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Twisted conjugacy in PL-homeomorphism groups of the circle

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Abstract
Given an automorphism $\phi : \Gamma \to \Gamma$ of a group, one has a left action of Γ on itself defined as $g.x = gx\phi(g^{-1})$. The orbits of this action are called the Reidemeister classes or ϕ-twisted conjugacy classes. We denote by $R(\phi) \in \mathbb{N} \cup \{\infty\}$ the Reidemeister number of ϕ, namely, the cardinality of the orbit space $\mathcal{R}(\phi)$ if it is finite and $R(\phi) = \infty$ if $\mathcal{R}(\phi)$ is infinite. The group Γ is said to have the R_∞-property if $R(\phi) = \infty$ for all automorphisms $\phi \in \mathrm{Aut}(\Gamma)$. We show that the generalized Thompson group $T(r, A, P)$ has the R_∞-property when the slope group $P \subset \mathbb{R}^\times_{>0}$ is not cyclic.

Keywords R. Thompson's groups · PL-homeomorphisms of the circle · Twisted conjugacy · Reidemeister number · R_∞-property

Mathematics Subject Classification (2010) 20E45 · 20E36

1 Introduction

Let $P \subset \mathbb{R}^\times_{>0}$ be a nontrivial subgroup of the multiplicative group of positive reals and let $A \subset \mathbb{R}$ be a nontrivial subgroup of the additive group of reals which is also a P-module, i.e., $t.A = A\ \forall t \in P$. Note that $A \subset \mathbb{R}$ is dense since P is nontrivial. Let $r \in A$ be positive. We shall denote by S_r the circle $\mathbb{R}/r\mathbb{Z}$. Let $T(r, A, P)$ denote the group of all PL-homeomorphisms of S_r which have slopes in P and break points (i.e., points of non-differentiability) in $A/r\mathbb{Z}$. Also let $G(r, A, P)$ denote the group of all PL-homeomorphisms of the interval $[0, r]$ with slopes in P and break points in A. We regard $G(r, A, P)$ as the subgroup of $T(r, A, P)$ consisting of elements which fix the trivial coset $r\mathbb{Z} =: \bar{0} \in S_r$. The family of groups $G(r, A, P)$, $T(r, A, P)$ were introduced by Bieri and Strebel [1], as a generalization of the

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Richard Thompson groups $\mathbf{G} = G(1, \mathbb{Z}[1/2], \langle 2 \rangle)$ and $\mathbf{T} = T(1, \mathbb{Z}[1/2], 2)$. The group $G(r, A, P)$ was denoted $G(I, A, P)$, where $I = [0, r]$, by Bieri and Strebel, who also considered similarly defined groups $G(I, A, P)$ where $I \subset \mathbb{R}$ is any interval, not necessarily compact.

When $P = \langle n_1, \ldots, n_k \rangle$ where $n_1, \ldots, n_k$ are multiplicatively independent integers (i.e., the subgroup generated by $n_1, \ldots, n_k$ is free abelian of rank k) and $A = \mathbb{Z}[1/n]$ where $n = \mathrm{lcm}\{n_j \mid 1 \leq j \leq k\}$, the group $T(r, A, P)$ is denoted by $T(r; n_1, \ldots, n_k)$. Stein [14] showed that $T(r; n_1, \ldots, n_k)$ are finitely presented groups, extending the same result for the groups $T(r; n)$ by Brown [4].

The R_∞-property for the Thompson's group $\mathbf{G}$ was established by Bleak et al. [2]. See also [6]. The same property for the groups $G(I, A, P)$, among others, was established in [9]. The case of the Thompson group $\mathbf{T}$ was settled by Burillo et al. [5] and also independently by Gonçalves and Sankaran [7]; see also [8, Sect. 4]. The following is the main result of this paper. The analogous result for $G(r, A, P)$ (among others) was established in [9].

Theorem 1.1 *Suppose that P is not cyclic, A is a non-zero P-module, $r \in A$ positive. Then $T(r, A, P)$ has the R_∞-property.*

We now outline the major steps in the proof. It is easy to see that $T(r, A, P)$ has infinitely many (untwisted) conjugacy classes and so the Reidemeister number of every inner automorphism is infinite. Hence it suffices to show that the Reidemeister number is infinite for a complete set of representatives of the outer automorphism group of $T(r, A, P)$. A basic result is that the automorphism group of $T(r, A, P)$ equals the normalizer of $T(r, A, P)$ in the group of all homeomorphisms of the circle S_r. Using this result, we show that the representative automorphisms $\{\alpha\}$ can always be chosen so that they restrict to automorphisms of $G(r, A, P)$. Denote by α_0 the restriction of the automorphism α to $G(r, A, P)$. The rest of the proof depends on two cases, depending on whether the homeomorphism of S_r that induces a given automorphism α is orientation preserving or reversing. In each case we show that, by appropriate construction of elements $\{f_n\}_{n \geq 1}$ of $G(r, A, P)$ belonging to distinct α_0-twisted conjugacy classes, the f_n belong to pairwise distinct Reidemeister classes of α. An important ingredient of the proof is the description of the automorphism group of $T(r, A, P)$, when P is not cyclic, as the normalizer of $T(r, A, P)$ in the group of all *piecewise linear* homeomorphisms of S_r. When P is cyclic, in general, there are *exotic automorphisms*—those which cannot be represented by PL homeomorphisms and our proof fails for such automorphisms.

