

ON PARABOLIC EXTERNAL MAPS

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ABSTRACT. We prove that any $C^{1+\text{BV}}$ degree $d \geq 2$ circle covering h having all periodic orbits weakly expanding, is conjugate by a $C^{1+\text{BV}}$ diffeomorphism to a metrically expanding map. We use this to connect the space of parabolic external maps (coming from the theory of parabolic-like maps) to metrically expanding circle coverings.

1. Introduction. In this paper we provide a connection between the worlds of real and complex dynamics by proving theorems on degree $d \geq 2$ circle coverings which are interesting in the world of real dynamics per se and interesting in the world of complex dynamics through quasi-conformal surgery. The main theorem states that any $C^{1+\text{BV}}$ degree $d \geq 2$ circle covering h (where $h \in C^{1+\text{BV}}$ means Dh is continuous and of bounded variation), all of whose periodic orbits are weakly expanding, is conjugate in the same smoothness class to a metrically expanding map. Here weakly expanding means that for any periodic point p of period s there exists a punctured neighborhood of p on which $Dh^s(x) > 1$. And metrically expanding means $Dh(x) > 1$ holds everywhere, except at parabolic points. This theorem strengthens a theorem by Mañé [6] who proved the same conclusion holds under the stronger assumption that h is C^2 with all periodic points hyperbolic repelling.

The real analytic version of the above theorem, which comes for the same price, provides a missing link between the space of parabolic external maps from the theory of parabolic-like maps and metrically expanding circle coverings. For an

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enlargement on the theory of parabolic-like maps and the role of parabolic external maps in this theory see the introduction to Section 4 and the paper by the first author [4].

2. Setting and statement of the results. Recall that a smooth covering map (this is, a local diffeomorphism which is a covering map) $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ has degree $d \neq 0$ if and only if $h(e^{i2\pi x})$ lifts to the exponential $e^{i2\pi x}$ as a diffeomorphism $H : \mathbb{R} \rightarrow \mathbb{R}$ with $H(x+1) = H(x) + d$. It will be convenient to work mostly with H and the induced map also denoted $H : \mathbb{T} \rightarrow \mathbb{T} := \mathbb{R}/\mathbb{Z}$. Denote by $\text{Par}(h)$ the set of parabolic periodic points for h , and note that for $d > 0$ the multiplier of any parabolic orbit is 1.

We denote by $\mathcal{F}_d^{1+\text{BV}}$ the set of smooth covering maps $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ of degree $d > 0$ with $h \in C^{1+\text{BV}}$ (i.e. writing Dh for the derivative of h , the function Dh is continuous and of bounded variation). We set also $\mathcal{F}_d^{r+\epsilon} := \mathcal{F}_d^{1+\text{BV}} \cap C^{r+\epsilon}$ with $r = 2, 3, \dots, \infty, \omega$, and $0 < \epsilon \leq 1$, where $C^{r+\epsilon}$ are the maps which have an ϵ -Hölder r -th derivative.

On the other hand, let us call *orbit expanding* and denote by $\mathcal{O}_d^{1+\text{BV}} \subset \mathcal{F}_d^{1+\text{BV}}$ (see Figure 1) the set of maps h for which for every periodic point p say of period s , there is a neighborhood $U(p)$ of p such that for all $x \in U(p) \setminus \{p\}$ we have $Dh^s(x) > 1$. This is, h is a degree d smooth covering of the circle, with continuous derivative of bounded variation, and with all periodic points either repelling or parabolic-repelling.

Finally a map $h \in \mathcal{F}_d^{1+\text{BV}}$ is called *metrically expanding* if for all x in $\mathbb{S}^1 \setminus \text{Par}(h)$, $Dh(x) > 1$. We shall see that for such maps $\text{Par}(h)$ is a finite set. We denote by $\mathcal{M}_d^{1+\text{BV}} \subset \mathcal{O}_d^{1+\text{BV}} \subset \mathcal{F}_d^{1+\text{BV}}$ (see Figure 1) the sub-class of metrically expanding maps.

In this paper we will prove the following result (see Section 3):

Theorem 2.1. *For each map $h \in \mathcal{O}_d^{1+\text{BV}}$, the set $\text{Par}(h)$ is finite and h is conjugate to a map $\tilde{h} \in \mathcal{M}_d^{1+\text{BV}}$ via a $C^{1+\text{BV}}$ diffeomorphism.*

Moreover, if h is $C^{r+\epsilon}$ for some $r = 2, 3, \dots, \infty, \omega$, and $0 < \epsilon \leq 1$, we can take the conjugacy map to be $C^{r+\epsilon}$.

A degree d circle map $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is *topologically expanding*, if every interval eventually expands onto the entire circle or equivalently the map is topologically conjugate to the map $P_d(z) = z^d$. We denote by $\mathcal{T}_d^{1+\text{BV}} = \{h \in \mathcal{F}_d^{1+\text{BV}} \mid h \text{ is topologically expanding}\}$. Theorem 2.1 implies the following:

Corollary 2.2. *We have $\mathcal{O}_d^{1+\text{BV}} = \mathcal{T}_d^{1+\text{BV}}$ and for all $h \in \mathcal{M}_d^{1+\text{BV}}$ the set $\text{Par}(h)$ is a finite set.*

We now restrict our attention to real analytic coverings maps with at least one parabolic orbit. So let us denote by $\mathcal{F}_d \subset \mathcal{F}_d^{1+\text{BV}}$ the set of all real analytic smooth covering maps $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ of degree d , by $\mathcal{T}_d = \mathcal{F}_d \cap \mathcal{T}_d^{1+\text{BV}}$, by $\mathcal{T}_{d,*} \subset \mathcal{T}_d$ the set of maps $h \in \mathcal{F}_d$ topologically expanding and for which $\text{Par}(h) \neq \emptyset$, and by $\mathcal{T}_{d,1} \subset \mathcal{T}_{d,*}$ the set of such h with $\text{Par}(h)$ a singleton. Similarly, denote by $\mathcal{M}_d = \mathcal{M}_d^{1+\text{BV}} \subset \mathcal{F}_d$ the set of degree d real analytic metrically expanding coverings $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$, by $\mathcal{M}_{d,*} = \mathcal{M}_d \cap \mathcal{T}_{d,*}$ the set of such h for which $\text{Par}(h) \neq \emptyset$, and by $\mathcal{M}_{d,1} \subset \mathcal{M}_{d,*}$ the set of such h with precisely one parabolic point.

An *external map* is a map $h \in \mathcal{F}_d$ with the following properties:

- $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is a degree $d \geq 2$ real analytic covering of the unit circle, with a finite set $\text{Par}(h)$ of parabolic points p of multiplier 1,

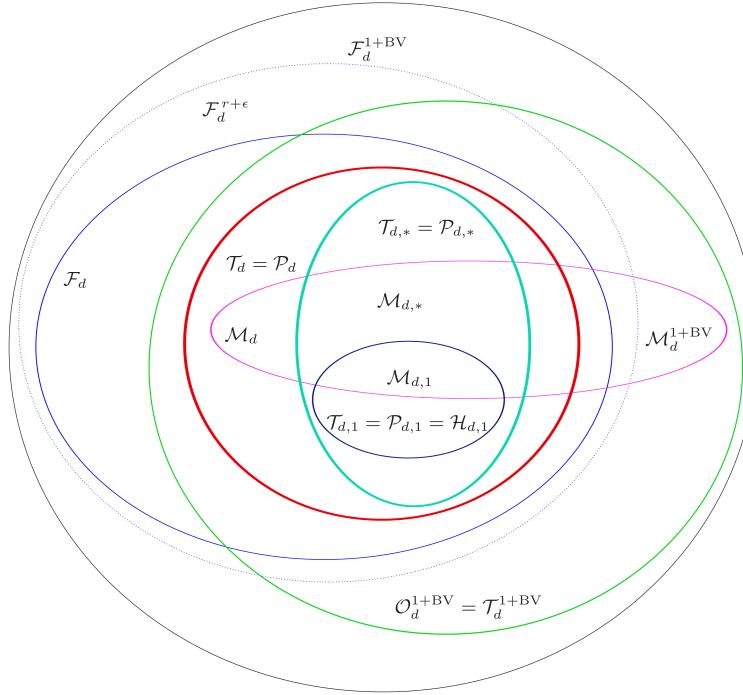
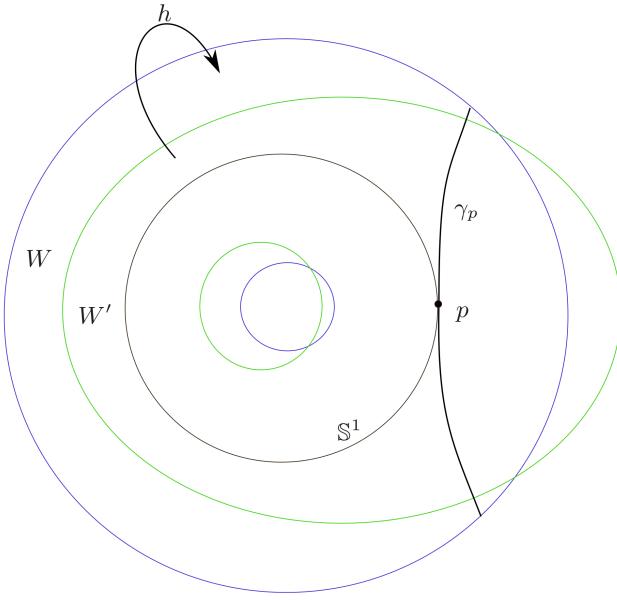


