress-strength reliability is defined by $R = P(X > Y)$. Since the integral on $(0, +\infty)$ of the pdf (2) with parameters $(\alpha, \lambda_1 + \lambda_2)$ denoted by $f_*(x)$ is equal to one, we have

$$\begin{aligned} R &= P(X > Y) = \int_0^{+\infty} f_X(x)F_Y(x)dx \\ &= 1 - \int_0^{+\infty} \lambda_1 \frac{\alpha x^{\alpha+1} - (\alpha+1)\, x^{\alpha} + 1}{(x-1)^2} e^{-(\lambda_1+\lambda_2)x\frac{x^{\alpha}-1}{x-1}} dx \\ &= 1 - \frac{\lambda_1}{\lambda_1 + \lambda_2}\int_0^{+\infty} f_*(x)dx = \frac{\lambda_2}{\lambda_1 + \lambda_2}. \end{aligned}$$

This result is of interest in a parametric estimation context; only λ_1 and λ_2 need to be estimated to have an estimation of R (the maximum likelihood estimators for λ_1 and λ_2 yield the maximum likelihood estimator for R by the plug-in method).

3.5. *Order statistics distributions*

We now introduce order statistics and present some of their properties in our mathematical framework (general results can be found, for instance, in David and Nagaraja (2003)). Let $X_1, X_2, \ldots, X_n$ be n i.i.d. random variables following the PE(α, λ) distribution with $\alpha > 0$ and $\lambda > 0$. Let us consider its order statistics as $X_{1:n}$, $X_{2:n}, \ldots, X_{n:n}$, *i.e.*, for any $i \in \{1, \ldots, n\}$, $X_{i:n} \in \{X_1, \ldots, X_n\}$ with $X_{1:n} \leq \ldots \leq X_{n:n}$

(so $X_{1:n} = X_{(1)} = \inf(X_1, \ldots, X_n)$ and $X_{n:n} = X_{(n)} = \sup(X_1, \ldots, X_n)$). Let us now present some important distributions related to $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$. Some important of them involving our distribution are presented below. The general expression of the cdf of $X_{i:n}$ is given by

$$F_{X_{i:n}}(x) = \frac{n!}{(i-1)!\,(n-i)!} \sum_{k=0}^{n-i} \binom{n-i}{k} \frac{(-1)^k}{i+k} \left[F\left(x\right)\right]^{i+k}, \quad x \in \mathbb{R}.$$

Hence, for any $x \in (0, +\infty)/\{1\}$, we have

$$F_{X_{i:n}}(x) = \frac{n!}{(i-1)!\,(n-i)!} \sum_{k=0}^{n-i} \binom{n-i}{k} \frac{(-1)^k}{i+k} \left[1 - e^{-\lambda x \frac{x^\alpha - 1}{x-1}}\right]^{i+k}.$$

For the case $x = 1$, we have

$$F_{X_{i:n}}(1) = \frac{n!}{(i-1)!\,(n-i)!} \sum_{k=0}^{n-i} \binom{n-i}{k} \frac{(-1)^k}{i+k} \left[1 - e^{-\lambda\alpha}\right]^{i+k}.$$

The general expression of the pdf of $X_{i:n}$ is given by

$$f_{X_{i:n}}(x) = \frac{n!}{(i-1)!\,(n-i)!}[F(x)]^{i-1}[1 - F(x)]^{n-i} f(x), \quad x \in \mathbb{R}.$$

Thus, for any $x \in (0, +\infty)/\{1\}$, we have

$$f_{X_{i:n}}(x) = \frac{n!}{(i-1)!\,(n-i)!}\lambda \left[1 - e^{-\lambda x \frac{x^\alpha - 1}{x-1}}\right]^{i-1} \frac{\alpha x^{\alpha+1} - (\alpha+1)\,x^\alpha + 1}{(x-1)^2} e^{-\lambda(n-i+1)x\frac{x^\alpha - 1}{x-1}}.$$

For the case $x = 1$, we have

$$f_{X_{i:n}}(1) = \frac{n!}{(i-1)!\,(n-i)!} \left[1 - e^{-\lambda\alpha}\right]^{i-1} \frac{\lambda\alpha(\alpha+1)}{2} e^{-\lambda\alpha(n-i+1)}.$$

As in Remark 1, one can express $f_{X_{i:n}}(x)$ in a one form using $\mathbf{1}_{(0,+\infty)\setminus\{1\}}(x)$ and $\mathbf{1}_{\{1\}}(x)$.

For $i < j$ and $x_i < x_j$, the general expression of the joint pdf of $(X_{i:n}, X_{j:n})$ is given by

$$\begin{aligned} f_{(X_{i:n},X_{j:n})}(x_i, x_j) &= \frac{n!}{(i-1)!\,(n-j)!\,(j-i-1)}[F(x_i)]^{i-1}[F(x_j) - F(x_i)]^{j-i-1} \times \\ &\times [1 - F(x_j)]^{n-j} f(x_i) f(x_j). \end{aligned}$$

For the case $(x_i, x_j) \in (0, +\infty)^2/\{(1,1)\}$, we have

$$f_{(X_{i:n},X_{j:n})}(x_i, x_j) = \frac{n!}{(i-1)!\,(n-j)!\,(j-i-1)}\lambda^2 \left[1 - e^{-\lambda x_i \frac{x_i^\alpha - 1}{x_i - 1}}\right]^{i-1} \left[e^{-\lambda x_i \frac{x_i^\alpha - 1}{x_i - 1}} - e^{-\lambda x_j \frac{x_j^\alpha - 1}{x_j - 1}}\right]^{j-i-1}$$
$$\times \quad \frac{(\alpha x_i^{\alpha+1} - (\alpha+1)\, x_i^\alpha + 1)(\alpha x_j^{\alpha+1} - (\alpha+1)\, x_j^\alpha + 1)}{(x_i - 1)^2\,(x_j - 1)^2} e^{-\lambda x_i \frac{x_i^\alpha - 1}{x_i - 1} - \lambda(n-j+1) x_j \frac{x_j^\alpha - 1}{x_j - 1}}.$$

The expression of $f_{(X_{i:n},X_{j:n})}(x_i, x_j)$ for $(x_i, x_j) \notin (0, +\infty)^2/\{(1,1)\}$ can be set in a similar manner, using the values of $F(1)$ and $f(1)$.

In the following proposition, we provide the asymptotic distributions of the extreme values $X_{1:n}$ and $X_{n:n}$, and show that they are exponential and Gumbel distributions, respectively, which adapt the standards of the asymptotic distribution of extremes.

Proposition 3.4. *Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. random variables following the $PE(\alpha, \lambda)$ distribution, then*

- $\left(nX_{(1)}\right)_{n\geq 1}$ *converges in distribution to a random variable X having the exponential distribution of parameter λ.*
- $\left(X_{(n)}^\alpha - \frac{\log(n)}{\lambda}\right)_{n\geq 1}$ *converges in distribution to a random variable X having the Gumbel distribution of parameters 0 and $\frac{1}{\lambda}$.*

Proof of Proposition 3.4. Let us prove the two points in turn.