2 The automorphism group of $T(r, A, P)$

McCleary and Rubin [12, Theorem 3] obtained a very general result concerning the automorphism group of a group G of orientation preserving homeomorphisms of a dense subset X of $\mathbb{S}^1$. Their result, applied to the case $X = \mathbb{S}^1$, states that if G (i) contains a nontrivial element whose support is not dense, and (ii) satisfies a certain interval-transitivity property—the $\mathcal{O} - 3$-transitivity on a dense subset—then the automorphism of the group G is equal to the normalizer of G in the group $\mathrm{Homeo}(\mathbb{S}^1)$ of all homeomorphisms of $\mathbb{S}^1$. Bieri and Strebel [1, Sect. 16-17] had shown earlier an analogous result when G is a subgroup of $G(r, A, P)$ satisfying certain axioms. (See also McCleary [11].) Applying these two results, Bieri and Strebel obtained the following.

Theorem 2.1 (Bieri-Strebel [1, Theorem N3.6]) *Suppose that $A \subset \mathbb{R}$ and $P \subset \mathbb{R}^\times_{>0}$ are non-trivial groups. Assume that $r \in A$ is positive. Let α be any automorphism of $T(r, A, P)$.*

Then there exists a unique homeomorphism $\phi : S_r \to S_r$ *such that* $\alpha(f) = \phi \circ f \circ \phi^{-1}$, $\forall f \in T(r, A, P)$. *Also* $\phi(A/r\mathbb{Z}) = A/r\mathbb{Z}$.

Definition 2.2 An automorphism α of $T(r, A, P)$ is said to be *orientation preserving* (resp. *orientation reversing*) if ϕ is orientation preserving (resp. orientation reversing).

When P is not a cyclic group, it is dense in $\mathbb{R}_{>0}$. We have the following theorem which guarantees that the homeomorphism ϕ in Theorem 2.1 is piecewise linear. Denote by $\mathrm{Aut}_0(A) \subset \mathbb{R}^{\times}_{>0}$ the subgroup $\{s > 0 \mid sA = A\}$. Since $P \subset \mathrm{Aut}_0(A)$, we have $T(r, A, P) \subset T(r, A, \mathrm{Aut}_0(A))$. In fact, $T(r, A, \mathrm{Aut}_0(A))$ normalises $T(r, A, P)$ (by the chain rule); that is, $gxg^{-1} \in T(r, A, P)$, $\forall g \in T(r, A, \mathrm{Aut}_0(A))$, $\forall x \in T(r, A, P)$. It is easily seen that the only element of $T(r, A, \mathrm{Aut}_0(A))$ that commutes with *every* element of $T(r, A, P)$ is the identity element. Thus, the homomorphism $T(r, A, \mathrm{Aut}_0(A)) \to \mathrm{Aut}(T(r, A, P))$ defined as $g \mapsto \iota_g$, the (restriction of) conjugation by g, is a monomorphism. Denote by $\rho_r : [0, r] \to [0, r]$ the reflection $t \mapsto r - t$ and by the same symbol the induced reflection of the circle S_r.

We have the following result, which had been obtained in the special case of $T(r; n_1, \ldots, n_k)$, $k \geq 3$, by Liousse [10].

Theorem 2.3 (Bieri-Strebel [1, Theorem N3.10]) *Suppose that* P *is not cyclic. Then the group* $Aut(T(r, A, P))$ *is isomorphic, via conjugation, to the subgroup of all homeomorphisms of* S_r *generated by* $T(r, A, \mathrm{Aut}_0(A))$ *and the reflection* ρ_r.

In particular, under the hypotheses of the theorem, there are no *exotic* automorphisms of $T(r, A, P)$, that is, every automorphism is realised as conjugation by a PL-homeomorphism of S_r. This is not the case in general when P is cyclic.

The group $T(n-1, \mathbb{Z}[1/n], \langle n \rangle) = T(n-1; n)$ is known to contain exotic automorphisms by the work of Brin and Guzmán [3] when $n > 2$ is an integer. The case $n = 2$ is the classical Richard Thompson group T and it is known that every automorphism of T is represented by a PL-homeomorphism—in fact $\mathrm{Out}(T)$ is cyclic of order 2, generated by the class of the reflection ρ_1.

