

Classification of closed sets and diffeos of
one-dimensional manifolds

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CLASSIFICATION OF CLOSED SETS AND DIFFEOS OF

ONE-DIMENSION MANIFOLDS.

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1. - INTRODUCTION.

Avoiding the classification of the closed sets of a manifold is one of the reasons for the generic study of Dynamical Systems.

The aim of this paper is to present a classification, by homeomorphism, of the closed sets of a differentiable manifold of one dimension (see 2.1, 2.3 and 4.1) and to all the diffeomorphisms of such a manifold (see 3.1, 3.2, 4.2, 4.4, 4.9).

In the case of closed sets of $[0,1]$ we obtain a topological conjugacy which is equivalent to the classification, by order type, of all the sequences of $[0,1]$ (see 2.4 and 2.5).

Except the results related to the diffeomorphisms of the circle S^1 ($\{z \in \mathbb{C} : |z| = 1\}$) without periodic points, everything else is in our Master Dissertation for the MPhil degree

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2. - CLOSED SETS OF M^1 .

M^1 is a differentiable manifold of one dimension.

2.1. DEFINITION.

Let A and B be closed sets in M^1 . We say that A is conjugate to B iff there exists a homeomorphism h on M^1 such that $h(A) = B$. Notation: $A \sim B$.

#

It is easy to check that \sim is an equivalence relation over the set of closed sets in M^1 and that the quotient space has the cardinality of the continuum. In particular, two perfect sets with empty interior (called Cantor sets) are conjugated by \sim ([2], page 38).

We are going to establish what means 2.1. over the set of closed sets of M^1 and we first present the case $M^1 = I = [0,1]$.

Let A be a closed set in I , then, $A = \partial A \cup \bigcup_{i \in K \subseteq \mathbb{N}} V_i$, where ∂A is the boundary of A and V_i is a connected component of $A \cup \overset{\circ}{(I \setminus A)}$ ($\overset{\circ}{A}$ is the interior of A and $I \setminus A$ is the complement of A in I).

2.2. DEFINITION.

Let $x, y \in I$. We say that x is conjugate to y iff either $x = y \in \partial A$ or $\exists i \in K$ such that $x, y \in V_i$.

Notations :

$x \sim_A y$ if x is conjugate to y ;

$[x]_A$ is the \sim_A - class of equivalence of x ;

$$X_{\overset{\circ}{A}} = \{[x]_A : x \in \overset{\circ}{A}\};$$

$$X_{IA} = \{[x]_A : x \in IA\};$$

$$X_{\partial A} = \{[x]_A : x \in \partial A\} = \{[x] : x \in \partial A\}$$

$$X_A = X_{\overset{\circ}{A}} \cup X_{IA} \cup X_{\partial A}$$

We observe that X_A has a natural order:

$$[x]_A < [y]_A \text{ iff } x < y \text{ and } x \not\sim_A y; [x]_A = [y]_A \text{ iff } x \sim_A y.$$

2.3. PROPOSITION.

Let A and B be closed sets in I with

$\{0, 1\} \subset A \cap B$. $A \sim B$ iff $\exists h: X_A \rightarrow X_B$, monotonic

bijection such that $h(X_{\overset{\circ}{A}}) = X_{\overset{\circ}{B}}$ and $h(X_{IA}) = X_{IB}$.

PROOF. see [3], page 49.

#

We denote by A' the set of accumulation points of A

2.4. COROLLARY.

Let S be a monotonic sequence of distinct elements of I . If $S - S' = \{r_n : n \in J \subseteq \mathbb{Z}\}$, then there exists a closed set A in I such that:

- a) $\{0, 1\} \subset A$;
- b) $r_n \notin_A r_m$ if $n \neq m$ (see page 3);
- c) $\{[r_n]_A : n \in J \subseteq \mathbb{Z}\} = X_A \cup X_{I-A}$ (see page 3).

PROOF.

We divide the proof in three parts:

- 1ª) $S' = \emptyset$;
- 2ª) S' has only one point;
- 3ª) S' has at least two points.

1ª) In this case, $S = \{r_1, \dots, r_n\}$ and $0 \leq r_1 < r_2 \dots < r_n \leq 1$. We define the following intervals:

$$I_m =](r_{m-1} + r_m)/2, (r_m + r_{m+1})/2 [\text{ if } m \in \{2, \dots, n-1\};$$

$$I_1 =] 0, (r_1 + r_2)/2 [;$$

$$I_n =](r_{n-1} + r_n)/2, 1[.$$

Then, if we take

$$A = \bigcup_{\substack{m \in \\ \text{par}}} I_m ; I-A = \bigcup_{\substack{m \in \\ \text{impar}}} I_m ; \text{ and}$$

$$A = \left(\bigcup_{\substack{m \in \\ \text{par}}} \bar{I}_m \right) \cup \{0, 1\},$$

we have the thesis.