- Since $X_1, \ldots, X_n$ are i.i.d., using standard mathematical arguments, for $x \in (0, +\infty)/\{n\}$, the cdf of $nX_{(1)}$ is given by

$$F_{nX_{(1)}}(x) = 1 - \left(1 - F\left(\frac{x}{n}\right)\right)^n = 1 - e^{-\lambda x \frac{(\frac{x}{n})^\alpha - 1}{\frac{x}{n} - 1}}.$$

  So $\lim_{n\to+\infty} F_{nX_{(1)}}(x) = 1 - e^{-\lambda x} = F_X(x)$. This ends the proof of the first point.
- Again, since $X_1, \ldots, X_n$ are i.i.d., using standard mathematical arguments, for $x \in \left(-\frac{\log(n)}{\lambda}, +\infty\right) / \left\{1 - \frac{\log(n)}{\lambda}\right\}$, the cdf of $X_{(n)}^\alpha - \frac{\log(n)}{\lambda}$ is given by

$$F_{X_{(n)}^\alpha - \frac{\log(n)}{\lambda}}(x) = \left(F\left(\left(x + \frac{\log(n)}{\lambda}\right)^{\frac{1}{\alpha}}\right)\right)^n = \left(1 - e^{-\lambda\left(x + \frac{\log(n)}{\lambda}\right)^{\frac{1}{\alpha}} \frac{x + \frac{\log(n)}{\lambda} - 1}{\left(x + \frac{\log(n)}{\lambda}\right)^{\frac{1}{\alpha}} - 1}}\right)^n.$$

  Therefore, when $n \to +\infty$, several equivalences give

$$F_{X_{(n)}^\alpha - \frac{\log(n)}{\lambda}}(x) \sim e^{n \log\left(1 - \frac{e^{-\lambda x}}{n}\right)} \sim e^{-e^{-\lambda x}}.$$

  Hence $\lim_{n\to+\infty} F_{X_{(n)}^\alpha - \frac{\log(n)}{\lambda}}(x) = e^{-e^{-\lambda x}} = F_X(x)$. The second point is proved.

□

3.6. *Stochastic ordering*

The ordering mechanism in life time distributions can be illustrate by the concept of stochastic ordering. See, for instance, (Shaked et al. 1995). This subsection presents the basic of this concept, with a result using the proposed distribution. A random variable X is said to be stochastically smaller than a random variable Y in the

- stochastic order ($X \leq_{st} Y$) if the associated cdfs satisfy: $F_X(x) \geq F_Y(x)$ for all x.
- hazard rate order ($X \leq_{hr} Y$) if the associated hrfs satisfy: $h_X(x) \geq h_Y(x)$ for all x.
- likelihood ratio order ($X \leq_{lr} Y$) if the ratio of the associated pdfs given by $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

Important equivalences exist; when the supports of X and Y have a common finite left end-point, then we have: $X \leq_{lr} Y \quad \Rightarrow \quad X \leq_{hr} Y \quad \Rightarrow \quad X \leq_{st} Y$.

Proposition 3.5. *Let X be a random variable following the PE(α, λ_1) distribution with pdf denoted by $f_X(x)$ and Y be a random variable following the PE(α, λ_2) distribution with pdf denoted by $f_Y(x)$, with $\alpha > 0$, $\lambda_1 > 0$ and $\lambda_2 > 0$. If $\lambda_1 > \lambda_2$, then we have $X \leq_{lr} Y$.*

Proof of Proposition 3.5. For any $x \in (0, +\infty)/\{1\}$, we have

$$\frac{f_X(x)}{f_Y(x)} = \frac{\lambda_1}{\lambda_2} e^{(\lambda_2 - \lambda_1) x \frac{x^\alpha - 1}{x - 1}}$$

and $\frac{f_X(1)}{f_Y(1)} = \frac{\lambda_1}{\lambda_2} e^{(\lambda_2 - \lambda_1)\alpha}$. As mentioned in Introduction, the function $H(x) = x\frac{x^\alpha - 1}{x-1}$ is increasing function of x. Indeed, we have $H'(x) = \frac{1 + \alpha x^{\alpha+1} - (1+\alpha) x^\alpha}{(x-1)^2}$ and a study of function shows that $1 + \alpha x^{\alpha+1} - (1+\alpha) x^\alpha \geq 0$ for any $\alpha > 0$. Hence, if $\lambda_1 > \lambda_2$, $\frac{f_X(x)}{f_Y(x)}$ decreases in x and $X \leq_{lr} Y$. Proposition 3.5 is proved. □

Remark 2. One can prove that $\lambda_1 > \lambda_2$ implies that $X \leq_{hr} Y$,which follows immediately from the definition (2): for any $x \in (0, +\infty)/\{1\}$, we have $h_X(x) = \lambda_1 \frac{\alpha x^{\alpha+1} - (\alpha+1) x^\alpha + 1}{(x-1)^2} \geq \lambda_2 \frac{\alpha x^{\alpha+1} - (\alpha+1) x^\alpha + 1}{(x-1)^2} = h_Y(x)$, and for $x = 1$, we have $h_X(1) = \frac{\lambda_1 \alpha(\alpha+1)}{2} \geq \frac{\lambda_2 \alpha(\alpha+1)}{2} = h_Y(1)$.

3.7. *Record values distributions*

Let us now focus our attention on record values and present some of their properties in our mathematical context (general results can be found, for instance, in Ahsanullah (1995)). Let $X_1, X_2, \ldots, X_n$ be n i.i.d. random variables following the PE(α, λ) distribution with $\alpha > 0$ and $\lambda > 0$. We define a sequence of record times $U(n)$ as follows: $U(1) = 1$, $U(n) = \min\{j;\ j > U(n-1),\ X_j > X_{U(n-1)}\}$ for $n \geq 2$. We define the i-th upper record value by $R_i = X_{U(i)}$, with $R_1 = X_1$. The general expression of the cdf of R_i is given by

$$F_{R_i}(x) = 1 - (1 - F(x)) \sum_{k=0}^{i-1} \frac{[-\log(1 - F(x))]^k}{k!}, \quad x \in \mathbb{R}.$$

Hence, for any $x \in (0, +\infty)/\{1\}$, we have

$$F_{R_i}(x) = 1 - e^{-\lambda x \frac{x^\alpha - 1}{x-1}} \sum_{k=0}^{i-1} \frac{1}{k!} \left[\lambda x \frac{x^\alpha - 1}{x - 1}\right]^k.$$

For the case $x = 1$, we have

$$F_{R_i}(1) = 1 - e^{-\lambda\alpha} \sum_{k=0}^{i-1} \frac{(\lambda\alpha)^k}{k!}.$$