FIGURE 1. A map of the maps we consider. \mathcal{F}_d^{1+BV} is the set of degree d smooth covering $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ with $h \in C^{1+BV}; \mathcal{O}_d^{1+BV}$ is the set of maps $h \in \mathcal{F}_d^{1+BV}$ for which for every periodic point p say of period s , there is a neighborhood $U(p)$ of p such that for all $x \in U(p) \setminus \{p\}$ we have $Dh^s(x) > 1$; while \mathcal{M}_d^{1+BV} and \mathcal{T}_d^{1+BV} are the class of respectively metrically and topologically expanding $h \in \mathcal{F}_d^{1+BV}$. \mathcal{F}_d is the class of real analytic degree d circle coverings, \mathcal{T}_d and \mathcal{M}_d the set of respectively topologically and metrically expanding $h \in \mathcal{F}_d$, and $\mathcal{T}_{d,*}$ and $\mathcal{M}_{d,*}$ the set of respectively topologically and metrically expanding $h \in \mathcal{F}_d$ for which $\text{Par}(h) \neq \emptyset$. Also, \mathcal{P}_d is the class of extenal maps and $\mathcal{P}_{d,*}$ the class of parabolic external maps. Finally, $\mathcal{H}_{d,1} = \{h \in \mathcal{F}_d \mid h \sim_{qs} h_d(z) = \frac{z^d + (d-1)/(d+1)}{(d-1)z^d/(d+1) + 1}\}$. By Corollary 2.2, $\mathcal{O}_d^{1+BV} = \mathcal{T}_d^{1+BV}$, and by Theorem 2.4, $\mathcal{M}_d \subset \mathcal{P}_d = \mathcal{T}_d$, $\mathcal{M}_{d,*} \subset \mathcal{P}_{d,*} = \mathcal{T}_{d,*}$ and $\mathcal{M}_{d,1} \subset \mathcal{P}_{d,1} = \mathcal{H}_{d,1} = \mathcal{T}_{d,1}$.

- the map h extends to a holomorphic covering map $h : W' \rightarrow W$ of degree d , where W' , W are reflection symmetric annular neighborhoods of \mathbb{S}^1 . We write $W_+ := W \setminus \overline{\mathbb{D}}$, and $W'_+ := W' \setminus \overline{\mathbb{D}}$,
- for each $p \in \text{Par}(h)$ there exists a *dividing arc* γ_p satisfying :
 - $p \in \gamma_p \subset \overline{W} \setminus \mathbb{D}$ and γ_p is smooth except at p ,
 - $\gamma_p \cap \gamma_{p'} = \emptyset$ for $p \neq p'$,
 - $h : \gamma_p \cap W' \rightarrow \gamma_{h(p)}$ is a diffeomorphism,
 - γ_p divides W and W' into Ω_p , Δ_p and Ω'_p , Δ'_p respectively, all connected, and such that $h : \Delta'_p \rightarrow \Delta_{h(p)}$ is an isomorphism and $\mathbb{D} \cup \Omega'_p \subset \mathbb{D} \cup \Omega_p$,

FIGURE 2. A parabolic external map in $\mathcal{P}_{d,1}$.

– calling $\Omega = \bigcap_p \Omega_p$ and $\Omega' = \bigcap_p \Omega'_p$, we have $\Omega' \cup \mathbb{D} \subset\subset W \cup \mathbb{D}$.

We denote by $\mathcal{P}_d \subset \mathcal{F}_d$ the set of external maps, by $\mathcal{P}_{d,*} \subset \mathcal{P}_d$ the set of *parabolic external maps* $h \in \mathcal{P}_d$ with $\text{Par}(h) \neq \emptyset$, and by $\mathcal{P}_{d,1} \subset \mathcal{P}_d$ the set of $h \in \mathcal{P}_d$ for which $\text{Par}(h)$ is a singleton (see Figure 2).

To emphasize the geometric properties of maps $h \in \mathcal{P}_d$ we shall also write (h, W', W, γ) for such maps, where $\gamma = \bigcup_p \gamma_p$, though neither the domain, range or dividing arcs are unique or in any way canonical. An external map for any parabolic-like map belongs to $\mathcal{P}_{d,*}$ (see Section 4.2). Note that the set \mathcal{P}_d is invariant under conjugacy by a real analytic diffeomorphism: for any $h \in \mathcal{P}_d$ and $\phi \in \mathcal{F}_1$, $\phi \circ h \circ \phi^{-1} \in \mathcal{P}_d$. It is easy to see that $\mathcal{P}_d \subset \mathcal{T}_d$ and $\mathcal{P}_{d,*} \subset \mathcal{T}_{d,*}$ (see Proposition 4.2). In particular, $\mathcal{P}_{d,1} \subset \mathcal{T}_{d,1}$, and so any two maps $h_1, h_2 \in \mathcal{P}_{d,1}$ are topologically conjugate by a unique orientation preserving homeomorphism sending the parabolic point to the parabolic point.

Consider the map $h_d(z) = \frac{z^d + (d-1)/(d+1)}{(d-1)z^d/(d+1) + 1}$. It has a parabolic fixed point at $z = 1$ of multiplier 1, and critical points at $z = 0$ and at $z = \infty$. Both \mathbb{D} and $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ are basins of attraction of the parabolic fixed point, while \mathbb{S}^1 is the Julia set for h_d . The map h_d plays in the parabolic-like map theory the same role the map $z \rightarrow z^d$ plays in the theory of polynomial-like maps; in particular, in degree 2 it is the external map for any member of the model family $P_A(z) = z + 1/z + A$, $A \in \mathbb{C}$, (see Proposition 4.2 in [4]). Define $\mathcal{H}_{d,1} = \{h \in \mathcal{F}_d \mid h \sim_{qs} h_d\}$ (where $h \sim_{qs} h_d$ means that h is quasi-symmetrically conjugate to the map h_d). It is rather easy to see that $h_d \in \mathcal{P}_{d,1}$, see Lemma 4.2. Moreover, clearly $h_d \in \mathcal{T}_{d,1}$, so that $\mathcal{H}_{d,1} \subseteq \mathcal{T}_{d,1}$. In Section 4.3 we prove:

Proposition 2.3. *Suppose $h_1, h_2 \in \mathcal{P}_{d,*}$ are topologically conjugate by an orientation preserving homeomorphism ϕ , which preserves parabolic points. Then ϕ is quasi-symmetric. In particular $\mathcal{P}_{d,1} \subseteq \mathcal{H}_{d,1} \subseteq \mathcal{T}_{d,1}$.*

Let $\widehat{\mathcal{F}}_d := \mathcal{F}_d/\mathcal{F}_1$ denote the set of conjugacy classes of maps in \mathcal{F}_d under real analytic diffeomorphism, and call $\pi_d : \mathcal{F}_d \rightarrow \widehat{\mathcal{F}}_d$ the natural projection. As a consequence of the above we have (see also page 5100):

Theorem 2.4. *For every $d \geq 2$ we have*

$$\mathcal{M}_d \subset \mathcal{P}_d = \mathcal{T}_d, \quad \mathcal{M}_{d,*} \subset \mathcal{P}_{d,*} = \mathcal{T}_{d,*} \quad \text{and} \quad \mathcal{M}_{d,1} \subset \mathcal{P}_{d,1} = \mathcal{H}_{d,1} = \mathcal{T}_{d,1}$$

Moreover

$$\begin{aligned} \pi_d(\mathcal{M}_d) &= \pi_d(\mathcal{P}_d) = \pi_d(\mathcal{T}_d), \\ \pi_d(\mathcal{M}_{d,*}) &= \pi_d(\mathcal{P}_{d,*}) = \pi_d(\mathcal{T}_{d,*}), \quad \text{and} \\ \pi_d(\mathcal{M}_{d,1}) &= \pi_d(\mathcal{P}_{d,1}) = \pi_d(\mathcal{H}_{d,1}) = \pi_d(\mathcal{T}_{d,1}). \end{aligned}$$

3. Proof of Theorem 2.1. Recall that for each integer $d \geq 2$, the set $\mathcal{O}_d^{1+\text{BV}}$ denotes the collection of all orientation preserving covering maps $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ with the following properties:

1. h has degree d ;
2. h is a C^1 local diffeomorphism and the derivative Dh has bounded variation;
3. If p is a periodic point of h with period s , then there is a neighborhood $U(p)$ of p such that $Dh^s(x) > 1$ holds for all $x \in U(p) \setminus \{p\}$.