2ª) $S' = \{a\}$ and we can have either $0 < a < 1$ or $a \in \{0, 1\}$. In the

first case, we have $0 \leq r_0 < r_1 < \dots < a < \dots < r_{-2} < r_{-1} \leq 1$

and we define the intervals I_m in the following way:

$$I_m =](r_m + r_{m-1})/2, (r_m + r_{m+1})/2[\text{ if } m \geq 1 \text{ or } m \leq -2 ;$$

$$I_0 =]0, (r_1 + r_0)/2[;$$

$$I_{-1} =](r_{-2} + r_{-1})/2, 1[.$$

Then, if we take

$$A = \left(\bigcup_{\substack{m \in \\ \text{par}}} \bar{I}_m \right) \cup \{0, 1, a\}$$

we have the thesis.

In the second case ($a \in \{0, 1\}$) we have either $0 \leq r_0 < r_1 < \dots < 1$ or $0 < \dots < r_{-1} < r_0 \leq 1$ and the procedure is the same than

anterior.

3a) Let a, b be two elements of S' such that $a < b$, $]a, b[\cap$

$$\cap S' = \emptyset \text{ and }]a, b[\cap S = \{r_i : i \in \mathbb{Z}\} \text{ with } r_i < r_{i+1}, \forall i \in \mathbb{Z}.$$

(If $S = S'$, then $\overset{\circ}{A} = \emptyset$ and $A = S' \cup \{0, 1\}$.)

In $]a, b[\cap S$ we define the following intervals:

$$I_m^{a,b} =](r_m + r_{m-1})/2, (r_m + r_{m+1})/2[, \forall m \in \mathbb{Z}$$

and we take

$$A \cap]a, b[= \bigcup_{\substack{m \in \\ \text{par}}} \overline{I_m^{a,b}}$$

And so on, for each two successive points of $S' \cup \{0, 1\}$.

Then, if we take

$$A = \underbrace{\{a, b \in S' \cup \{0, 1\} :]a, b[\cap S' \neq \emptyset\}}_{\text{}} \cup \left(\bigcup_{\substack{m \in \\ \text{par}}} \overline{I_m^{a,b}} \right) \cup S' \cup \{0, 1\}$$

we have the thesis.

#

Let A be a closed set in I and we consider the following sequences:

- $S_{\overset{\circ}{A}}$, formed by one element of each connected component of $\overset{\circ}{A}$;
- $S_{I \setminus A}$, formed by one element of each connected component of $I \setminus A$

$$- S_A = (S_{O_A} \times \{0\}) \cup (S_{I-A} \times \{1\}).$$

2.5. COROLLARY.

Let A and B be closed sets in I . $A \sim B$ iff there exists a monotonic bijection $\phi: S_A \rightarrow S_B$ such that $(\pi_1 \circ \phi)(S_{O_A} \times \{0\}) = S_{O_B} \times \{0\}$ and $(\pi_1 \circ \phi)(S_{I-A} \times \{1\}) = S_{I-B} \times \{1\}$.

(Obs.: $\pi_1(x, y) = x$ and we say that $\phi: S_A \rightarrow S_B$ is monotonic iff $\pi_1 \circ \phi$ is monotonic in the sense of real numbers.)

#

3. - DIFFEOMORPHISMS.

Let $\text{Dif}^r(M^1)$ be the set of C^r diffeomorphisms on M^1 ; we denote by $\text{per}_n f$ the set of periodic points of f with period n and $\text{per}_1 f = \text{fix } f$.

3.1. DEFINITION.

Let $f, g \in \text{Dif}^r(M^1)$, $0 \leq r \leq \infty$. f is conjugate to g iff there exists a homeomorphism h on M^1 which takes orbits of f onto orbits of g (i.e., $\forall x \in M^1, h(O_f(x)) = O_g(h(x))$). Notation: $f \sim g$.