The general expression of the pdf of R_i is given by

$$f_{R_i}(x) = \frac{[-\log(1 - F(x))]^{i-1}}{(i-1)!} f(x), \quad x \in \mathbb{R}.$$

Hence, for any $x \in (0, +\infty)/\{1\}$, we have

$$f_{R_i}(x) = \frac{1}{(i-1)!} \lambda^i \frac{x^{i-1}(x^\alpha - 1)^{i-1}(\alpha x^{\alpha+1} - (\alpha + 1)\, x^\alpha + 1)}{(x-1)^{i+1}} e^{-\lambda x \frac{x^\alpha - 1}{x-1}}.$$

Note that, for $x = 1$, we have

$$f_{R_i}(1) = \frac{1}{(i-1)!} (\lambda\alpha)^i \frac{\alpha + 1}{2} e^{-\lambda\alpha}.$$

The general expression of the joint pdf of $(R_1, \ldots, R_n)$ is given by, for $(x_1, \ldots, x_n) \in \mathbb{R}^n$ with $x_1 < \ldots < x_n$,

$$f_{(R_1,\ldots,R_n)}(x_1, \ldots, x_n) = f(x_n) \prod_{k=1}^{n-1} h(x_k).$$

For the case $(x_1, \ldots, x_n) \in (0, +\infty)^n/\{(1, \ldots, 1)\}$, we have

$$f_{(R_1,\ldots,R_n)}(x_1, \ldots, x_n) = \lambda^{n+1} \frac{\alpha x_n^{\alpha+1} - (\alpha + 1)\, x_n^\alpha + 1}{(x_n - 1)^2} e^{-\lambda x_n \frac{x_n^\alpha - 1}{x_n - 1}} \prod_{k=1}^{n-1} \frac{\alpha x_k^{\alpha+1} - (\alpha + 1)\, x_k^\alpha + 1}{(x_k - 1)^2}.$$

The expression of $f_{(R_1,\ldots,R_n)}(x_1, \ldots, x_n)$ for $(x_1, \ldots, x_n) \notin (0, +\infty)^n/\{(1, \ldots, 1)\}$ can be set in a similar manner, using the values of $f(1)$ and $h(1)$.

For $i < j$ and $x_i < x_j$, the general expression of the joint pdf of (R_i, R_j) is given by

$$f_{(R_i,R_j)}(x_i, x_j) = \frac{[-\log(1 - F(x_i))]^{i-1}}{(i-1)!} \frac{\left[\log\left(\frac{1-F(x_i)}{1-F(x_j)}\right)\right]^{j-i-1}}{(j-i-1)!} h(x_i) f(x_j).$$

For the case $(x_i, x_j) \in (0, +\infty)^2/\{(1, 1)\}$, we have

$$
\begin{aligned}
f_{(R_i,R_j)}(x_i,x_j) &= \frac{[-\log(1-F(x_i))]^{i-1}}{(i-1)!}\frac{\left[\log\left(\frac{1-F(x_i)}{1-F(x_j)}\right)\right]^{j-i-1}}{(j-i-1)!}h(x_i)f(x_j)\\
&= \frac{1}{(i-1)!}\lambda^j\left[x_i\frac{x_i^\alpha-1}{x_i-1}\right]^{i-1}\frac{1}{(j-i-1)!}\left[x_j\frac{x_j^\alpha-1}{x_j-1}-x_i\frac{x_i^\alpha-1}{x_i-1}\right]^{j-i-1}\\
&\times \frac{(\alpha x_i^{\alpha+1}-(\alpha+1)\,x_i^\alpha+1)(\alpha x_j^{\alpha+1}-(\alpha+1)\,x_j^\alpha+1)}{(x_i-1)^2\,(x_j-1)^2}e^{-\lambda x_j\frac{x_j^\alpha-1}{x_j-1}}.
\end{aligned}
$$

The expression of $f_{(R_i,R_j)}(x_i,x_j)$ for $(x_i,x_j)\notin(0,+\infty)^2/\{(1,1)\}$ can be set in a similar manner, using the values of $F(1)$, $f(1)$ and $h(1)$.

4. Inference

In this section, we discuss different estimation methods for the parameters of the PE distribution.

4.1. *Weighted least-square estimate*

Let $t_{(1)},t_{(2)},\cdots,t_{(n)}$ be the order statistics of the random sample of size n from the cdf $F(x)=F(x|\alpha,\lambda)$. The weighted least-squares estimates (WLSE), $\widehat{\alpha}_{WLSE}$ and $\widehat{\lambda}_{WLSE}$, can be obtained by minimizing

$$
W(\alpha,\lambda)=\sum_{i=1}^{n}\frac{(n+1)^2\,(n+2)}{i\,(n-i+1)}\left[F\left(t_{(i)}\mid\alpha,\lambda\right)-\frac{i}{n+1}\right]^2.
$$

These estimates can also be obtained by solving the non-linear equations:

$$
\begin{aligned}
&\sum_{i=1}^{n}\frac{(n+1)^2\,(n+2)}{i\,(n-i+1)}\left[F\left(x_{(i)}\mid\alpha,\lambda\right)-\frac{i}{n+1}\right]\Delta_1\left(x_{(i)}\mid\alpha,\lambda\right)=0,\\
&\sum_{i=1}^{n}\frac{(n+1)^2\,(n+2)}{i\,(n-i+1)}\left[F\left(x_{(i)}\mid\alpha,\lambda\right)-\frac{i}{n+1}\right]\Delta_2\left(x_{(i)}\mid\alpha,\lambda\right)=0,
\end{aligned}
$$

where

$$
\Delta_1\left(x_{(i)}\mid\alpha,\lambda\right)=\begin{cases}\frac{\lambda x^{\alpha+1}\log(x)e^{-\lambda x\frac{x^\alpha-1}{x-1}}}{x-1}, & x\in(0,+\infty)\backslash\{1\},\\ \lambda e^{-\lambda\alpha}, & x=1\end{cases}
$$

and

$$
\Delta_2\left(x_{(i)}\mid\alpha,\lambda\right)=\begin{cases}\frac{x(x^\alpha-1)e^{-\lambda x\frac{x^\alpha-1}{x-1}}}{x-1}, & x\in(0,+\infty)\backslash\{1\},\\ \alpha e^{-\lambda\alpha}, & x=1.\end{cases}
$$

4.2. *Maximum likelihood estimation*

Let $X_1, X_2, \ldots, X_n$ be a random sample with common distribution the PE(α, λ) distribution with $\alpha > 0$ and $\lambda > 0$. Let $\boldsymbol{\theta} = (\alpha, \lambda)$ be the parameter vector and $x_1, x_2, \ldots, x_n$ be the observed values. Then the likelihood function associated to $x_1, \ldots, x_n$ is given by