In the next section 3.1 we will prove the following

Theorem 3.1. *For each map $h \in \mathcal{O}_d^{1+\text{BV}}$, $\text{Par}(h)$ is finite. Moreover, there exists a positive integer N and a real analytic function $\rho : \mathbb{S}^1 \rightarrow \mathbb{R}^+$ such that*

$$|Dh^N(x)|_\rho := \frac{\rho(h^N(x))}{\rho(x)} Dh^N(x) > 1$$

holds for all $x \in \mathbb{S}^1 \setminus \text{Par}(h)$.

In particular, the theorem claims that a map $h \in \mathcal{O}_d^{1+\text{BV}}$ without neutral cycles is uniformly expanding on the whole phase space \mathbb{S}^1 , a result proved by Mañé [6] under a stronger assumption that h is C^2 . Some partial result on the validity of Mañé's theorem under the $C^{1+\text{BV}}$ condition was obtained in [5].

Recall that a map $h \in \mathcal{O}_d^{1+\text{BV}}$ is called *metrically expanding* if $Dh(x) > 1$ holds for $x \in \mathbb{S}^1 \setminus \text{Par}(h)$. Theorem 3.1 implies the following

Theorem 3.2. *Each map $h \in \mathcal{O}_d^{1+\text{BV}}$ is conjugate to a metrically expanding map via a $C^{1+\text{BV}}$ diffeomorphism. Moreover, if h is $C^{r+\epsilon}$, $r = 2, 3, \dots, \infty, \omega$, $0 < \epsilon \leq 1$, we can take the conjugacy map to be $C^{r+\epsilon}$.*

Proof. Let ρ and N be given by Theorem 3.1 and set

$$\rho_*(x) = \sum_{j=0}^{N-1} \rho(h^j(x)) Dh^j(x)$$

which is a continuous function with bounded variation. Then a computation shows that

$$|Dh(x)|_{\rho_*} = Dh(x) \cdot \frac{\rho_*(h(x))}{\rho_*(x)} = \frac{Dh^N(x)\rho(h^N(x)) + \sum_{j=1}^{N-1} Dh^j(x)\rho(h^j(x))}{\rho(x) + \sum_{j=1}^{N-1} Dh^j(x)\rho(h^j(x))}$$

which is strictly greater than 1 for $x \in \mathbb{S}^1 \setminus \text{Par}(h)$.

Let us complete the proof. Identify \mathbb{S}^1 with \mathbb{R}/\mathbb{Z} via $e^{i2\pi x} \mapsto x \bmod 1$. Then the map

$$\phi(x) := \int_0^x C\rho_* dx = C \int_0^x \rho_* dx, \text{ with } \frac{1}{C} = \int_0^1 \rho_* dx$$

defines a $C^{1+\text{BV}}$ diffeomorphism of \mathbb{S}^1 , and setting $g := \phi \circ h \circ \phi^{-1}$, we obtain that $Dg(\phi(x)) = |Dh(x)|_{\rho_*} > 1$ for all $x \in \mathbb{S}^1 \setminus \text{Par}(h)$.

Clearly, if h is C^r then so is ϕ . \square

The condition that Dh has bounded variation is used to control the distortion. Recall that the distortion of h on an interval $J \subset \mathbb{S}^1$ is defined as

$$\text{Dist}(h, J) = \sup_{x,y \in J} \log \frac{|Dh(x)|}{|Dh(y)|}.$$

For proving Theorem 3.1 we will use the following

Lemma 3.1. *There exists a $C_0 > 0$ such that, for any interval $J \subset \mathbb{S}^1$ and $n \geq 1$, if $J, h(J), \dots, h^{n-1}(J)$ are intervals with pairwise disjoint interiors, then*

$$\text{Dist}(h^n, J) \leq C_0.$$

Proof. Since h is a C^1 covering and Dh has bounded variation, $\log Dh$ also has bounded variation. For each $x, y \in J$,

$$\left| \log \frac{Dh^n(x)}{Dh^n(y)} \right| \leq \sum_{i=0}^{n-1} \left| \log \frac{Dh(h^i(x))}{Dh(h^i(y))} \right| \leq \sum_{i=0}^{n-1} \text{Var}(\log Dh, h^i(J))$$

is bounded from above by the total variation of $\log Dh$. \square

3.1. Proof of Theorem 3.1. The main step is to prove that a map $h \in \mathcal{O}_d^{1+\text{BV}}$ has the following expanding properties, which we will assume first, and prove in the next subsection 3.1.1.

Proposition 3.3. *For each $h \in \mathcal{O}_d^{1+\text{BV}}$ the following properties hold:*

- (a) *Par(h) is a finite set.*
- (b) *There exists a constant $K_0 > 0$ such that $Dh^k(x) \geq K_0$ holds for each $x \in \mathbb{S}^1$ and $k \geq 1$.*
- (c) *For each $x \notin \bigcup_{k=0}^{\infty} h^{-k}(\text{Par}(h))$, $Dh^n(x) \rightarrow \infty$ as $n \rightarrow \infty$.*
- (d) *Let p be a fixed point and let $\delta_0 > 0$, $K > 0$ be constants. Then there exists $\delta = \delta(p, \delta_0, K) > 0$ such that if*

$$d(x, p) < \delta \text{ and } \max_{j=1}^k d(h^j(x), p) \geq \delta_0,$$

then $Dh^k(x) \geq K$.

- (e) *For any $K > 0$, there exists a positive integer n_0 such that for each $n > n_0$ and $x \in h^{-n}(\text{Par}(h)) \setminus h^{-n+1}(\text{Par}(h))$, we have $Dh^n(x) \geq K$.*

Proof of Theorem 3.1 assuming Proposition 3.3 Replacing h by an iterate if necessary, we may assume that *all points in $\text{Par}(h)$ are fixed points* (since $\text{Par}(h)$ is finite). We say that a function $\rho : \mathbb{S}^1 \rightarrow (0, \infty)$ is *admissible* if the following properties are satisfied:

- (A1) *there is $\delta_0 > 0$ such that whenever $x \in B(p, \delta_0) \setminus \{p\}$ for some $p \in \text{Par}(h)$, we have $\rho(h(x)) > \rho(x)$;*
- (A2) *for any $x \in \mathbb{S}^1 \setminus \text{Par}(h)$ and $s \geq 1$ with $h^s(x) \in \text{Par}(h)$, we have*

$$|Dh^s(x)|_{\rho} \geq 2.$$

We will first prove the existence of an admissible function ρ (Lemma 3.2) real analytic (Lemma 3.2 together with Lemma 3.3), and then we will prove that every $x_0 \in \mathbb{S}^1$ has a neighborhood $U(x_0)$ such that for all $x \in U(x_0) \setminus \text{Par}(h)$, we have $|Dh^k(x)|_\rho > 1$, where $k = k(U(x_0)) > 0$ (see Completion of proof of Theorem 3.1). By compactness, this gives us Theorem 3.1.

Lemma 3.2. *There exists a real analytic admissible function ρ .*

Proof. Let $X_0 = \text{Par}(h)$ and $X_k = h^{-k}(\text{Par}(h)) \setminus h^{-k+1}(\text{Par}(h))$ for each $k \geq 1$. By Proposition 3.3 (e), there exists n_0 such that $Dh^n(x) \geq 4$ holds for $x \in X_n$, $n > n_0$. Let $\rho_0 = \min\{Dh^k(x) : x \in X_k \text{ for some } k = 1, 2, \dots, n_0\}$. Let $\pi : \mathbb{R} \rightarrow \mathbb{S}^1$ be the universal covering $\pi(t) = e^{2\pi it}$. Let $\hat{\rho} : \mathbb{R} \rightarrow (0, \infty)$ be a real analytic function of period 1 with the following properties:

- (i) $\hat{\rho}(\hat{p}) = 1$, $\hat{\rho}'(\hat{p}) = 0$ and $\hat{\rho}''(\hat{p}) > 0$ for each $\hat{p} \in \pi^{-1}(\text{Par}(h))$;
- (ii) $\hat{\rho}(\hat{x}) < \rho_0/2$ holds for each $\hat{x} \in \pi^{-1}(X_1 \cup X_2 \cup \dots \cup X_{n_0})$;
- (iii) $0 < \hat{\rho}(\hat{x}) < 2$ for all $\hat{x} \in \mathbb{R}$.

It is easy to see that there is a smooth function $\check{\rho}$ satisfying all the requirements. To get a real analytic one, choose $\epsilon > 0$ such that (ii) holds for $\check{\rho}$ on a 2ϵ -neighbourhood of $\pi^{-1}(X_1 \cup X_2 \cup \dots \cup X_{n_0})$ and $\check{\rho}''(x) > 0$ on a ϵ -neighbourhood of $\pi^{-1}(\text{Par}(h))$. Write $\pi^{-1}(\text{Par}(h)) \cap [0, 1[:= \{\hat{p}_1 < \dots < \hat{p}_n\}$ and let $\delta > 0$ be given by Lemma 3.3 below. And let $\tilde{\rho}$ be a partial sum of the Fourier series of $\check{\rho}$ satisfying $\tilde{\rho}'(y_j) = 0$ for some y_j with $|y_j - \hat{p}_j| < \delta$ for each j , $\tilde{\rho}''(x) > 0$ on a ϵ -neighbourhood of $\pi^{-1}(\text{Par}(h))$ and $\tilde{\rho} < \rho_0/2$ on an ϵ -neighbourhood of $\pi^{-1}(X_1 \cup X_2 \cup \dots \cup X_{n_0})$. Let ϕ be the corresponding real analytic diffeomorphism given by Lemma 3.3. Then $\hat{\rho} = \tilde{\rho} \circ \phi$ is the desired real analytic function.