If \sim denotes the usual conjugacy (i.e., $f \sim g$ iff $\exists h \in \text{Dif}^0(M^1)$ such that $h \circ f = g \circ h$), it is easy to check that

$$f \sim g \implies f \sim g;$$

but the converse is not true as may be seen from the following

counter-example:

let $f, g \in \text{Dif}^0([0, 1])$ with $f(x) \neq g(x) = \{0, 1/2, 1\}$ and

$$g(x) = \begin{cases} f(x) & 0 \leq x \leq 1/2 ; \\ f^{-1}(x), & 1/2 \leq x \leq 1 . \end{cases}$$

For example, like in the picture.

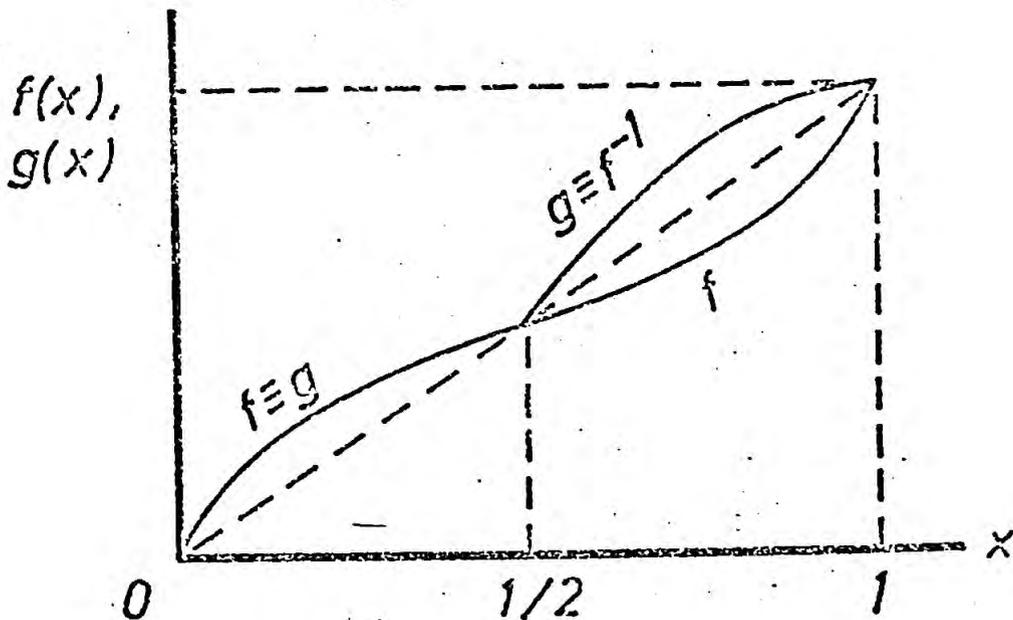


Fig. 1

We have $0_f(x) = 0_g(x)$, $\forall x \in I$, and this means that $f \circ g$ by the identity map; however, there does not exist a homeomorphism on $I = [0, 1]$ such that either $h \circ f = g \circ h$ or $h \circ f = g^{-1} \circ h$, because $(1/2, 1/2)$ is a *sink* for g but it is *not* for f .

3.2. PROPOSITION.

Let $f, g \in \text{Dif}^r(I)$, $0 \leq r \leq \infty$. $f \approx g$ iff $f \circ g$ is order-preserving and there exists $h \in \text{Dif}^0(I)$ such that $h(\text{per}_n f) = \text{per}_n g$, $n=1,2$.

PROOF. see [3], page 82.

#

It is possible to prove, with the help of 3.2, the following statements.

1. $\forall f \in \text{Dif}^0(I)$, f is not $\mathcal{A} - C^0$ - structurally stable.
2. Let $f \in \text{Dif}^r(I)$; $1 \leq r \leq \infty$; f is $\mathcal{A} - C^r$ - structurally stable iff its periodic points are hyperbolic. ([3], pages 95 and 140, respectively).

Afterwards, we can prove, by using Urysohn's differentiable lemma (see any book in General Topology) that: if A is a closed set on I with $\{0,1\} \subset A$, there exists $f \in \text{Dif}^\infty(I)$, order-preserving, such that $\text{fix } f = A$; particularly, if A is finite, then f can be a hyperbolic one.

All the preceding results have an immediate generalization to that cases in which M^1 is either $]0,1[$ or $[0,1[$ (the case $M^1 =]0,1]$ is similar to $[0,1[$) ([3], page 142).

There is only one more case in which M^1 is connected:
 $M^1 = S^1 = \{z \in \mathbb{C} : |z| = 1\}$.

4. - DIFFEOMORPHISMS OF THE CIRCLE S^1 .

We denote by j the map $j: \mathbb{R} \rightarrow S^1$ defined by $j(x) = \exp(2\pi i x)$.