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} \left(\lambda \frac{\alpha x_i^{\alpha+1} - (\alpha+1)\, x_i^{\alpha} + 1}{(x_i - 1)^2} e^{-\lambda x_i \frac{x_i^{\alpha}-1}{x_i-1}} \right)^{\mathbf{1}_{(0,+\infty)\setminus\{1\}}(x_i)} \left(\frac{\lambda\alpha(\alpha+1)}{2} e^{-\lambda\alpha} \right)^{\mathbf{1}_{\{1\}}(x_i)}.$$

For the set of simplicity, let us set $u_i = \mathbf{1}_{(0,+\infty)\setminus\{1\}}(x_i)$, $v_i = \mathbf{1}_{\{1\}}(x_i)$, and $\sum\limits_{i=1}^{n} u_i + \sum\limits_{i=1}^{n} v_i = n$. The log-likelihood function can be expressed as

$$\begin{aligned} \ell(\boldsymbol{\theta}) = \log(\lambda) \sum_{i=1}^{n} u_i &+ \sum_{i=1}^{n} u_i \log\left(\frac{\alpha x_i^{\alpha+1} - (\alpha+1)\, x_i^{\alpha} + 1}{(x_i - 1)^2} \right) \\ &- \lambda \sum_{i=1}^{n} u_i x_i \frac{x_i^{\alpha} - 1}{x_i - 1} + \sum_{i=1}^{n} v_i \left(\log(\lambda) + \log(\alpha) + \log(\alpha+1) - \log(2) \right). \end{aligned}$$

The nonlinear log-likelihood equations $\frac{\partial \ell(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = 0$ are given by

$$\begin{aligned} \frac{\partial \ell(\boldsymbol{\theta})}{\partial \alpha} &= \sum_{i=1}^{n} u_i x_i^{\alpha} \frac{(x_i - 1)[1 + \alpha \log(x_i)] - \log(x_i)}{\alpha x_i^{\alpha+1} - (\alpha+1)\, x_i^{\alpha} + 1} - \lambda \sum_{i=1}^{n} u_i x_i \frac{x_i^{\alpha}}{x_i - 1} \log(x_i) \\ &+ \left(\frac{1}{\alpha} + \frac{1}{\alpha+1} \right) \sum_{i=1}^{n} v_i = 0, \end{aligned} \tag{6}$$

and

$$\frac{\partial \ell(\boldsymbol{\theta})}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} u_i x_i \frac{x_i^{\alpha} - 1}{x_i - 1} = 0. \tag{7}$$

Note that after some algebraic manipulations in (7) we have that

$$\lambda = \frac{n}{\sum_{i=1}^{n} u_i x_i \frac{x_i^{\alpha}-1}{x_i-1}}. \tag{8}$$

Replacing (8) in (6) the maximum likelihood estimates of α and λ are determined by solving the one linear equation (6). Since it does not admit any explicit solution, numerical procedures can be used. Under mild conditions the maximum likelihood estimators are asymptotically normal, with an asymptotic variance-covariance matrix depending on the Fisher information matrix. Crucial quantities to determine the entries of this matrix are the second partial derivatives of the log-likelihood function given by

$$\begin{aligned}
\frac{\partial \ell^2(\boldsymbol{\theta})}{\partial \alpha^2} &= \sum_{i=1}^{n} u_i x_i^{\alpha} \frac{\log(x_i)((x_i-1)[2+\alpha\log(x_i)]-\log(x_i))(\alpha x_i^{\alpha+1}-(\alpha+1)\,x_i^{\alpha}+1)}{(\alpha x_i^{\alpha+1}-(\alpha+1)\,x_i^{\alpha}+1)^2} \\
&- \sum_{i=1}^{n} u_i x_i^{\alpha} \frac{x_i^{\alpha}[(x_i-1)[1+\alpha\log(x_i)]-\log(x_i)]^2}{(\alpha x_i^{\alpha+1}-(\alpha+1)\,x_i^{\alpha}+1)^2} \\
&- \lambda \sum_{i=1}^{n} u_i x_i \frac{x_i^{\alpha}}{x_i-1}(\log(x_i))^2 - \left(\frac{1}{\alpha^2}+\frac{1}{(\alpha+1)^2}\right)\sum_{i=1}^{n} v_i,
\end{aligned}$$

and

$$\frac{\partial \ell^2(\boldsymbol{\theta})}{\partial \lambda^2} = -\frac{n}{\lambda^2}, \qquad \frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \lambda \partial \alpha} = -\sum_{i=1}^{n} u_i x_i \frac{x_i^{\alpha}}{x_i-1}\log(x_i).$$

Bias corrected MLEs Reduce the bias of the MLEs is an important procedure. A possibility is to consider the bootstrap resampling method proposed by Efron (1979) in order to obtain bias corrected MLEs. The basic idea is to generate pseudo-samples from the original to estimate the bias of the MLEs. Then, the bias-corrected MLEs (CMLE) is obtained by removing the estimated bias in relation to the original MLEs.

Let $\mathbf{x} = (x_1, \ldots, x_n)^\top$ be a random sample selected from cdf $F(x)$. The pseudo-samples $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)^\top$ are obtained from the original sample $\mathbf{x}$ through resampling with replacement. Then, the bootstrap bias estimate is given by

$$\hat{B}_{F_{\hat{\theta}}}(\hat{\theta}, \theta) = E_{F_{\hat{\theta}}}[\hat{\theta}^*] - \hat{\theta},$$

where the bootstrap expectations $E_{F_{\hat{\theta}}}[\hat{\theta}^*]$ can approximately by

$$\hat{\theta}^{*(\cdot)} = \frac{1}{M}\sum_{i=1}^{M} \hat{\theta}^{*(i)}$$

and M is number of replications through resampling approach. Finally, the bias corrected estimators obtained through by bootstrap resampling method that is given by

$$\theta^B = \hat{\theta} - \hat{B}_F(\hat{\theta}, \theta) = 2\hat{\theta} - \hat{\theta}^{*(\cdot)}.$$

In our case, we have θ^B denoted by $\hat{\theta}_{BOOT} = (\hat{\alpha}_{BOOT}, \hat{\lambda}_{BOOT})^\top$.