The function $\hat{\rho}$ induces a function $\rho : \mathbb{S}^1 \rightarrow \mathbb{R}$ by the formula $\rho(e^{2\pi t}) = \hat{\rho}(t)$. The property (A1) follows from (i) immediately. Let us check the property (A2). Of course it suffices to show $|Dh^n(x)|_\rho \geq 2$ for each $x \in X_n$, $n \geq 1$. If $n \leq n_0$, then $Dh^n(x) \geq \rho_0$, $\rho(x) \leq \rho_0/2$ and $\rho(h^n(x)) = 1$, hence $|Dh^n(x)|_\rho \geq 2$. If $n > n_0$, then $Dh^n(x) \geq 4$, $\rho(x) < 2$ and $\rho(h^n(x)) = 1$, hence again $|Dh^n(x)|_\rho \geq 2$. \square

The following Lemma completes the previous one, expliciting the details used for obtaining the admissible function ρ (induced by $\hat{\rho} = \tilde{\rho} \circ \phi$) real analytic.

Lemma 3.3. *Given $\epsilon > 0$ and $n \geq 1$ distinct $x_1 < \dots < x_n < x_1 + 1$ there exists $\delta > 0$ such that for any set of n points y_1, \dots, y_n with $|y_j - x_j| < \delta$ for each j , $1 \leq j \leq n$ there exists a real analytic diffeomorphism $\phi : \mathbb{R} \rightarrow \mathbb{R}$, satisfying for all $x \in \mathbb{R}$: $\phi(x+1) = \phi(x) + 1$, $|\phi(x) - x| < \epsilon$, and $|\phi'(x) - 1| < \epsilon$ and for each j : $\phi(x_j) = y_j$.*

Proof. If $n = 1$ set $\delta = \epsilon$ and $\phi(x) = x + y_1 - x_1$. Otherwise set

$$m = \min\{(x_2 - x_1), \dots, (x_n - x_{n-1}), (1 + x_1 - x_n)\}$$

and define $g_j(x) := \sin^2(\pi(x - x_j))$ for each j , $1 \leq j \leq n$. Then g_j is 1-periodic, $0 \leq g_j(x) \leq 1$ for all x with $g_j(x) = 0$ only at x_j , and the absolute value of $g'_j(x) = \pi \sin(2\pi(x - x_j))$ is bounded by π . Set

$$G_j(x) := \prod_{i, i \neq j} g_i(x)$$

So that $0 \leq G_j(x) \leq 1$, $G_j(x_i) = 0$ for $i \neq j$, $|G'_j(x)| \leq \pi(n-1)$ and

$$G_j(x_j) = \prod_{i, i \neq j} g_i(x_j) \geq K(m)$$

where $K(m)$ is a constant depending only on m . Define

$$\phi(x) = x + \sum_{j=1}^n (y_j - x_j) \frac{G_j(x)}{G_j(x_j)}$$

so that $\phi(x_j) = y_j$ for each j . Then for $\delta = m\epsilon/n^2$ and for each $j : |y_j - x_j| < \delta$ the map ϕ is the desired diffeomorphism. \square

Fix an admissible function ρ as above and let

$$\eta = \inf_{y \in \mathbb{S}^1} \rho(y) / \sup_{y \in \mathbb{S}^1} \rho(y) \quad (1)$$

Note that $|Dh^k(x)|_\rho \geq \eta Dh^k(x)$ holds for any $x \in \mathbb{S}^1$ and any $k \geq 1$.

We say that a set U is *eventually expanding* if there exists a positive integer $k(U)$ such that whenever $k \geq k(U)$ and $x \in U \setminus \text{Par}(h)$, we have $|Dh^k(x)|_\rho > 1$. The assertion of Theorem 3.1 is that \mathbb{S}^1 is eventually expanding.

Completion of proof of Theorem 3.1. By compactness, it suffices to show that each $x_0 \in \mathbb{S}^1$ has an eventually expanding neighborhood $U(x_0)$.

Case 1. Assume $h^k(x_0) \notin \text{Par}(h)$ for each $k \geq 0$. Then by Proposition 3.3 (c), $Dh^k(x_0) \rightarrow \infty$ as $k \rightarrow \infty$, so by continuity, there exists a k_0 and a neighborhood $U(x_0)$ of x_0 such that, for $x \in U(x_0)$, $Dh^{k_0}(x) \geq \frac{2}{K_0\eta}$. By Proposition 3.3 (b), for all $k \geq k_0$ and $x \in U(x_0)$,

$$Dh^k(x) = Dh^{k_0}(x) Dh^{k-k_0}(h^{k_0}(x)) \geq K_0 Dh^{k_0}(x) \geq \frac{2}{\eta},$$

hence

$$|Dh^k(x)|_\rho \geq \eta Dh^k(x) \geq 2.$$

Thus $U(x_0)$ is eventually expanding.

Case 2. Assume that $h^k(x_0) \in \text{Par}(h)$ for some $k \geq 0$. By (A2), it suffices to consider the case $x \in \text{Par}(h_0)$. Reducing δ_0 in (A1) if necessary, we may assume that $Dh(x) > 1$ holds on $B(x_0, \delta_0) \setminus \{x_0\}$. Let $K = 2/\eta$ and let $\delta = \delta(x_0, \delta_0, K) > 0$ be a small constant given by Proposition 3.3 (d). Let us prove that $|Dh^k(x)|_\rho > 1$ holds for all $x \in B(x_0, \delta) \setminus \{x_0\}$ and $k \geq 1$, so in particular, $B(x_0, \delta)$ is eventually expanding. Indeed, if $x, h(x), \dots, h^k(x) \in B(x_0, \delta_0)$, then $\rho(h^k(x)) > \rho(x)$ and $Dh^k(x) > 1$, hence $|Dh^k(x)|_\rho > 1$. Otherwise, we have $Dh^k(x) > 2/\eta$ which implies that $|Dh^k(x)|_\rho \geq \eta Dh^k(x) \geq 2$. \square

3.1.1. Geometric expanding properties of maps in $\mathcal{O}_d^{1+\text{BV}}$:

Proof of Proposition 3.3. This section is devoted to the proof of Proposition 3.3. Throughout, fix $h \in \mathcal{O}_d^{1+\text{BV}}$. We shall first establish lower bounds on the derivative of first return maps to small nice intervals.

Recall an open interval $A \subset \mathbb{S}^1$ is *nice* if $h^n(\partial A) \cap A = \emptyset$ for all $n \geq 0$. Let

$$D(A) = \{x \in \mathbb{S}^1 : \exists k \geq 1 \text{ such that } h^k(x) \in A\}.$$

For each $x \in D(A)$, the *first entry time* $k(x)$ is the minimal positive integer such that $h^{k(x)}(x) \in A$. The *first entry map* $R_A : D(A) \rightarrow A$ is defined by $x \mapsto h^{k(x)}(x)$. For $x \in D(A) \cap A$, the entry time is also called the *first return time* and the map $R_A|_{D(A) \cap A}$ is called the *first return map*. For a nice interval A and any component J of $D(A)$, the entry time $k(x)$ is independent of $x \in J$, and if we denote the common

entry time by k , then the intervals $J, h(J), \dots, h^{k-1}(J)$ are pairwise disjoint and $h^k : J \rightarrow A$ is a diffeomorphism.

In order to prove Proposition 3.3, we will first prove lower bounds for the derivative of the first entry map on small nice intervals around periodic points (Lemma 3.4). We will then use it for proving that for any $K \geq 1$, there is a s_0 such that $Dh^s(p) \geq K$ for any p periodic point of period at least s_0 (Lemma 3.5). This implies that $\text{Par}(h)$ is finite, which is the first statement of Proposition 3.3. Then, we will prove the existence of a lower bound $\lambda_0 > 1$ for the derivative of the first return map on small nice intervals about any point $x \in \mathbb{S}^1 \setminus \text{Par}(h)$ (Lemma 3.6). Finally, we will prove Proposition 3.3 using these properties, Lemma 3.1 and the compactness of \mathbb{S}^1 .

Before proceeding with the plan described above, note that there is an arbitrarily small nice interval around any point $z_0 \in \mathbb{S}^1$. Indeed, let O be an arbitrary periodic orbit such that $h^k(z_0) \notin O$ for all $k \geq 0$. Then for any n , any component of $\mathbb{S}^1 \setminus h^{-n}(O)$ is a nice interval. By [7], h has no wandering interval which implies that $h^{-n}(O)$ is dense in \mathbb{S}^1 . The statement follows.