We give a brief outline of the proof of Theorem 2.3, referring the reader to [1, Sect. N] for further details. Suppose that $\alpha \in \mathrm{Aut}(T(r, A, P))$ is represented by a homeomorphism $\phi : S_r \to S_r$ as in Theorem 2.1. Let $u = \phi(\bar{0}) \in A/r\mathbb{Z}$. Then the rotation map $\tau = \tau_{-u}$ defined as $t \mapsto t - u$ of S_r is piecewise linear and belongs to $T(r, A, P)$ as $u \in A/r\mathbb{Z}$ and has constant slope $1 \in P$. Note that τ maps u to $\bar{0}$ and so $\psi := \tau \circ \phi$ fixes $\bar{0}$. It follows that the automorphism $\iota_\tau \circ \alpha =: \beta$, represented by ψ, stabilizes the subgroup $G(r, A, P) \hookrightarrow T(r, A, P)$. (This assertion holds even if P is infinite cyclic.) Note that β is orientation preserving if and only if α is. It suffices to show that either ψ or $\psi \circ \rho_r$ is a PL-homeomorphism in $T(r, A, \mathrm{Aut}_0(P))$. Let $\tilde{\beta} \in \mathrm{Aut}(G(r, A, P))$ be the restriction of β to $G(r, A, P)$. Then $\tilde{\psi} : [0, r] \to [0, r]$, the 'lift' of ψ, represents the automorphism $\tilde{\beta}$ (i.e., $\tilde{\beta} = \iota_{\tilde{\psi}}$). We observe that $\tilde{\psi}(0) = 0$ or $\tilde{\psi}(0) = r$ according as ψ is orientation preserving or not. Under the hypothesis that P is dense, Bieri and Strebel [1, Corollary E17.8] showed that $\tilde{\psi}$ is piecewise linear and in fact either $\tilde{\psi}$ or $\tilde{\psi} \circ \rho_r$ is in $G(r, A, \mathrm{Aut}_0(A))$ according as whether $\tilde{\psi}$ is orientation preserving or not. It follows that ψ or $\psi \circ \rho_r$ belongs to $T(r, A, \mathrm{Aut}_0(A))$ as was to be shown.

We record below an observation made in the course of the above discussion.

Theorem 2.4 *Let* $P \subset \mathbb{R}^{\times}_{>0}$ *be any nontrivial subgroup and* $A \subset \mathbb{R}$ *be any non-zero* P*-submodule. Let* $r \in A_{>0}$. *Any outer automorphism of* $T(r, A, P)$ *is represented by an automorphism* $\beta : T(r, A, P) \to T(r, A, P)$ *which restricts to an automorphism of* $G(r, A, P)$. □

3 Twisted conjugacy classes in $G(r, A, P)$ and $T(r, A, P)$

We shall continue to assume that $P \subset \mathbb{R}^{\times}_{>0}$ is a nontrivial subgroup, possibly infinite cyclic, and that $A \subset \mathbb{R}$ is a nontrivial P-submodule. Also we assume that $r \in A$ is positive. We regard S_r as the quotient space $[0, r]/\{0, r\}$. Observe that $T(r, A, P)$ contains infinitely many (untwisted) conjugacy classes. For example, this may be seen by noting that, for each $n \geq 2$, there exist elements of $T(r, A, P)$ whose support is a union of n pairwise disjoint open arcs in S_r. (Recall that the support of a homeomorphism $f : X \to X$ is defined as $\mathrm{supp}(f) := \{x \in X \mid f(x) \neq x\}$.)

Suppose that $\alpha \in \mathrm{Aut}(T(r, A, P))$ is induced by a homeomorphism $\phi : S_r \to S_r$ and that $f, g \in T(r, A, P)$ are ϕ-twisted conjugates. Let $z \in T(r, A, P)$ be such that

$$f = z.g.\alpha(z^{-1}) = z.g.\phi z^{-1}\phi^{-1}. \tag{1}$$

This implies that

$$f\phi = z(g\phi)z^{-1}. \tag{2}$$

Therefore $f\phi$ and $g\phi$ are conjugates in $\mathrm{Homeo}(S_r)$.

Suppose that $f_n, n \geq 1$, is a sequence of elements in $T(r, A, P)$ such that $f_n\phi$ are in pairwise distinct conjugacy classes of $\mathrm{Homeo}(S_r)$. Then it follows from (2) that $R(\alpha) = \infty$.

For example, it is easy to see that there is an element $f_n \in T(r, A, P)$ whose support is a disjoint union of n arcs. Evidently the $f_n, n \geq 1$, are in pairwise distinct conjugacy classes of $\mathrm{Homeo}(S_r)$. Taking α to be identity, we have $\phi = id$ and so we conclude that $R(id) = \infty$. It follows that $R(\beta) = \infty$ for any inner automorphism β of $T(r, A, P)$.

More generally, let α be any automorphism of a group Γ and let $\beta = \iota_g \circ \alpha$ where $g \in \Gamma$ and ι_g is the inner automorphism $h \mapsto ghg^{-1}$. One has a well-defined bijection $\mathcal{R}(\beta) \to \mathcal{R}(\alpha)$ defined as $[x]_\beta \mapsto [xg]_\alpha$ where $[x]_\alpha$ denotes the α-twisted conjugacy class of $x \in \Gamma$. Hence $R(\alpha) = \infty$ if and only if $R(\beta) = \infty$. (See [7, Sect. 3].)

It follows that, in order to show the R_∞-property for $T(r, A, P)$, it suffices to show that that $R(\alpha) = \infty$ for a set of coset representatives for $\mathrm{Out}(T(r, A, P))$.

In view of Theorem 2.4, we may choose a representative automorphism α that restricts to an automorphism α_0 of $G(r, A, P)$. This is equivalent to the requirement that $\phi(\bar{0}) = \bar{0}$.