5. Simulation analysis

In this section a simulation study is presented to compare the efficiency of the maximum likelihood method. The comparison is performed by computing the Bias and the mean square errors (MSE) given by

$$\text{Bias}(\alpha_i)=\frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i-\alpha)\ ,\quad \text{MSE}(\alpha_i)=\frac{1}{N}\sum_{j=1}^{N}(\hat{\alpha}_i-\alpha)^2,$$

$$\text{Bias}(\lambda_i)=\frac{1}{N}\sum_{i=1}^{N}(\hat{\lambda}_i-\lambda)\ ,\quad \text{MSE}(\lambda_i)=\frac{1}{N}\sum_{j=1}^{N}(\hat{\lambda}_i-\lambda)^2,$$

where N is the number of estimates obtained through the MLE, CMLE and WLSE. The 95% coverage probability of the asymptotic confidence intervals are also evaluated. Here we expect that the most efficient estimation method returns both Bias and MSE closer to zero. Additionally, for a large number of experiments, using a 95% confidence level, the frequencies of intervals that covered the true values of α and λ should be closer to 95%. The CMLE is obtained by considering $m = 1000$. The programs can be obtained, upon request. The values of the PE were generated considering the following algorithm:

(1) Generate $U_i \sim \text{Uniform}(0, 1), i = 1, \ldots, n$;
(2) Find x_i from the solution of $F(x_i) - u_i = 0\ , i = 1, \ldots, n$;

The simulation study is performed under the assumption $(0.7, 0.4)$ and $(1.5, 0.4)$, $N = 10,000$ and $n = (20, 40, \ldots, 300)$. The chosen values allow us to obtain data with both increasing ($\alpha < 1$) and decreasing ($\alpha > 1$) hazard rate. It is important to point out that, the results of this simulation study were similar for different choices of α and λ. Figures 3 and 4 present the Bias, the MSE and the coverage probability with a 95% confidence level of the estimates obtained through the MLE, CMLE and the WLSE for different samples of size.

From the obtained results, we can conclude that as there is an increase of n both Bias and MSE tend to zero, i.e., the estimators are asymptotic efficiency. Moreover, the coverage probability of the confidence levels tend to the nominal value assumed 0.95. Note that we have not computed the coverage probabilities of the WLSE since we can not construct the WLSE confidence levels using the asymptotic theory similar that was used in the MLEs. Nevertheless, comparing the estimation procedures we observed that the CMLE returned better estimates than obtained from the other estimation methods in terms of Bias and MSE. Therefore, the CMLE showed to be a good estimator for the parameters of the PE distribution.

6. Application to real data

In this section, we illustrate the flexibility of our proposed distribution by considering four real data sets. The results obtained from the PE distribution are compared with ones of the Weibull, Gamma, Lognormal and the EE distributions, and nonparametric survival function.

To discriminate the different models we will consider the most common discrimination criterion. Let k be the number of parameters to be fitted and $\hat{\boldsymbol{\theta}}$ the MLEs of $\boldsymbol{\theta}$, the discrimination criterion methods are respectively: Akaike information criterion AIC $= -2l(\hat{\boldsymbol{\theta}};\boldsymbol{x})+2k$; Corrected Akaike information criterion AICC $=$ AIC $+(2\,k\,(k+1))/(n-k-1)$; Hannan-Quinn information criterion HQIC $= -2\,l(\hat{\boldsymbol{\theta}};\boldsymbol{x}) + 2\,k\,\log(\log(n))$; Consistent Akaike information criterion CAIC $= -2\,l(\hat{\boldsymbol{\theta}};\boldsymbol{x}) + k\,(\log(n) + 1)$. The best

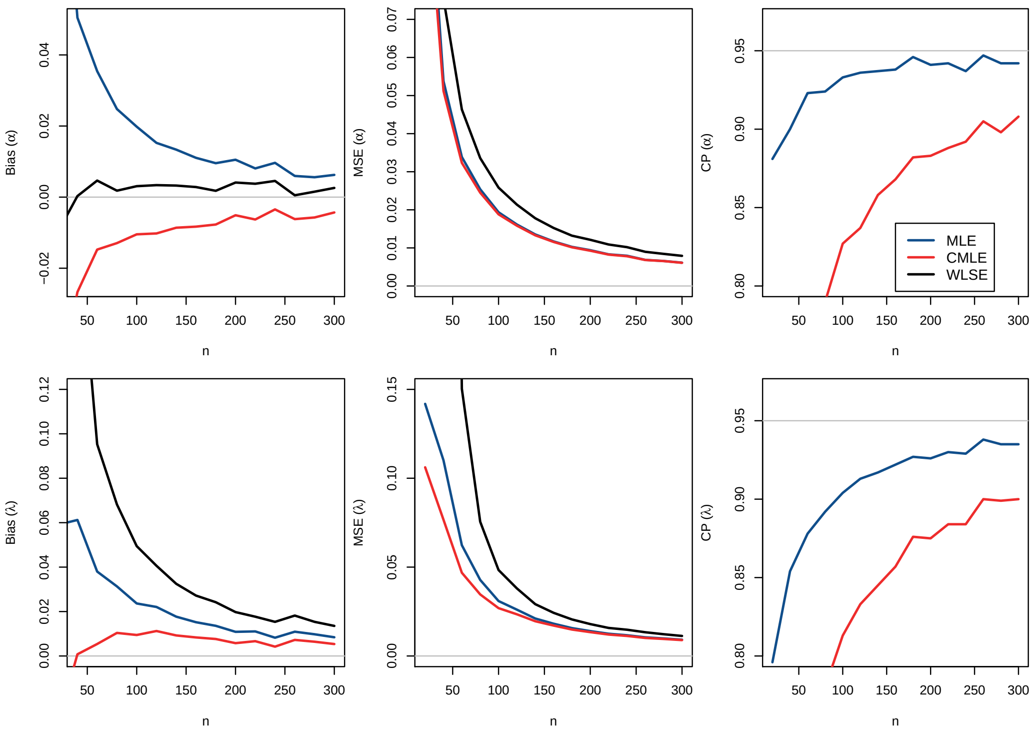

Figure 3. Bias, MSEs related from the estimates of $\alpha = 0.7$ and $\lambda = 0.4$ for N simulated samples under the MLE, CMLE and WLSE.