Lemma 3.4. *For any periodic point p and any constant $K > 0$, there exists an arbitrarily small nice interval $A \ni p$ with the following property. Denote by A' the component of $D(A)$ which contains p . Then*

$$DR_A(x) > 1 \text{ for all } x \in A' \setminus \{p\}$$

and

$$DR_A(x) \geq K \text{ for all } x \in D(A) \cap (A \setminus A').$$

Proof. Let s_0 be the period of p . Let $B_0 \ni p$ be an arbitrary nice interval such that $B \cap \text{orb}(p) = \{p\}$. For each $n \geq 1$, define inductively B_n to be the component of $h^{-s_0}(B_{n-1})$ which contains p . Then B_n is a nice interval for each n and $|B_n| \rightarrow 0$ as $n \rightarrow \infty$. Let

$$\varepsilon_n = \sup\{|J| : J \text{ is a component of } h^{-i}(B_n) \text{ for some } i \geq 0\}.$$

Since h has no wandering intervals, $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$.

Let δ_0 be the minimum of the length of the components of $B_0 \setminus B_1$. Choose n large enough such that

- $\varepsilon_n \leq e^{-2C_0} \delta_0 / K$; (where C_0 is the total variation of $\log Dh$.)
- $Dh^{s_0} > 1$ on $B_{n+1} \setminus \{p\}$ (according to the third property defining $\mathcal{O}_d^{1+\text{BV}}$).

Let us verify that $A := B_n$ satisfies the desired properties. So let $x \in A \setminus A' = B_n \setminus B_{n+1}$ and let $k \geq 1$ be the first return time of x into A . We need to prove that $Dh^k(x) \geq K$.

To this end, let T be the component of $B_n \setminus B_{n+1}$ which contains x and let J be the component of $h^{-k}(B_n)$ which contains x . Then $J \subset T$ and $k > ns_0$. Note that $h^{js_0}(T)$ is a component of $B_{n-j} \setminus B_{n-j+1}$ for each $0 \leq j \leq n$. Since the first return time of p to B_0 is equal to s_0 , the intervals $B_1, h(B_1), \dots, h^{s_0-1}(B_1)$ are pairwise disjoint. Therefore, the intervals $h^j(T)$, $0 \leq j < ns_0$, are pairwise disjoint. By Lemma 3.1, $h^{s_0 n}|T$ has distortion bounded by C_0 . Since $h^{ns_0}(J)$ is a component of $h^{-k+son}(B_n)$, we have $|h^{ns_0}(J)| \leq \varepsilon_n$. Therefore,

$$\frac{|J|}{|T|} \leq e^{C_0} \varepsilon_n / \delta_0.$$

Since $J, h(J), \dots, h^{k-1}(J)$ are pairwise disjoint, by Lemma 3.1 again, we obtain

$$Dh^k(x) \geq e^{-C_0} \frac{|B_n|}{|J|} \geq e^{-C_0} \frac{|T|}{|J|} \geq K.$$

□

Lemma 3.5. *For any $K \geq 1$, there exists s_0 such that if p is a periodic point with period $s \geq s_0$ then $Dh^s(p) \geq K$. In particular, $\text{Par}(h)$ is finite.*

Proof. Let p_0 be an arbitrary fixed point of h and for each $n = 1, 2, \dots$, let

$$\varepsilon_n = \min\{|J| : J \text{ is a component of } \mathbb{S}^1 \setminus h^{-n}(p_0)\}.$$

Then $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$.

By Lemma 3.4, there is a small nice interval $A \ni p_0$ such that $DR_A \geq 1$ holds on A' and $DR_A > K \geq 1$ holds on $D(A) \cap (A \setminus A')$, where A' is the component of $h^{-1}(A)$ which contains p_0 . Let δ be the minimum of the length of the components of $A \setminus \{p_0\}$ and let $s_0 \geq 2$ be so large that $\varepsilon_s \leq \delta/(e^{C_0} K)$ for all $s \geq s_0$.

Now let p be a periodic point with period $s \geq s_0$. We shall prove that $Dh^s(p) \geq K$. Assume first that there exists $p' \in \text{orb}(p) \cap A$. Let $0 = t_0 < t_1 < t_2 < \dots < t_n = s$ the consecutive returns of p' into A . Note that there exists $0 \leq i_0 < n$ such that $h^{t_{i_0}}(p') \in A \setminus A'$, so

$$Dh^s(p) = Dh^s(p') = \prod_{i=0}^{n-1} DR_A(h^{t_i}(p')) \geq K.$$

Now assume that $\text{orb}(p) \cap A = \emptyset$. Let I be an open interval bounded by p_0 and some point p' in $\text{orb}(p)$ with the property that $I \cap \text{orb}(p) = \emptyset$. Then I is a nice interval and $|I| \geq \delta$. Let J be a component of $h^{-s}(I)$ which has p' as a boundary point. Then $h^j(J) \cap I = \emptyset$ for $j = 1, 2, \dots, s-1$ and $|J| \leq \varepsilon_s$. By Lemma 3.1, we have

$$Dh^s(p) = Dh^s(p') \geq e^{-C_0} \frac{|I|}{|J|} \geq e^{-C_0} \delta / \varepsilon_s \geq K.$$

This proves the first statement. As fixed points of h^n are isolated for each $n \geq 1$, it follows that $\text{Par}(h)$ is finite. □

Lemma 3.6. *For each $h \in \mathcal{O}_d^{1+\text{BV}}$, there exists a constant $\lambda_0 > 1$ such that for any $x \in \mathbb{S}^1 \setminus \text{Par}(h)$, if A is a sufficiently small nice interval containing x , then $DR_A \geq \lambda_0$ holds on $D(A) \cap A$.*

Proof. By Lemma 3.5, there exists s_0 such that if p is a periodic point with period $s > s_0$ then $Dh^s(p) \geq 2e^{C_0}$. Let $1 < \lambda_0 < \lambda_1 < 2$ be a constant such that if $p \notin \text{Par}(h)$ is a periodic point of period $s \leq s_0$, then $Dh^s(p) > \lambda_1$. Let $\delta > 0$ be a small constant such that $|Dh^s(x_1) - Dh^s(x_2)| < \lambda_1 - \lambda_0$ whenever $s \leq s_0$ and $\text{dist}(x_1, x_2) < \delta$.

Now let $x \in \mathbb{S}^1 \setminus \text{Par}(h)$ and let $A \ni x$ be a nice interval such that $|A| < \delta$ and $\overline{A} \cap \text{Par}(h) = \emptyset$. Now consider $y \in A$ with $k \geq 1$ as the first return time of y to A . Let J be the component of $h^{-k}(A)$ which contains y . Then $h^k : J \rightarrow A$ is a diffeomorphism with distortion bounded by C_0 (by Lemma 3.1). Since $J \subset A$, there is a fixed point p of h^k in \overline{J} . Note $p \notin \text{Par}(h)$. Since $h^j|J$ is monotone increasing for all $0 \leq j \leq k$, k is equal to the period of p . If $k \leq n_0$ then $Dh^k(p) \geq \lambda_1$, and since $|J| \leq |A| < \delta$, we have $Dh^k(y) \geq Dh^k(p) - (\lambda_1 - \lambda_0) \geq \lambda_0$. If $k > n_0$, then $Dh^k(p) \geq 2e^{C_0}$, and hence $Dh^k(y) \geq e^{-C_0} Dh^k(p) \geq 2 > \lambda_0$. □

Proof of Proposition 3.3. (a). This property was proved in Lemmas 3.5.

(b). By Lemmas 3.4 and 3.6, for any $y \in \mathbb{S}^1$ there is a nice interval $A(y) \ni y$ such that the derivative of the first return map is at least 1. By compactness, there exist $y_1, y_2, \dots, y_r \in \mathbb{S}^1$ such that $\bigcup_{i=1}^r A(y_i) = \mathbb{S}^1$. Now consider an arbitrary $x \in \mathbb{S}^1$ and $k \geq 1$. Define a sequence $\{i_n\} \subset \{1, 2, \dots, r\}$ and $\{k_n\}$ as follows. First let $k_0 = -1$, take i_0 such that $x \in A(y_{i_0})$ and let $k_1 = \max\{1 \leq j \leq k : h^j(x) \in A(y_{i_0})\}$. If $k_1 = k$ then we stop. Otherwise, take $i_1 \subset \{1, 2, \dots, r\} \setminus \{i_0\}$ be such that $h^{k_1+1}(x) \in A(y_{i_1})$ and let $k_2 = \max\{k_1 < j \leq k : h^j(x) \in A(y_{i_1})\}$. Repeat the argument until we get $k_n = k$. Then $n \leq r$ and $Dh^{k_{j+1}-k_j-1}(h^{k_j+1}(x)) \geq 1$. It follows that

$$Dh^k(x) \geq \prod_{i=1}^{n-1} Dh(h^{k_i}(x)) \geq \left(\min_{y \in \mathbb{S}^1} Dh(y) \right)^{r-1}.$$

This proves the property (b).