3.1 Strategy of proof

Since $G(r, A, P)$ has the R_∞-property by [9], we are guaranteed of a sequence of elements $f_n, n \geq 1$, in $G(r, A, P) \subset T(r, A, P)$ which are in pairwise distinct α_0-twisted conjugacy classes. Suppose that α is induced by $\phi \in \mathrm{Homeo}(S_r)$ (via conjugation). Then $f_n\phi, n \geq 1$, are in pairwise distinct $G(r, A, P)$-conjugacy classes, that is, the $f_n\phi$ are in pairwise distinct orbits for the conjugacy action of $G(r, A, P)$ on $\mathrm{Homeo}(S_r)$. We shall choose f_n so that when P is not cyclic, the elements $f_n\phi, n \geq 1$, remain in pairwise distinct $T(r, A, P)$-conjugacy classes. Indeed, we shall choose our $f_n \in G(r, A, P)$ so that $f_n\phi, n \geq 1$, are in pairwise distinct $\mathrm{Homeo}(S_r)$-conjugacy classes. Our choice of the sequence $\{f_n\}$ will depend on whether α is orientation preserving or orientation reversing.

Assume that $\phi(\bar{0}) = \bar{0}$ and let $\tilde{\phi} : [0, r] \to [0, r]$. Suppose that α is orientable. Then $\tilde{\phi}(0) = 0$. Evaluating both sides of (2) at $z(\bar{0})$ we obtain $f.\phi(z(\bar{0})) = z(\bar{0})$. Thus $z(\bar{0})$ is a fixed point of $f \circ \phi$. By appropriate choices for f, g, if one can arrange so that $\bar{0}$ is the *only* fixed point of $f \circ \phi$, then we can conclude that $z(\bar{0}) = \bar{0}$. This would force z to be in $G(r, A, P)$ since z is orientation preserving. Hence f, g must be in the same α_0-twisted conjugacy class. This will be our strategy of proof when ϕ is orientation preserving. We will achieve this assuming that ϕ has non-vanishing one-sided derivative at $\bar{0}$. This assumption is always valid when P is not cyclic in view of Theorem 2.3.

In the case when ϕ is orientation reversing, we consider the squares $f\phi f\phi = f\alpha(f).\phi^2$ and $g\phi g\phi = g\alpha(g).\phi^2$. It is immediate from (2) that $f\alpha(f)\phi^2$ and $g\alpha(g)\phi^2$ are conjugates. Note that ϕ^2 is orientation preserving. Again we will choose $f_n \in G(r, A, P)$ so that $f_n\alpha(f_n)\phi^2$ are in pairwise distinct conjugacy classes in $\mathrm{Homeo}(S_r)$. There will be two cases to consider, depending on whether ϕ^2 fixes point-wise a non-degenerate interval or not. It will be assumed that ϕ is piecewise linear. This is not a restriction when P is non-cyclic in view of Theorem 2.3.

3.2 Orientation preserving automorphisms

Let α be an orientation preserving automorphism of $T(r, A, P)$ represented by a homeomorphism $\phi : S_r \to S_r$. We assume that ϕ has non-vanishing one-sided derivatives at a point $a \in A/r\mathbb{Z}$. Since A is dense in $\mathbb{R}$, this is automatically valid when ϕ is piecewise linear; this is so when P is non-cyclic by Theorem 2.3.

Denote by $\tau_c : S_r \to S_r$ the rotation $t \mapsto t + c \in S_r$. If $c \in A$, then $\tau_c \in T(r, A, P)$. Also, by Theorem 2.1, ϕ maps $A/r\mathbb{Z}$ to itself. Therefore conjugation by $\psi := \tau_{-b} \circ \phi \circ \tau_a$ where $b = \phi(a)$ defines an automorphism of $T(r, A, P)$. Moreover we have $\psi(\bar{0}) = \bar{0}$ and ψ has non-vanishing one-sided derivative at $\bar{0}$. Note that since $\tau_a, \tau_{-b} \in T(r, A, P)$, both ϕ and ψ determine the same outer automorphism of $T(r, A, P)$. Thus we may (and do) assume, without loss of generality, that $\phi(\bar{0}) = \bar{0}$ and that ϕ has non-vanishing one sided derivatives at $\bar{0}$. We shall denote by the same symbol ϕ its lift $\phi : [0, r] \to [0, r]$. The (one-sided) derivatives at the end points will be denoted $\phi'(0), \phi'(r)$.

Let $f_\lambda : [0, r] \to [0, r]$ be the unique PL-homeomorphism with exactly one break-point ξ_λ in $(0, r)$ and having slopes λ and λ^{-1} at the end points 0 and r respectively. Explicitly, $\xi_\lambda = r/(\lambda + 1)$, and,

$$f_\lambda(t) = \begin{cases} \lambda t, & 0 \le t \le r/(\lambda + 1), \\ \lambda^{-1} t + r(1 - \lambda^{-1}), & r/(1 + \lambda) \le t \le r. \end{cases}$$

If $x, y : [0, r] \to [0, r]$ we declare that $x < y$ if $x(t) < y(t)$ for all $0 < t < r$. Thus $f_\lambda > f_\mu$ if $\lambda > \mu$.

Lemma 3.1 *Suppose that $\phi : [0, r] \to [0, r]$ is an orientation preserving homeomorphism with non-vanishing derivatives at end points. Then, for any $p \in P$ sufficiently large, there exists an $h \in G(r, A, \langle p \rangle)$ such that $\phi < h$ with $h'(0) = p$.*

Proof First we show the existence of a $\lambda > 1$ such that $\phi < f_\lambda$. Then we shall show that, for $p > \lambda$ in P, there exists an $h = h_p \in G(r, A, \langle p \rangle)$ such that $f_\lambda < h$ and $h'(0) = p$.