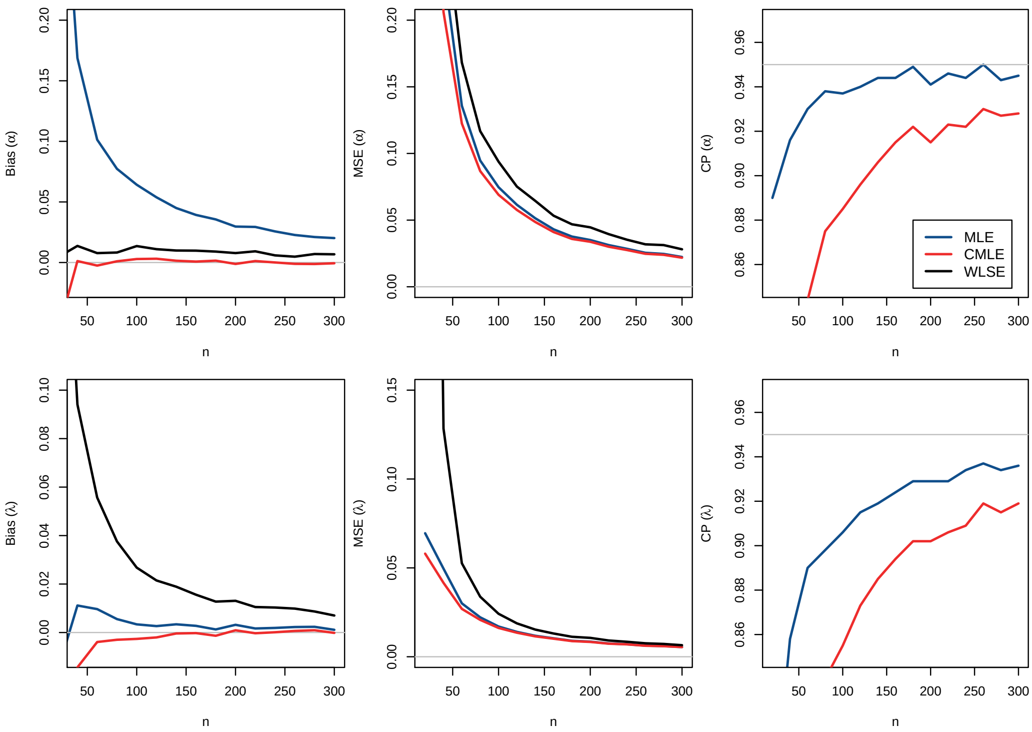

Figure 4. Bias, MSEs related from the estimates of $\alpha = 1.5$ and $\lambda = 0.4$ for N simulated samples under the MLE, CMLE and WLSE.

model is the one which provides the minimum values of these criteria. It is important to point out that these discrimination criteria are the most commonly used in the literature and are founded on information theory (see for instance, Anderson and Burnham (2004) and the references therein).

The discrimination criteria used above can only be used to discriminate the most adequate model among the chosen distribution. However, in some situations none of the selected distribution are satisfactory to describe the proposed data set. To overcome this problem, the Kolmogorov-Smirnov (KS) test is also considered aiming to check the goodness of the fit for the models. This procedure is widely known and based on the KS statistic $D_n = \sup_x |F_n(x) - F(x;\boldsymbol{\theta})|$, where $\sup x$ is the supremum of the set of distances, $F_n(x)$ is the empirical distribution function and $F(x;\boldsymbol{\theta})$ is the cdf of the fitted distribution. Under a significance level of 5% if the data comes from $F(x;\boldsymbol{\theta})$ (null hypothesis), the hypothesis is rejected if the P-value is smaller than 0.05.

The next subsections give a description of the used data and their analysis under the mentioned distributions above.

6.1. *Machine failure*

The proposed dataset is related to the 64 failure times of the motor of an agricultural machine. Such machine works on extreme environmental conditions and its operation is in a regime of 24 hours on the workdays, the study of the malfunction of its components are of main interesting. The time is presented in hours divided by 24 and are summarized in the histogram available in Figure 5. Additionally, Figure 5 presents the survival function fitted by different distributions and the hazard function adjusted by the PE distribution.

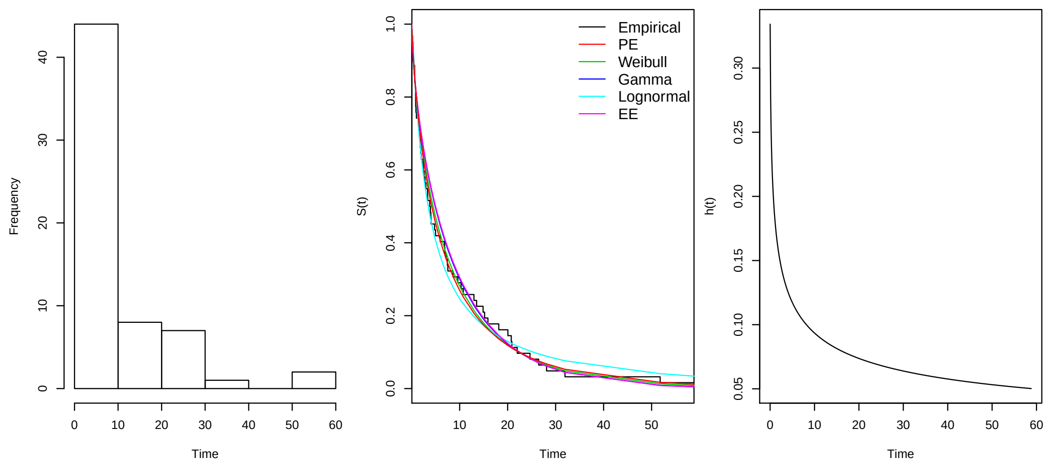

Figure 5. Histogram, survival function for different distributions and the hazard function adjusted by the PE distribution

Table 1 presents the results of AIC, AICc, HQIC, CAIC and the p-value related to the KS test in order to discriminate the best fit.

As shown in Figure 5 we observe a goodness of the fit for the PE distribution, which is confirmed from different discrimination criterion methods as the PE distribution has the minimum value for all statistics and the largest for the P-value. Consequently, we conclude that the data related to the number of successive failures of the motor of the agricultural machine can be described by the PE distribution. Table 2 displays the

Table 1. Results of AIC, AICc, HQIC, CAIC criteria and the p-value for the KS test for all fitted distributions considering the machine component.

Criteria	PE	Weibull	Gamma	Lognormal	EE
AIC	**385.76**	386.92	388.42	388.28	388.66
AICc	**381.96**	383.12	384.62	384.49	384.86
CAIC	**392.01**	393.17	394.67	394.54	394.91
HAIC	**387.43**	388.59	390.09	389.95	390.33
KS	0.9906	0.8703	0.5756	0.8661	0.5255

MLE, standard-error and 95% confidence intervals for α and λ.

Table 2. MLE, Standard-error (SE) and 95% confidence intervals for α and λ.

$\boldsymbol{\theta}$	MLE	SE	$CI_{95\%}(\boldsymbol{\theta})$
α	0.6245	0.0127	(0.4033; 0.8457)
λ	0.3697	0.0209	(0.0866; 0.6528)

6.2. *Breaker failure*

Here, we also considered the problem related to the failure time of an agricultural machine component. The lifetime is related to 48 failure of the breaker in hours per day. Figure 6 presents the histogram of the data, the survival function fitted by different distributions and the hazard function adjusted by the PE distribution.