(c). Assuming $h^k(x) \notin \text{Par}(h)$ for all $k \geq 0$, let us prove that $Dh^k(x) \rightarrow \infty$ as $k \rightarrow \infty$. By (b), it suffices to show that $\limsup_{k \rightarrow \infty} Dh^k(x) = \infty$. Let $y \in \omega(x) \setminus \text{Par}(h)$ (where $\omega(x)$ is the ω -limit set for x) and consider a small nice interval A containing y for which the conclusion of Lemma 3.6 holds. Since $y \in \omega(x)$ there exist $n_1 < n_2 < \dots$ such that $h^{n_k}(x) \in A$. By Lemma 3.6, $Dh^{n_{k+1}-n_k}(h^{n_k}(x)) \geq \lambda_0 > 1$ for all k . Thus $Dh^{n_{k+1}}(x) \geq Dh^{n_1}(x)\lambda_0^k \rightarrow \infty$ as $k \rightarrow \infty$.

(d). The proof repeats part of the proof of Lemma 3.4. Let B_0 be a nice interval such that $B_0 \subset B(p, \delta_0)$, $B_0 \cap \text{orb}(p) = \{p\}$. Define B_n to be the component of $h^{-n}(B_0)$ which contains p . Let $\tau > 0$ be the minimal length of the components of $B_0 \setminus B_1$. Given $K > 0$ let n_0 be so large that $|B_{n_0}| < e^{-C_0} \tau K_0 / K$. Choose $\delta > 0$ such that $B(p, \delta) \subset B_{n_0}$.

Now assuming that $d(x, p) < \delta$ and $\max_{j=1}^k d(h^j(x), p) \geq \delta_0$, let us prove $Dh^k(x) \geq K$. Let $n \geq n_0$ be such that $x \in B_n \setminus B_{n+1}$. Note that $k > n$. Let J be the component of $B_n \setminus B_{n+1}$ which contains x , then the intervals $J, h(J), \dots, h^{(n-1)}(J)$ are pairwise disjoint, $h^n(J)$ is a component of $B_0 \setminus B_1$. Thus by Lemma 3.1,

$$Dh^n(x) \geq e^{-C_0} \frac{|h^n(J)|}{|J|} \geq e^{-C_0} \frac{\tau}{|B_{n_0}|} \geq K/K_0.$$

By (b), it follows that $Dh^k(x) \geq K_0 Dh^n(x) \geq K$.

(e). Without loss of generality, we may assume that all periodic points in $\text{Par}(h)$ are fixed points. Let $X_0 = \text{Par}(h)$ and for $n \geq 1$, let $X_n = h^{-n}(\text{Par}(h)) \setminus h^{-n+1}(\text{Par}(h))$. So for each $y \in X_n$, n is the minimal integer such that $h^n(y) \in \text{Par}(h)$.

Let $\delta_0 > 0$ be a small constant such that $h|_{B(p, \delta_0)}$ is injective and $B(p, \delta_0) \cap \text{Par}(h) = \{p\}$ for each $p \in \text{Par}(h)$. Note that this choice of δ_0 implies the following: if $y \in B(p, \delta_0) \cap X_m$ for some $m \geq 1$, then $\max_{j=1}^m d(h^j(y), p) \geq \delta_0$. Thus by (d), there is a constant $\delta > 0$ with the following property: if $y \in B(p, \delta) \cap X_m$ for some $m \geq 1$, then $Dh^m(y) \geq K/K_0$.

Now for each $p \in \text{Par}(h)$, fix a nice interval $A_p \ni p$ such that $A_p \subset B(p, \delta)$. Given $x \in X_n$ with $n \geq 1$, we shall estimate $Dh^n(x)$ from below. Let $p = f^n(x)$.

Case 1. Assume that there exists $0 \leq j < n$ such that $y := h^j(x) \in B(p, \delta)$. Then $y \in X_{n-j} \cap B(p, \delta)$ and hence $Dh^{n-j}(x) \geq K/K_0$. By (b), it follows that $Dh^n(x) \geq K_0 Dh^{n-j}(y) \geq K$.

Case 2. Assume now that $h^j(x) \notin B(p, \delta)$ for all $0 \leq j < n$. Then n is the first entry time of x into A_p . Let J be the component of $h^{-n}(A_p)$ which contains x . Then

$J, h(J), \dots, h^{n-1}(J)$ are pairwise disjoint. By Lemma 3.1, $Dh^n(x) \geq e^{-C_0}|A_p|/|J|$. Provided that n is large enough, $|J|$ is small so that $Dh^n(x) \geq K$. \square

4. Parabolic external maps. In this section we will prove Theorem 2.4, which relates parabolic external maps to topologically expanding maps and to metrically expanding maps, and which completes the theory of parabolic-like maps. We will start by giving an introduction to parabolic-like maps. We will always assume the degree $d \geq 2$, if not specified otherwise.

4.1. Parabolic-like maps. The notion of parabolic-like maps is modeled on the notion of polynomial-like maps and can be thought of as an extension of the later theory. A polynomial-like map is an object which encodes the dynamics of a polynomial on a neighborhood of its filled Julia set. We recall that the filled Julia set for a polynomial is the complement of the basin of attraction of the superattracting fixed point ∞ , and therefore the dynamics of a polynomial is expanding on a neighborhood of its filled Julia set.

A (degree d) polynomial-like mapping is a (degree d) proper holomorphic map $f : U' \rightarrow U$, where $U', U \approx \mathbb{D}$ and $\overline{U'} \subset U$. The filled Julia set for a polynomial-like map (f, U', U) is the set of points which never leave U' under iteration. Any polynomial-like map is associated with an external map, which encodes the dynamics of the polynomial-like map outside of its filled Julia set, so that a polynomial-like map is determined (up to holomorphic conjugacy) by its internal and external classes together with their matching number in $\mathbb{Z}/(d-1)\mathbb{Z}$. By replacing the external map of a degree d polynomial-like map with the map $z \rightarrow z^d$ (which is an external map of a degree d polynomial) via surgery, Douady and Hubbard proved that any degree d polynomial-like map can be straightend (this is, hybrid conjugate) to a degree d polynomial (see [3]).

On the other hand, in degree 2 a parabolic-like map is an object encoding the dynamics of a member of the family $P_A(z) = z + 1/z + A \in \text{Per}_1(1)$, where $A \in \mathbb{C}$, on a neighborhood of its filled Julia set K_A . This family can be characterized as the quadratic rational maps with a parabolic fixed point of multiplier 1 at ∞ , and critical points at ± 1 . The filled Julia set K_A of P_A is defined to be the complement of the parabolic basin of attraction of ∞ (see [4]). So on a neighborhood of the filled Julia set K_A of a map P_A there exist an attracting and a repelling direction.

A degree d parabolic-like map is a 4-tuple (f, U', U, γ) where $U', U, U \cup U' \approx \mathbb{D}$, $U' \not\subseteq U$, $f : U' \rightarrow U$ is a degree d proper holomorphic map with a parabolic fixed point at $z = z_0$ of multiplier 1, and with a forward invariant arc $\gamma : [-1, 1] \rightarrow \overline{U}$, which we call dividing arc, emanating from z_0 such that:

- γ is C^1 on $[-1, 0]$ and on $[0, 1]$, and $\gamma(\pm 1) \in \partial U$,
- $f(\gamma(t)) = \gamma(dt)$, $\forall -\frac{1}{d} \leq t \leq \frac{1}{d}$, and $\gamma([\frac{1}{d}, 1) \cup (-1, -\frac{1}{d}]) \subseteq U \setminus U'$,
- it divides U', U into Ω', Δ' and Ω, Δ respectively, such that $\Omega' \subset\subset U$ (and $\Omega' \subset \Omega$) and $f : \Delta' \rightarrow \Delta$ is an isomorphism.

The filled Julia set is defined in the parabolic-like case to be the set of points which do not escape $\Omega' \cup \gamma$ under iteration. As for polynomial-like maps, any parabolic-like map is associated with an external map (see [4]), so that a parabolic-like map is determined (up to holomorphic conjugacy) by its internal and external classes. By replacing the external map of a degree 2 parabolic-like map with the map $h_2(z) = \frac{z^2+1/3}{z^2/3+1}$, (which is an external map of any member of the family $\text{Per}_1(1)(z) = \{[P_A]|P_A(z) = z + 1/z + A\}$, as shown in [4]) one can prove that any degree 2 parabolic-like map is hybrid equivalent to a member of the family $\text{Per}_1(1)$ (see [4]).

The notion of parabolic-like map can be generalized to objects with a finite number of parabolic cycles. More precisely, let us call *simply parabolic-like maps* the objects defined before, which have a unique parabolic fixed point. Then a *parabolic-like map* is a 4-tuple (f, U', U, γ) where $U', U, U \cup U', \approx \mathbb{D}$, $U' \not\subseteq U$, $f : U' \rightarrow U$ is a degree d proper holomorphic map with a finite set $\text{Par}(f)$ of parabolic points p of multiplier 1, such that for all $p \in \text{Par}(f)$ there exists a dividing arc $\gamma_p \subset \overline{U}$, $p \in \gamma_p$, smooth except at p , $\gamma = \bigcup_p \gamma_p$, and such that:

- for $p \neq p'$, $\gamma_p \cap \gamma_{p'} = \emptyset$ and $f : \gamma_p \cap U' \rightarrow \gamma_{f(p)}$ is a diffeomorphism,
- it divides U and U' in Ω_p , Δ_p and Ω'_p , Δ'_p respectively, all connected, and such that $f : \Delta'_p \rightarrow \Delta_{f(p)}$ is an isomorphism and $\Omega'_p \subset \Omega_p$,
- calling $\Omega = \bigcap_p \Omega_p$ and $\Omega' = \bigcap_p \Omega'_p$, we have $\Omega' \subset \subset U$.