Since ϕ is orientation preserving, one-sided derivatives, when they exist, are positive if non-zero. Choose $\mu > \max\{1, \phi'(0), 1/\phi'(1)\}$. Then, for some $0 < \epsilon < \min\{r/\mu, r/2\}$, we have $\phi(t) < \mu t$, for $0 < t \le \epsilon$ and $\phi(t) < \mu^{-1} t + (1 - \mu^{-1})r$ for $\eta := r - \epsilon \le t < r$. We choose $\lambda \ge \mu$ sufficiently large so that $\lambda r/(\lambda + 1) > \mu^{-1}\eta + r(1 - \mu^{-1}) > \phi(\eta)$. Then $\phi < f_\lambda$ for all $0 < t < r$ (Fig. 1).

Let $\xi = \xi_\lambda = r/(\lambda + 1)$ be the break point of f_λ. Choose $p \in P, p > \lambda + 1$ so that $p\xi > r$. We pick an $a \in A$ such that $\lambda\xi/p < a < r/p < \xi$. Let k be sufficiently large so that, writing $q := p^{-k}$, we have $(r - ap)/(r - a) > q$ so that $ap < aq + r(1 - q)$. Finally, choose $b \in A$ such that $a < b < \xi$. Then the slope of the straight line joining the

Fig. 1 Choice of f_λ

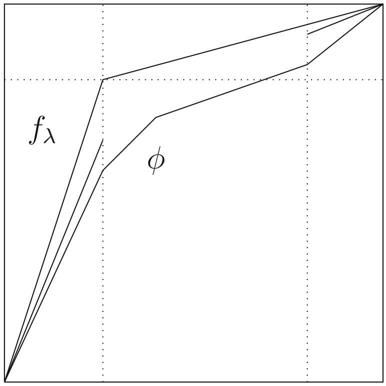
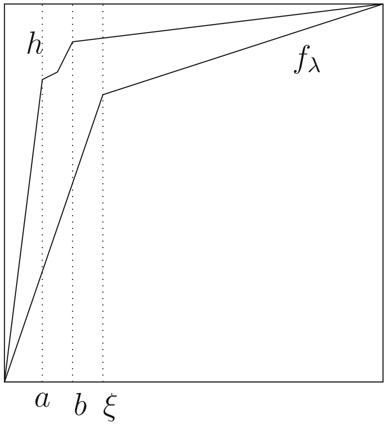


Fig. 2 Choice of h



points (b, ap) and (r, r) is less than that of the line joining $(\xi, \lambda\xi)$ and (r, r). Moreover, $ap < aq + r(1-q) < bq + r(1-q) < r$. So the slope of the straight line joining the points $(b, bq + r(1-q))$ and (r, r) is less than that of the line joining (b, ap) and (r, r), which in turn implies that the slope of the straight line joining the points $(b, bq + r(1-q))$ and (r, r) is less than that of the line joining $(\xi, \lambda\xi)$ and (r, r) (Fig. 2).

We claim that there is a PL-homeomorphism $h_1 : [a, b] \to [ap, bq + r(1-q)]$ with slopes in the cyclic group $P_0 := \langle p \rangle \subset P$ and break-points in A. In view of [1, TheoremA4.1], such a homeomorphism exists if $a - b \equiv ap - bq + r(1-q) \mod IP_0.A$. Indeed, $ap - a, bq - b, r(1-q) \in IP_0.A$, and so such a h_1 exists. Pasting this with the linear isomorphisms $[0, a] \to [0, ap]$ and $[b, r] \to [bq + r(1-q), r]$ fixing end points, yields a PL-homeomorphism $h \in G(r, A, P_0)$. By the very construction, it is clear that $\phi < h$ and that $h'(0) = p$. □

3.3 Orientation reversing automorphisms

Let $\alpha \in T(r, A, P)$ be orientation reversing, represented by a homeomorphism $\phi : S_r \to S_r$. Again, as already observed in Sect. 2, we may (and do) assume without loss of generality that $\phi(\bar{0}) = \bar{0}$. Then its lift to $[0, r]$, also denoted ϕ, satisfies $\phi(0) = r, \phi(r) = 0$. It is clear that $\phi \circ \rho_r$ and ϕ^2 are orientation *preserving*. (Recall that $\rho_r(t) = r - t$ is the reflection of $[0, r]$ about the midpoint $r/2$.)

We assume that $\phi : [0, r] \to [0, r]$ *is piecewise linear.*

Since ϕ is orientation reversing, there is a *unique* $t_0 \in (0, r)$ such that $\phi(t_0) = t_0$. Suppose that $f \in G(r, A, P) \subset T(r, A, P)$ has support $\mathrm{supp}(f) \subset (0, t_0)$. Then $\phi f \phi^{-1}$ has support in (t_0, r). It follows that each of the homeomorphisms $\phi^2, f, \phi f \phi^{-1}, u := f\phi f \phi^{-1}$ maps $[0, t_0] =: J_0$ (resp. $[t_0, r] = J_1$) to itself, fixing the end points. Also $\phi(J_i) = J_{1-i}$ and so, if $z \in T(r, A, P)$, then $zu\phi^2 z^{-1}(z(J_i)) = z(J_i)$.