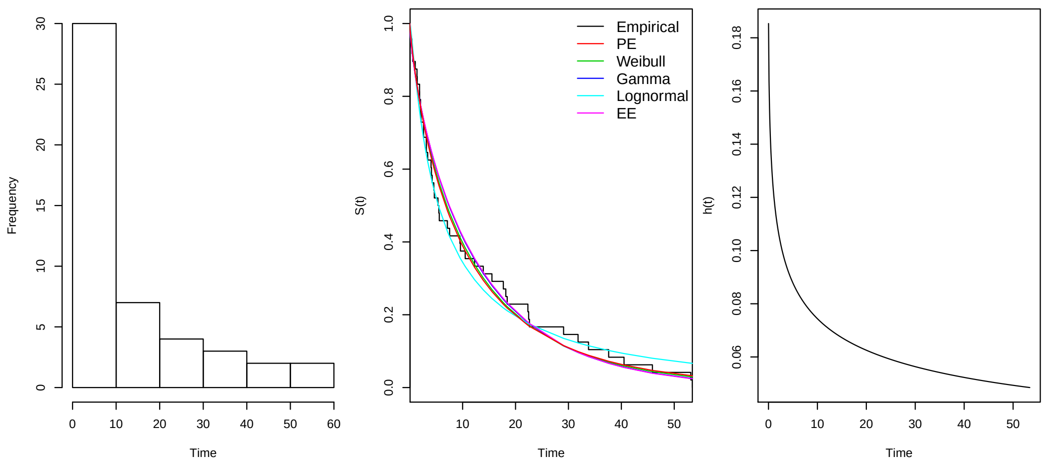

Figure 6. Histogram, survival function for different distributions and the hazard function adjusted by the PE distribution

Table 3 presents the results of AIC, AICc, HQIC, CAIC and the p-value related to the KS test in order to discriminate the best fit.

Analogously to the previous case, comparing the empirical survival function with the adjusted models we observe a goodness of the fit for the PE distribution, which is confirmed from different discrimination criterion methods. Therefore, we conclude that the data related to the number of successive failures of the breaker of the agricultural machine can be described by the PE distribution. Table 4 displays the MLE, standard-error and 95% confidence intervals for α and λ.

Table 3. Results of AIC, AICc, HQIC, CAIC criteria and the p-value for the KS test for the for all fitted distributions considering the pricker component.

Criteria	PE	Weibull	Gamma	Lognormal	EE
AIC	**338.20**	338.44	338.73	342.90	338.75
AICc	**334.47**	334.70	335.00	339.17	335.01
CAIC	**343.94**	344.18	344.48	348.64	344.49
HAIC	**339.61**	339.85	340.15	344.31	340.16
KS	0.7558	0.6525	0.4924	0.9339	0.4713

Table 4. MLE, Standard-error (SE) and 95% confidence intervals for α and λ.

$\boldsymbol{\theta}$	MLE	SE	$CI_{95\%}(\boldsymbol{\theta})$
α	0.7325	0.0155	(0.4883; 0.9767)
λ	0.1931	0.0073	(0.0258; 0.3604)

6.3. *Monthly rainfall data*

In this subsection, we considered the data set firstly presented in Bakouch et al. (2017). The data set is related to the total monthly rainfall l (mm) during June at São Carlos located in southeastern Brazil. Such city has an active industrial profile and high agricultural importance where the study of the behavior of dry and wet periods has proved to be strategic and economically significant its development.

Nadarajah and Haghighi (2011) observed that maximum likelihood estimate of the shape parameter is non-unique for the Gamma, Weibull and Generalized exponential distributions if data set consists of zeros and therefore none of these three distributions can fit this kind of data set. On the other hand, the PE distribution is defined as $x \geq 0$, which allow us to use the original values in the presence of zero. Since it not possible to compute the MLE of the distributions cited above we will consider the Kuş and Nadarajah-Haghighi distributions, as mentioned in the introduction to fit the proposed dataset. Table 5 displays the MLE, standard-error and 95% confidence intervals for α and λ.

Table 5. MLE, Standard-error (SE) and 95% confidence intervals for α and λ.

$\boldsymbol{\theta}$	MLE	SE	$CI_{95\%}(\boldsymbol{\theta})$
α	0.4556	0.0159	(0.2085; 0.7028)
λ	0.3027	0.0329	(0.0001; 0.6579)

Table 6 presents the results of AIC, AICc, HQIC, CAIC and the p-value related to the KS test in order to discriminate the best fit.

In the Figure 7, the survival function adjusted by the compared distributions and the non-parametric estimator.

The adjusted models when compared to the empirical survival show a goodness of the fit for the PE distribution, which is confirmed from different discrimination criterion methods. Therefore, our proposed distribution can be used to describe the data related to the total monthly rainfall during June at São Carlos. It is worth mentioning that other distributions may be studied further to very if they can be used to describe zero occurrence data (see for instance Habibi and Asgharzadeh (2018)).

Table 6. Results of AIC, AICc, HQIC, CAIC criteria and the p-value for the KS test for the for all fitted distributions considering the total monthly rainfall during June at São Carlos.

Criteria	PE	Kus	NH
AIC	**473.21**	489.40	486.85
AICc	**473.45**	489.63	487.09
CAIC	**479.19**	495.38	492.83
HAIC	**474.74**	490.93	488.39
KS	0.0706	0.1758	0.2113

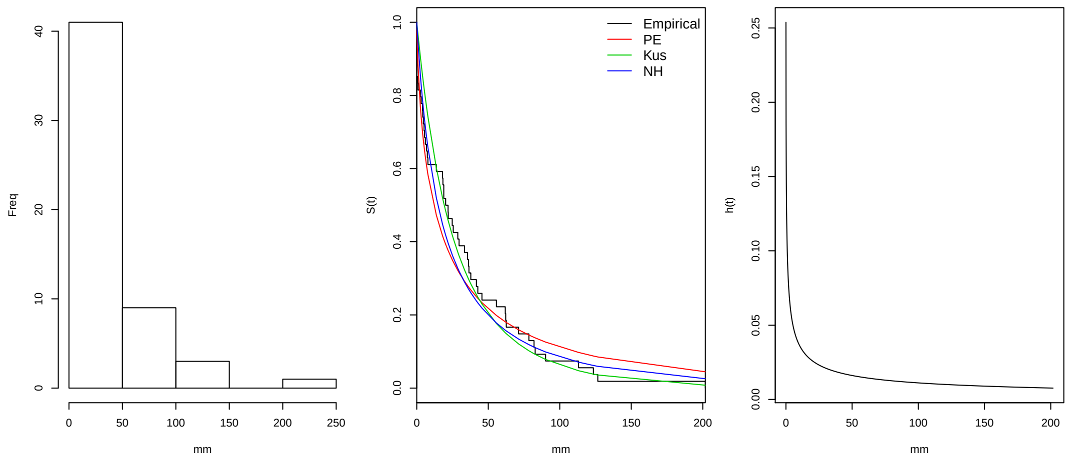

Figure 7. Histogram, survival function adjusted by the compared distributions and a non-parametric method considering the data set related to the total monthly rainfall during June at São Carlos.

6.4. *Air conditioning system*

The data have been presented by Proschan (1963) and further analyzed by Adamidis and Loukas (1998) and consists of the number of successive failures of the air conditioning system of each member of a fleet of 13 Boeing 720 jet airplanes. Figure 8, we have the histogram to summarize the data, the survival function adjusted by the compared distributions and the non-parametric survival function.

Table 7 presents the results of AIC, AICc, HQIC, CAIC and the p-value related to the KS test for the compared distributions.