The filled Julia set for a parabolic-like map (f, U', U, γ) is (again) the set of points that never leave $\Omega' \cup \gamma$ under iteration.

4.2. External maps for parabolic-like maps. The construction of an external map for a simply parabolic-like map (f, U', U, γ) with connected filled Julia set K_f is relatively easy, and it shows that this map belongs to $\mathcal{P}_{d,1}$. Indeed, consider the Riemann map $\alpha : \widehat{\mathbb{C}} \setminus K_f \rightarrow \widehat{\mathbb{C}} \setminus \mathbb{D}$, normalized by fixing infinity and by setting $\alpha(\gamma(t)) \rightarrow 1$ as $t \rightarrow 0$. Setting $W_+ = \alpha(U \setminus K_f)$ and $W'_+ = \alpha(U' \setminus K_f)$, we can define a degree d covering $h^+ := \alpha \circ f \circ \alpha^{-1} : W'_+ \rightarrow W_+$, reflect the sets and the map with respect to the unit circle, and the restriction to the unit circle $h : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is an external map for f . An external map for a parabolic-like map is defined up to real analytic diffeomorphism. From the construction it is clear that $h \in \mathcal{P}_{d,1}$. The construction of an external map for a simply parabolic-like map with disconnected filled Julia set is more elaborate (see [4]), and still produces a map in $\mathcal{P}_{d,1}$. Repeating the constructions handled in [4] for (generalized) parabolic-like maps, one can see that the external map for a degree d parabolic-like map belongs to $\mathcal{P}_{d,*}$.

On the other hand, it comes from the Straightening Theorem for parabolic-like mappings (see [4]) that a map in $\mathcal{P}_{d,1}$ is the external map for a parabolic-like map (with a unique parabolic fixed point) of same degree (and the proof is analogous in case of several parabolics fixed points and parabolic cycles).

While the space of external classes of polynomial-like mappings is easily characterized as those circle coverings which are q-s.-conjugate to $z \mapsto z^d$ for some $d \geq 2$, this is not the case for parabolic external classes. Theorem 2.4 gives a characterization for these maps.

4.3. Proof of Theorem 2.4. The main technical difficulty for proving Theorem 2.4 is to prove the following property for maps in $\mathcal{M}_{d,*}$:

Lemma 4.1. *For any $h \in \mathcal{M}_{d,*}$ there is a map $\phi \in \mathcal{F}_1$ such that the map $\check{h} := \phi \circ h \circ \phi^{-1}$ also belongs to $\mathcal{M}_{d,*}$ and in addition for every orbit $\check{p}_0, \check{p}_1, \dots, \check{p}_s = \check{p}_0 \in \text{Par}(\check{H})$ say of parabolic multiplicity $2n$, the power series developments of $\check{H} : \mathbb{T} \rightarrow \mathbb{T}$ at the points \check{p}_k , $k \in \mathbb{Z}/s\mathbb{Z}$, take the form*

$$\check{H}(x) = \check{p}_{k+1} + (x - \check{p}_k)(1 + (x - \check{p}_k)^{2n} \cdot \check{P}(x - \check{p}_k) + \mathcal{O}(x - \check{p}_k)^{6n}) \quad (2)$$

for some fixed polynomial \check{P} (i.e. \check{P} depends on the cycle, but not on k) with non-zero constant term and degree at most $4n - 1$.

We will first prove the Theorem assuming the Lemma, which we will prove in Subsection 4.3.1. In order to prove Theorem 2.4 (assuming Lemma 4.1), we will

first prove that $\mathcal{M}_d \subset \mathcal{P}_d$ (Proposition 4.1), then that $h_d \in \mathcal{P}_{d,1}$ (Lemma 4.2), and later that two maps in $\mathcal{P}_{d,*}$ topologically conjugate by a conjugacy preserving the parabolic points are quasi-symmetrically conjugate (Proposition 2.3). Finally, we will prove that $\mathcal{P}_d \subseteq \mathcal{O}_d^{1+\text{BV}} \cap \mathcal{F}_d = \mathcal{T}_d$ (Proposition 4.2) and put together all these bits for obtaining Theorem 2.4.

Proposition 4.1. *For every $h \in \mathcal{M}_d$ there exists $\epsilon_0 > 0$ such that for every $0 < \epsilon \leq \epsilon_0$ the map h has a holomorphic extension (h, W', W, γ) as an external map with range $W \subseteq \{z : |\log|z|| < \epsilon\}$. In particular, a map $h \in \mathcal{M}_{d,*}$ has a holomorphic extension (h, W', W, γ) as a parabolic external map with range $W \subseteq \{z : |\log|z|| < \epsilon\}$. So $\mathcal{M}_d \subset \mathcal{P}_d$, $\mathcal{M}_{d,*} \subset \mathcal{P}_{d,*}$ and any map which is conjugate to $h \in \mathcal{M}_d$ by $\phi \in \mathcal{F}_1$ also belongs to \mathcal{P}_d .*

Proof. If $h \in \mathcal{M}_d \setminus \mathcal{M}_{d,*}$ the result is obvious. So let us consider maps $h \in \mathcal{M}_{d,*}$. It suffices to consider maps $h \in \mathcal{M}_{d,*}$ satisfying the properties of \check{h} in the Lemma above. Also it suffices to work with the representative $H : \mathbb{T} \rightarrow \mathbb{T}$ of h . Since H is a real analytic covering map, it extends to a holomorphic isomorphism $H : V' \rightarrow V$ between reflection symmetric neighborhoods of \mathbb{R} and satisfying $H(z+1) = H(z)+d$. Set $E(x) := e^{i2\pi x}$. For each $p \in P = E^{-1}(\text{Par}(h))$, choose a pair of repelling Fatou coordinates $\phi_p^\pm : \Xi_p^\pm \rightarrow \mathcal{H}_l := \{z : \Re(z) < 0\}$ such that each ϕ_p^\pm and each Ξ_p^\pm is symmetric with respect to \mathbb{R} , and $\phi_{p+1}^\pm(x) = \phi_p^\pm(x-1)$. Possibly restricting the ϕ_p^\pm we can suppose all the domains Ξ_p^\pm , with p ranging over P , are disjoint for each choice of sign, that $H(\Xi_p^\pm) \supset \Xi_{h(p)}^\pm$, and that H^s is univalent on Ξ_p^\pm , where $s = s_p$ denotes the period of $E(p)$.

For each orbit in $\text{Par}(h)$ choose a representative $p_i \in P$, and call $2n_i$ the parabolic multiplicity of the orbit. Define $S_\epsilon := \{x + iy : |y| < \epsilon\}$ for $\epsilon > 0$. For p in the orbit of p_i , call C_p the double cone, symmetric with respect to the real line, such that $C_p \cap \mathbb{R} = \{p\}$ and the angle between \mathbb{R} and ∂C_p is $1/(16n_i)$. Call $X_\epsilon = S_\epsilon \setminus \bigcup_{p \in P} C_p$. By a compactness argument, since $DH(z) > 1$ for $z \in \mathbb{R} \cap X_\epsilon$, and $\limsup_{z \in \partial X_\epsilon, z \rightarrow p} |\text{Arg}(DH(z) - 1)| = \pi/4$, there exists an $\epsilon_0 > 0$ such that, for all $z \in X_{\epsilon_0}$, $\Re(DH(z)) > 1$. Possibly decreasing ϵ_0 , we can assume that for all $p, p' \in P$, $\overline{S_{\epsilon_0}} \cap C_p \cap C_{p'} = \emptyset$. Since h satisfies the conclusion of Lemma 4.1, the curves $(\phi_p^\pm)^{-1}(\mp a \cdot i + \mathbb{R}_-)$ intersect the boundary of C_p at angle $\pi/4$ asymptotically as $a \rightarrow \infty$, and moreover for $E(p)$ and $E(p')$ in the same orbit this happens asymptotically at the same imaginary height. Thus, possibly decreasing ϵ_0 and fixing any $\epsilon, 0 < \epsilon < \epsilon_0$, we may choose $a_i > 0$ (depending on ϵ) such that, for all i and all p with $E(p)$ in the orbit of $E(p_i)$, the arcs $\gamma_p^\pm = (\phi_p^\pm)^{-1}(\mp a_i \cdot i + \mathbb{R}_-)$ exits X_ϵ through ∂S_ϵ transversally (see Figure 3).

Let Δ_p^u be the closed connected component in S_ϵ bounded by $\gamma_p^- \cup p \cup \gamma_p^+$ and containing C_p , and set $\Delta_p = \Delta_p^u \cup \tau(\Delta_p^u)$, where $\tau(z) = \bar{z}$. Define $\hat{X}'_\epsilon = S_\epsilon \setminus \bigcup_p \Delta_p$ (note that $\hat{X}'_\epsilon \subset X_\epsilon$), and $\hat{X}_\epsilon = S_\epsilon \setminus \bigcup_p \Delta_{H(p)}$.