Let $b_0(X)$ denote the 0-th Betti number of X. Note that if $\psi : [0, r] \to [0, r]$ is a PL-homeomorphism then $b_0(\mathrm{supp}(\psi)), b_0(\mathrm{Fix}(\psi))$ are finite. We will construct a sequence of elements $f_m \in G(r, A, P)$ such that $\{b_0(\mathrm{supp}(f_m\alpha(f_m)\phi^2))\}_{m\geq 1}$ (resp. $\{b_0(\mathrm{Fix}(f_m\alpha(f_m)\phi^2))\}_{m\geq 1}$) is an unbounded sequence, when $\mathrm{supp}(\phi^2)$ is not dense (resp. when $\mathrm{supp}(\phi^2)$ is dense).

Case 1. Suppose that ϕ^2 is identity in an interval J. If $t \in \phi(J)$, write $t = \phi(s), s \in J$. Now $\phi^2(t) = \phi^3(s) = \phi(\phi^2(s)) = \phi(s) = t$. So $\phi^2 \restriction \phi(J)$ is also identity. Let $J = (a, b)$, where $0 < a < b \leq t_0$. Thus ϕ^2 is identity in $J \cup \phi(J) = (a, b) \cup (\phi(b), \phi(a))$. We choose $f_m \in G(r, A, P)$ to have support a union of m pairwise disjoint intervals $I_1, \ldots, I_m$ contained in J. Then $u_m = f_m.\phi f_m \phi^{-1}$ has support $U_m := \mathrm{supp}(f_m) \cup \phi(\mathrm{supp}(f_m))$ and moreover, $\mathrm{supp}(u_m\phi^2)$ equals $U_m \cup \mathrm{supp}(\phi^2)$. Note that $\mathrm{supp}(\phi^2)$ is a disjoint union of *finitely many*—say k—intervals, in view of our assumption that ϕ is a PL-homeomorphism. Since $J \cup \phi(J)$ is disjoint from the support of ϕ^2, the support of $u_m\phi^2$ is a disjoint union of exactly $2m + k$ intervals.

Case 2. Suppose that $\mathrm{supp}(\phi^2)$ is a dense open subset of $(0, r)$. Since ϕ^2 is piecewise linear it follows that $\mathrm{Fix}(\phi^2)$ is a finite set.

First we make a preliminary observation.

Lemma 3.2 *Suppose that* $\psi : [a, b] \to [a, c]$ *is the affine isomorphism fixing* a*, namely,* $\psi(t) = \lambda(t - a) + a$ *where* $\lambda = (c - a)/(b - a) \neq 1$*. Let* $m \geq 1$*. Then there exists a PL-homeomorphism* $f : [a, c] \to [a, c]$ *such that (i)* f *is identity near the end points and* $2m \leq \#Fix(f \circ \psi) < \infty$,
(ii) slopes of f *are in* P *and break points of* f *are in* A.

Proof We will assume that $\lambda > 1$; the case when $\lambda < 1$ being similar. Thus we have $t < \psi(t)$ for $a < t < b$.

Step 1 First we prove the lemma for $m = 1$. The required f will have support equal to an interval $(a_0, c_0) \subset (a, b)$ and will map a sub interval (a_0, b_0) into an interval (a_0, b_1) by an affine map with sufficiently small slope so that $b_1 < \psi^{-1}(b_0)$. Then $f \circ \psi([\psi^{-1}(a_0), \psi^{-1}(b_0)]) = f([a_0, b_0]) = [a_0, b_1] \subset [\psi^{-1}(a_0), \psi^{-1}(b_0)]$ and so $f \circ \psi$ fixes a point in $(a, \psi^{-1}(b_0))$.

Choose $a_0 \in A \cap (a, b)$ so that $a < a_0 < \psi(a_0) < c$. Choose $b_0, c_0 \in A$ such that $a_0 < b_0 < c_0 < \psi(a_0)$. Choose $p \in P$, $p > \lambda$; we shall presently refine our choice of p. Set $b_1 := p^{-1}(b_0 - a_0) + a_0 \in A$. Then $t \mapsto a_0 + p^{-1}(t - a_0)$ defines a PL-homeomorphism $h_0 : [a_0, b_0] \to [a_0, b_1]$. We choose p so large that $a + (b_0 - a)/\lambda = \psi^{-1}(b_0) > b_1$—in fact any $p \in P$ such that $p > \frac{b_0 - a_0}{\psi^{-1}(b_0) - a_0}$ will do. Now we choose a PL-homeomorphism $h_1 : [b_0, c_0] \to [b_1, c_0]$ with slopes in $P_0 := \langle p \rangle \subset P$ and break-points in A. The existence of such a homeomorphism follows from [1, TheoremA4.1] in view of the fact that $c_0 - b_1 = c_0 - b_0 + (1 - p^{-1})(b_0 - a_0) \in IP_0.A$.

We piece together the two homeomorphisms h_0, h_1 to obtain $f : [a, c] \to [a, c]$ with support in (a_0, c_0) slopes in $P_0 = \langle p \rangle$, break points in A. Explicitly, we define as follows: $f \restriction [a_0, b_0] = h_0$, $f \restriction [b_0, c_0] = h_1$ and is identity on $[a, a_0] \cup [c_0, c]$. We claim that $f \circ \psi$ has at least two fixed points: one in $I_0 := (\psi^{-1}(a_0), \psi^{-1}(b_0))$ and one in $I_1 := (\psi^{-1}(b_0), a_0)$. This is because $f \circ \psi(I_0) \subset [f(a_0), f(b_0)] \subset [a_0, b_1] \subset [\psi^{-1}(a_0), \psi^{-1}(b_0)] = I_0$. Similarly, $f \circ \psi(I_1) \subset f([b_0, \psi(a_0)]) = [b_1, \psi(a_0)] \subset [\psi^{-1}(b_0), a_0] = I_1$. Thus $f \circ \psi$ has at least 2 fixed points in $[a, b]$. The slopes of $f \circ \psi$ are all in λP_0. Since $1 < \lambda < p$ we have $1 \notin \lambda P_0$, (as $\lambda \notin P_0$) and we see that $f\psi$ has only *finitely* many fixed points in $[a, b]$.