Table 7. Results of AIC, AICc, HQIC, CAIC criteria and the p-value for the KS test for the compared distributions considering the number of successive failures of the air conditioning system of each member of a fleet of 13 Boeing 720 jet airplanes.

Test	PE	Weibull	Gamma	Lognormal	EE
AIC	**2358.61**	2359.17	2360.58	2361.76	2360.81
AICc	**2354.67**	2355.23	2356.64	2357.82	2356.86
CAIC	**2367.34**	2367.89	2369.30	2370.48	2369.52
HQIC	**2361.33**	2361.89	2363.30	2364.470	2363.52
P-value	**0.63435**	0.61336	0.37719	0.56520	0.33913

Table 8 displays the MLEs, standard-error and 95% confidence intervals for α and λ.

Comparing the empirical survival function with the adjusted models we observe a goodness of the fit for the PE distribution, which is confirmed from different discrimi-

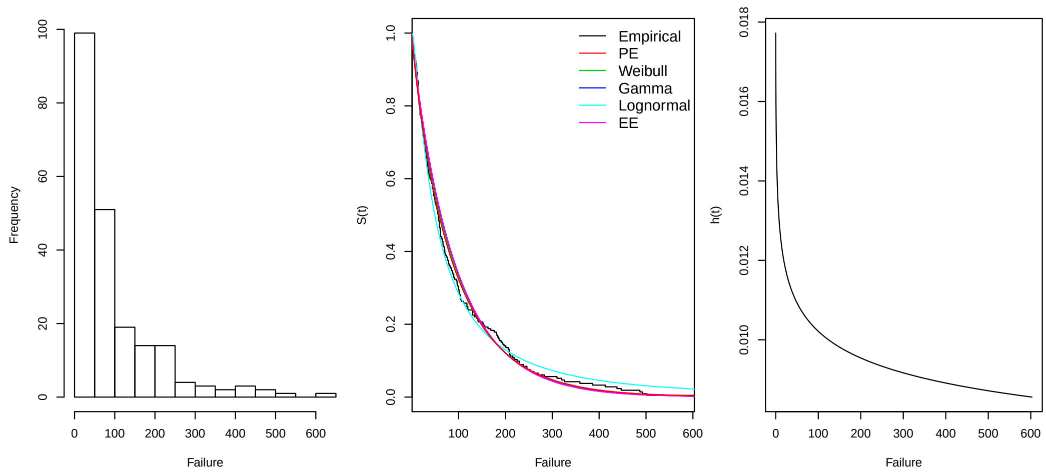

Figure 8. Histogram, survival function adjusted by the compared distributions and a non-parametric method considering the data sets related to the number of successive failures of the air conditioning system of each member of a fleet of 13 Boeing 720 jet airplanes.

Table 8. MLE, Standard-error (SE) and 95% confidence intervals for α and λ.

$\boldsymbol{\theta}$	MLE	SE	$CI_{95\%}(\boldsymbol{\theta})$
α	0.9010	0.01791	(0.7906; 1.0113)
λ	0.0032	0.00003	(0.0076; 0.0282)

nation criterion methods as the PE distribution has the minimum value for all statistics and the largest for the P-value. Consequently, we conclude that the data related to the number of successive failures of the air conditioning system of each member of a fleet of 13 Boeing 720 jet airplanes can be described by the PE distribution.

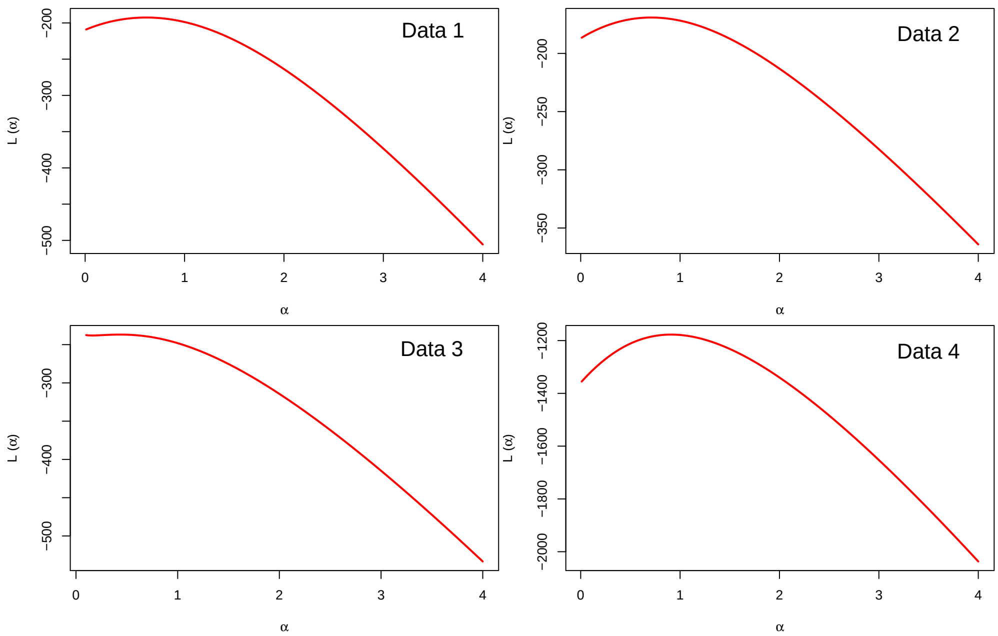

Figure 9. Plot for the profile likelihood considering different values of α for the datasets.

It is important to point out that although it was not possible to prove that the MLE is unique theoretically, the likelihood function is well-behaved, i.e., a unique maximum was achieved for the different datasets (see Figure 9) as well as for all simulation data. Since we have closed-form estimator for λ, we only have to maximize α, hence it was considered different initial values for the parameter (from 0.01 up to 1000) which lead to the same estimates. Hence, the inference for the parameters can be performed without any problem.

7. Concluding remarks

In this paper, we introduced a new two-parameter distribution namely polynomial-exponential distribution, which generalizes the ordinary exponential, linear exponential and other combinations of Weibull distributions, once the survival function of the PE distribution represents the product of the survival functions of Weibull distributions with parameters $(\lambda, 1)$, $(\lambda, 2), \ldots,$ and (λ, α). The new distribution could be an alternative model for lifetime data, specially for the presence of instantaneous failures (inliers), since standard distributions such as Gamma, Weibull, Lognormal and exponentiated exponential may not be suitable. We provided a mathematical treatment of the new distribution. The estimation of parameters was discussed by the maximum likelihood approach and a bias corrective approach was presented. Simulation studies were performed to assess the performance of the maximum likelihood estimators and the corrected MLEs. We fitted the proposed distribution to four real data sets and compared its fit to those of commonly known lifetime distributions, establishing that the new model can be a good competitor for the latter. We hope that the proposed distribution may be used in wide applications as well as lifetime modeling.

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