4.1. PROPOSITION.

Let A, B be closed sets on S^1 ; $A \sim B$ iff there exists $z_0 \in S^1$ such that $j^{-1}(A) \cap [0, 1] \sim (z_0 j)^{-1}(B) \cap [0, 1]$.

(Obs.: $(z_0 j)(x) = z_0 \exp(2\pi i x)$).

PROOF.

Immediate. ([3], page 159)

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4.2. PROPOSITION.

Let $f, g \in \text{Dif}^n(S^1)$, $0 \leq n \leq \infty$ with $\text{per } f \neq \emptyset \neq \text{per } g$; $f \approx g$ iff $f \circ g$ is order-preserving and there exists $h \in \text{Dif}^0(S^1)$ such that $h(\text{per}_n f) = \text{per}_n g$, $\forall n \in \mathbb{N}$.

PROOF. see [3], page 168.

#

Let $f \in \text{Dif}^r(S^1)$, $0 \leq r \leq \infty$, order-preserving, with $\text{per } f \neq \emptyset$; it is known ([2], page 38) that the rotation number of f , $\rho(f)$, is a rational number of the form p/q , where p and q are positive, co-prime and q represents the period of the periodic points of f .

4.3. COROLLARY.

Let $f, g \in \text{Dif}^n(S^1)$, $0 \leq n \leq \infty$, f and g order-preserving

and $\rho(f) = p/q$, $\rho(g) = p'/q'$. If $f \sim g$, then $q = q'$ and there is no conditions over p and p' .

PROOF.

Immediate from 4.2.

#

The converse of 4.3 is true if $\text{per}_q f \sim \text{per}_q g$.

The last case to be considered in S^1 is when $\text{per} f = \emptyset$. In this case, it is known ([2], page 38), that $\rho(f)$ is a irrational number and the non-wandering set of f , $\Omega(f)$, is either S^1 or a Cantor set of S^1 .

The Denjoy's theorem ([1], page 372) establishes that: if $f \in \text{Diff}^n(S^1)$, $n \geq 2$, then $\Omega(f) = S^1$. The Denjoy's example ([1], page 348, [3]; page 209) shows that: there is a $f \in \text{Diff}^n(S^1)$ $n \in \{0, 1\}$, with $\text{per} f = \emptyset$ and $\Omega(f)$ a Cantor set of S^1 .

4.4. PROPOSITION.

Let $f, g \in \text{Diff}^n(S^1)$, $0 \leq n \leq \infty$, with $\text{per} f = \text{per} g = \emptyset$ and $\Omega(f) = \Omega(g) = S^1$; $f \sim g$ if $\rho(f) = \rho(g)$.

PROOF.

Let R_α be the rotation of S^1 by angle $2\pi\alpha$.

The sufficient condition comes from the fact that

$f \circ R_{\rho(f)}$ and $g \circ R_{\rho(g)}$ ([2], page 41 and [3], page 197).

#

I couldn't prove any converse yet, but I think the

converse is true. ($f \sim g \implies \rho(f) = \rho(g)$.)

4.5. PROPOSITION.

Let $f \in \text{Dif}^r(S^1)$, $r \in \{0, 1\}$, with $\text{per } f = \emptyset$ and $\Omega(f)$ a Cantor set of S^1 ; then, each orbit of f meets at most once each connected component of $S^1 \setminus \Omega(f)$.

PROOF. see [3], page 203.

#

We remember that a Cantor set is a perfect set with empty interior.

Let C be a Cantor set either of $M^1 = I = [0, 1]$ or of $M^1 = S^1$; then,

$$M^1 \setminus C = \bigcup_{i \in \mathbb{N}}] a_i, b_i [.$$

Since $M^1 \setminus C$ has \mathbb{N} connected components, we can construct a map $p: M^1 \rightarrow M^1$, continuous, onto and such that: $p(x) = p(y)$ iff $x, y \in] a_i, b_i [$ for some $i \in \mathbb{N}$; furthermore, if $f \in \text{Dif}^r(S^1)$, $r \in \{0, 1\}$, with $\text{per } f = \emptyset$ and $\Omega(f)$ a Cantor set of S^1 , then the map $\tilde{f}: S^1 \rightarrow S^1$ defined by $\tilde{f} \circ p = p \circ f$ satisfies:

a) $\tilde{f} \in \text{Dif}^0(S^1)$;

b) $\rho(\tilde{f}) = \rho(f)$;

c) $\Omega(\tilde{f}) = S^1$.