Step 2 Let m be any positive integer. Consider the points a_j, $1 \leq j \leq m$ in $A \cap (a, b)$ such that $a_{j+1} < \psi(a_{j+1}) < a_j$ for all j. Choose $b_j, c_j \in A$ such that $a_j < b_j < c_j < \psi(a_j)$. Proceeding as in step 1, we obtain a PL-homeomorphism $f_j : [a, b] \to [a, c]$ with support in (a_j, c_j) such that $f_j \circ \psi$ has (at least) two fixed points. Since the f_j have disjoint support we see that $f := f_1 \circ \cdots \circ f_m : [a, c] \to [a, c]$ is identity near the end points and $f \circ \psi$ has finitely many fixed points, the number of fixed points being at least $2m$. □

We are now ready to construct, in the lemma below, a sequence $\{f_m\}_{m \geq 1}$ in $G(r, A, P)$ with the asserted property.

Lemma 3.3 *Let $\phi : S_r \to S_r$ be an orientation reversing PL-homeomorphism that fixes $\bar{0}$ and induces an automorphism $\alpha \in T(r, A, P)$. Suppose that support of $\phi^2 : [0, r] \to [0, r]$ is dense. Then there exist a sequence of elements $\{f_m\}_{m \geq 1}$ in $G(r, A, P)$ such that $2m \leq b_0(Fix(u_m \circ \phi^2)) < \infty$ where $u_m = f_m \alpha(f_m) = f_m \phi f_m \phi^{-1}$.*

Proof Clearly ϕ has a unique fixed point, denoted t_0, in $(0, r)$. Our assumption on ϕ^2 implies that t_0 is an isolated fixed point of ϕ^2. Let $t_1 > t_0$ be sufficiently close to t_0 so that $\phi^2(t) = \lambda(t - t_0) + t_0$ for $t_0 \leq t \leq t_1$. Taking $\psi := \phi^2 \restriction [t_0, t_1]$ we are in the situation of Lemma 3.2 and we obtain PL-homeomorphisms $g_m : [t_0, \lambda t_1] \to [t_0, \lambda t_1]$ which is supported in (t_0, t_1) has break points in A, slopes in P, such that $g_m \circ \psi$ has at least $2m$ fixed points in $[t_0, t_1]$. We extend g_m to an element $f_m \in G(r, A, P)$ with support the same as that of g_m. Then the support of $\phi f_m \phi^{-1}$ equals $\phi(\text{supp}(f_m)) \subset (0, t_0)$ and hence disjoint from $\text{supp}(f_m)$. Now let $u_m = f_m . \phi f_m \phi^{-1} = \phi f_m \phi^{-1} f_m$. Then $u_m \phi^2$ has at least $2m$ *isolated* fixed points in (t_0, t_1). It follows that $b_0(\text{Fix}(u_m \phi^2)) \geq 2m$. □

4 Proof of Theorem 1.1

Let α be an automorphism of $T(r, A, P)$. It is represented by a homeomorphism $\phi : S_r \to S_r$. As already observed in Sect. 1, it suffices to show that $R(\alpha) = \infty$ when α restricts to an automorphism α_0 of $G([0, r], A, P)$. So we assume that $\phi(\bar{0}) = \bar{0}$. Our hypothesis P is non-cyclic implies, by Theorem 2.3, that ϕ is piecewise linear.

Suppose that ϕ is orientation preserving. For each $p \in P$ sufficiently large, we constructed in Lemma 3.1 an element $h_p \in G(r, A, P)$ with slope $p \in P$ near 0 and such that $\phi < h_p$ (that is, $\phi(t) < h_p(t), 0 < t < r$). Let $f_p = h_p^{-1}$. Then $f_p \phi < id$ for all p sufficiently large. This implies that $\bar{0}$ is the *only* fixed point of $f_p \phi : S_r \to S_r$.

Suppose that $R(\alpha) < \infty$. Choose $p, q \in P$ sufficiently large and distinct such that f_p and f_q must be α-twisted conjugates. From Equation (2), we have that $f_q \phi = z f_p \phi z^{-1}$ for some $z = z_{p,q} \in T(r, A, P)$. Evaluating at $z(\bar{0})$ we obtain $f_q \phi(z(\bar{0})) = z(\bar{0})$. This forces that $z(\bar{0}) = \bar{0}$ since $f_q \phi$ fixes no other point of S_r. Hence $z \in G(r, A, P)$ and f_p, f_q are α_0 conjugates where $\alpha_0 : G(r, A, P) \to G(r, A, P)$ is the restriction of α.

On the other hand, by [9, Sect. 3], the homomorphism $\sigma_\ell : G(r, A, P) \to P$ defined as $f \mapsto f'(0)$ is invariant under α_0, that is, $\sigma_\ell = \sigma_\ell \circ \alpha_0$ (since ϕ is orientation preserving). So $\sigma_\ell(h_p) = p$ and we have $q^{-1} = \sigma_\ell(f_q) = \sigma_\ell(zf_p\alpha_0(z^{-1})) = \sigma_\ell(z)\sigma_\ell(f_p)\sigma_\ell(\alpha_0(z^{-1})) = \sigma_\ell(z)\sigma_\ell(f_p)\sigma_\ell(z^{-1}) = \sigma_\ell(f_p) = p^{-1}$. Therefore $p = q$ which contradicts our choice.

Next assume that ϕ is orientation reversing. There are two cases to consider depending on whether $\text{supp}(\phi^2)$ is dense or not.

Suppose that $\text{supp}(\phi^2)$ is not dense. In Sect. 3.3 we constructed a sequence of elements $f_m, m \geq 1$, in $G(r, A, P)$ such that, denoting by u_M the element $f_m\alpha(f_m) = f_m.\phi f_m \phi^{-1}$, the sequence $\{b_0(\text{supp}(u_m\phi^2))\}, m \geq 1$, is an unbounded sequence of natural numbers. (Recall that $b_0(X)$ is the number of path components of X.) By passing to a subsequence we may assume that the sequence $b_m := b_0(u_m\phi^2), m \geq 1$, consists of pairwise *distinct* positive integers.

If $R(\alpha) < \infty$, then, by (2), there would be some pairs of distinct integers $m, n \geq 1$, and elements $z = z_{m,n} \in T(r, A, P)$ such that $u_m\phi^2 = z.u_n\phi^2.z^{-1}$. This means that $\text{supp}(u_m\phi^2)$ and $\text{supp}(u_n\phi^2)$ are homeomorphic. Therefore $b_m = b_0(\text{supp}(u_m\phi^2)) = b_0(\text{supp}(u_n\phi^2)) = b_n$, a contradiction since b_n are pairwise distinct and $m \neq n$ by our choice.

Finally suppose that $\text{supp}(\phi^2)$ is dense. Consider the sequence of elements $f_m, m \geq 1$ in $G(r, A, P)$ constructed in Lemma 3.3 with the property that $b_0(\text{Fix}(u_m\phi^2))$ is an unbounded sequence of natural numbers. We proceed exactly as in the previous case, replacing $b_0(\text{supp}(u_m\phi^2))$ by $b_0(\text{Fix}(u_m\phi^2))$ throughout, we arrive at a contradiction in case $R(\alpha) < \infty$.

Thus we conclude that $R(\alpha) = \infty$ and so $T(r, A, P)$ has the R_∞-property.

Remark 4.1 Suppose that P is cyclic and that $\phi \in \text{Homeo}(S_r)$ represents a given automorphism α of $T(r, A, P)$. A basic fact is that the set of singular points of ϕ (where the derivative does not exist) has Lebesgue measure 0. However if A is countable (eg. $A = \mathbb{Z}[1/p]$ where $P = \langle p \rangle$) it could so happen that every point of A is singular and it is not possible to replace ϕ by $x\phi y$ for any $x, y \in T(r, A, P)$ so as to make the resulting homeomorphism to fix $\bar{0}$ and to have (non-vanishing) one-sided derivative there. In the case when ϕ is orientable, Lemma 3.1 can be extended to the case when $\phi : [0, r] \to [0, r]$ has finite non-vanishing (one-sided) Dini numbers $D^+(\phi; 0), D_+(\phi; 0), D^-(\phi; r), D_-(\phi; r)$ at the end points. (See [13, Sect. 3, Chapter 3].) But it is possible that $D^+(\phi; 0) = \infty, D_+(\phi, 0) = 0$ and in such a case it is impossible to find a $\lambda > 0$ such that $\phi < f_\lambda$ or $f_\lambda < \phi$. For example, if the graph of ϕ meets both arcs as in Fig. 3 below arbitrarily close to 0. The arcs are tangential to horizontal (resp. vertical) axis at 0.

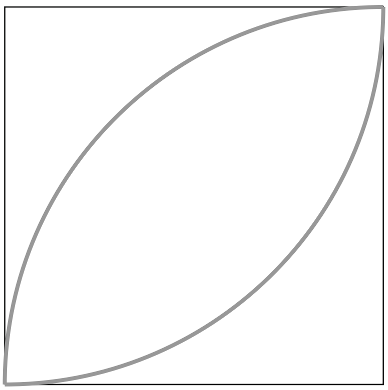


Fig. 3 Non-existence of f_λ

When $\phi : [0, r] \to [0, r]$ is orientation reversing, we used the fact that $\text{supp}(\phi^2)$ has only finitely many components for PL-homeomorphisms. This is evidently false even if the restriction of ϕ to $(0, r)$ is piecewise linear. Also the topology of $\text{Fix}(\phi)$ is possibly very complicated, containing infinitely many disjoint intervals and infinitely many discrete points. There is some room for improvement in our results of Sect. 3.3 since any $\text{Fix}(u\phi^2)$ and $\text{Fix}(zu\phi^2z^{-1})$ are *order isomorphic* as subspaces of $[0, r]$. This is a much stronger statement than the equality of their 0-th Betti numbers. But the general situation is too complex that we have not been able to exploit this.

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