([4], page 693).

Now we can state the central theorem about conjugacy of diffeomorphisms of S^1 without periodic points and with a

Cantor set as non-wandering set.

4.6. PROPOSITION.

Let $f, g \in \text{Dif}^n(S^1)$, $n \in \{0, 1\}$, with $\text{per } f = \text{per } g = \emptyset$, $\Omega(f) = C_1 \subsetneq S^1$ and $\Omega(g) = C_2 \subsetneq S^1$; if $f \sim g$ then $\rho(f) = \rho(g)$.

PROOF.

We first prove that : if $f \sim g$ then $\tilde{f} \sim \tilde{g}$ (see page 12 for statements about \tilde{f} and \tilde{g}).

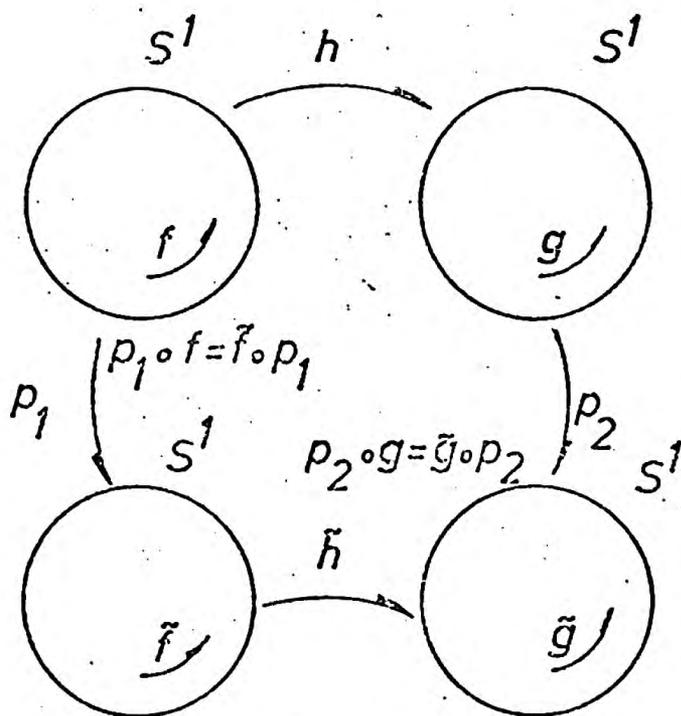


Fig. 2

If $f \sim g$, there exists $h \in \text{Dif}^0(S^1)$ such that $h(O_f(z)) =$

$= 0_g(h(z)), \forall z \in S^1$; to prove that $\tilde{f} \approx \tilde{g}$, we need to construct $\tilde{h} \in \text{Dif}^0(S^1)$ such that $\tilde{h}(0_{\tilde{f}}(z)) = 0_{\tilde{g}}(\tilde{h}(z)), \forall z \in S^1$.

It is not hard to check that \tilde{h} , related to h by $\tilde{h} \circ p_1 = p_2 \circ h$ (where $\tilde{f} \circ p_1 = p_1 \circ f$ and $\tilde{g} \circ p_2 = p_2 \circ g$; see page 12) is well defined and satisfies the required conditions; then, $\tilde{f} \approx \tilde{g}$.

Since $\Omega(\tilde{f}) = \Omega(\tilde{g}) = S^1$ (see page 12), then $\tilde{f} \approx_{R_\rho} \tilde{f}$ and $\tilde{g} \approx_{R_\rho} \tilde{g}$ ([2], page 41 and [3], page 197); but $R_\rho(\tilde{f}) \approx R_\rho(\tilde{g})$ iff $\rho(\tilde{f}) = \rho(\tilde{g})$ (see 4.4) and we can conclude, using: $\rho(f) = \rho(\tilde{f})$; $\rho(g) = \rho(\tilde{g})$; and if $f \approx g$ then $\tilde{f} \approx \tilde{g}$, that

if $f \approx g$ then $\rho(f) = \rho(g)$.

If we identify $S^1 \setminus \Omega(f)$ by the orbit relation, we can see the same \tilde{f} ; then it isn't true the strict converse for 4.6. and I don't yet know what kind of converse we can prove for it. #

Finally, we remember that:

- if M^1 is not-connected, then M^1 is the disjoint union of its connected components;
- the closed sets of M^1 have closed intersection with each connected component of M^1 ;
- the homeomorphisms of M^1 preserve each connected component of M^1 .

Thus, with the results on M^1 connected, we have the analogous results on M^1 not-connected.

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