Then, by construction $H^{-1}(\hat{X}_\epsilon) \subset \hat{X}'_\epsilon$ and $\overline{H^{-1}(\hat{X}_\epsilon)} \subset S_\epsilon$. Thus, taking $W := \exp(S_\epsilon)$, $W' := h^{-1}(W)$ and the multi arc γ as the family $\exp(\gamma_p)$, $p \in P$, we have constructed an extension (h, W', W, γ) of h in \mathcal{P}_d . \square

Lemma 4.2. *The map h_d is Möbius conjugate to a map in $\mathcal{M}_{d,1}$, so $h_d \in \mathcal{P}_{d,1}$.*

Proof. For $0 < r < 1$ define $M_r(z) = (z+r)/(1+rz)$. Then, $|M'_r(z)|$ is a monotone decreasing function of $\Re(z)$ with $|M'_r(-1)| = (1+r)/(1-r)$ and $|M'_r(1)| = (1-r)/(1+r)$.

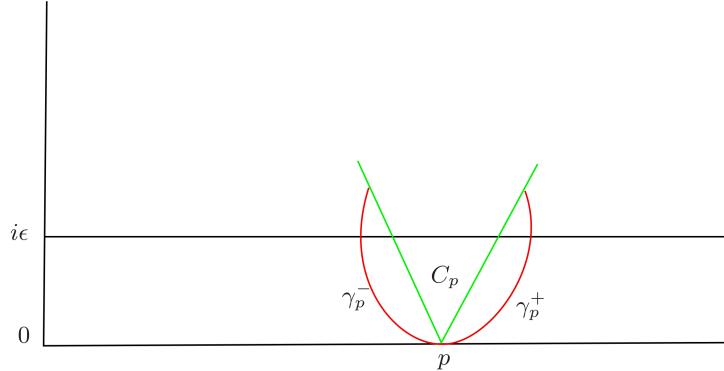


FIGURE 3. Construction

$r)/(1+r)$. Note that for $r = (d-1)/(d+1)$ we have $h_d = M_r(z^d)$. Thus $M_r^{-1} \circ h_d \circ M_r = (M_r)^d$ and this map evidently belongs to $\mathcal{M}_{d,1}$. \square

Proposition 2.3: *Suppose $h_1, h_2 \in \mathcal{P}_{d,*}$ are topologically conjugate by an orientation preserving homeomorphism ϕ , which preserves parabolic points. Then ϕ is quasi-symmetric.*

Proof. Let $(h_i, W'_i, W_i, \gamma^i)$, $i = 1, 2$ be holomorphic extensions with W'_i and W_i bounded by C^1 Jordan curves intersecting γ^i transversely. The case $h_i \in \mathcal{P}_2^1$ is handled in Lomonaco, [4]. The general case is completely analogous, we include the details for completeness. It suffices to construct a quasi-conformal extension, $\phi : \overline{W}_1^+ \rightarrow \overline{W}_2^+$, with $\phi(\gamma_p^1(t)) := \gamma_{\phi(p)}^2(t)$ for each $p \in \text{Par}(h_1)$ and with $\phi \circ h_1 = h_2 \circ \phi$ on Ω'_1 .

For each $p \in \text{Par}(h_1)$, extend ϕ so that $\phi(\gamma_p^1(t)) := \gamma_{\phi(p)}^2(t)$. It is proved in [4] that the arcs γ_p^1 and $\gamma_{\phi(p)}^2$ are quasi-arcs, and that this extension, which is C^1 for $z \neq p$, is quasi-symmetric. Next, extend ϕ as a diffeomorphism between the outer boundary of W_1^1 and W_2^2 respecting the intersections with γ^i , i.e. besides being a diffeomorphism it satisfies $\phi(\gamma_p^1(\pm 1)) = \gamma_{\phi(p)}^2(\pm 1)$. Then ϕ is defined as a quasi-symmetric homeomorphism from the quasi-circle boundary of Δ_p^1 to the quasi-circle boundary of $\Delta_{\phi(p)}^2$ for each $p \in \text{Par}(h_1)$. We extend ϕ as a quasi-conformal homeomorphism $\phi : \Delta_p^1 \rightarrow \Delta_{\phi(p)}^2$. Next, consider the C^1 lift $\tilde{\phi} : \partial W'_1 \rightarrow \partial W'_2$ of $\phi \circ h_1$ to h_2 respecting the dividing multi arcs. We next extend ϕ by $\tilde{\phi}$ on $\partial W'_1 \cap W_1^+$. For each $i = 1, 2$, the connected components of $W_i^+ \setminus \overline{W}'_i$ are quadrilaterals Q_p^i indexed by the $p \in \text{Par}(h_i)$, preceding Q_p^i in the counter-clockwise ordering. Moreover, ϕ thus defined restricts to a piecewise C^1 and hence quasi-symmetric homeomorphism from the boundary of Q_p^1 to the boundary of $Q_{\phi(p)}^2$. Extend this boundary homeomorphism to a quasi-conformal homeomorphism between Q_p^1 and $Q_{\phi(p)}^2$.

Call the thus extended map ϕ_1 and its domain and range U_1^1 and U_2^1 respectively. Define recursively, for $i = 1, 2$ and $n \geq 1$:

$$U_i^{n+1} := U_i^n \cup (h_i^{-1}(U_i^n) \cap \Omega'_i).$$

Moreover, define recursively $\phi_{n+1} : U_1^{n+1} \rightarrow U_2^{n+1}$ as the quasi-conformal extension of ϕ_n which on $h_1^{-1}(U_1^n) \cap \Omega'_1$ satisfies

$$\phi_n \circ h_1 = h_2 \circ \phi_{n+1}$$

(i.e. it is the lift of $\phi_n \circ h_1$ to h_2). Then, $\phi_n \cup \phi$ converges uniformly to a quasi-conformal homeomorphism $\phi_\infty : \overline{W_1^+} \rightarrow \overline{W_2^+}$, which conjugates dynamics except on Δ'_1 . Thus, ϕ is the restriction to \mathbb{S}^1 of a quasi-conformal homeomorphism, and thus it is a quasi-symmetric map. \square

Proposition 4.2. $\mathcal{P}_d \subseteq \mathcal{O}_d^{1+\text{BV}} \cap \mathcal{F}_d = \mathcal{T}_d$, and in particular $\mathcal{P}_{d,*} \subset \mathcal{T}_{d,*}$.

Proof. Let $h \in \mathcal{P}_d$ and let (h, W', W, γ) be a degree d holomorphic extension of h as a parabolic external map with dividing multi arc γ and associated sets Δ' and Δ . We shall first redefine Ω and Ω' so as to be reflection-symmetric: $\Omega = W \setminus \overline{\Delta \cup \tau(\Delta)}$ and $\Omega' = W' \setminus \overline{\Delta' \cup \tau(\Delta')}$ (where $\tau(z) = 1/\bar{z}$), then $\Omega'' := h^{-1}(\Omega) \subset \Omega' \subsetneq \Omega$. It follows that each $p \in \text{Par}(h)$, say of period n , admits the circle as repelling directions. Indeed, if not, then it would have a τ -symmetric attracting petal along \mathbb{S}^1 to one or both sides. However, since $\Omega'' \subset \Omega$, the parabolic basin for h^n containing such a petal would be a proper basin and thus would contain a critical point.

To prove that all other periodic orbits are repelling, let ρ denote the hyperbolic metric on Ω . Then, each connected component V of Ω'' is a subset of $U \cap W'$ for some connected component U of Ω . Thus, h is expanding with respect to the conformal metric ρ . Since any non parabolic orbit is contained in $\Omega'' \cap \mathbb{S}^1$, it follows that all non parabolic orbits are repelling. This proves the first inclusion. The equality sign is immediate from Corollary 2.2. \square

Completion of proof of Theorem 2.4: By Proposition 4.1, $\mathcal{M}_d \subset \mathcal{P}_d$ and $\mathcal{M}_{d,*} \subset \mathcal{P}_{d,*}$ (and so $\mathcal{M}_{d,1} \subset \mathcal{P}_{d,1}$), and by Proposition 4.2, $\mathcal{P}_d \subseteq \mathcal{T}_d$ and $\mathcal{P}_{d,*} \subseteq \mathcal{T}_{d,*}$. Since h_d is topologically expanding we have that $\mathcal{H}_{d,1} \subset \mathcal{T}_{d,1}$, and combining Lemma 4.2 and Proposition 2.3 we obtain $\mathcal{P}_{d,1} \subset \mathcal{H}_{d,1}$. So:

$$\mathcal{M}_d \subset \mathcal{P}_d \subset \mathcal{T}_d, \quad \mathcal{M}_{d,*} \subset \mathcal{P}_{d,*} \subset \mathcal{T}_{d,*}, \quad \text{and} \quad \mathcal{M}_{d,1} \subset \mathcal{P}_{d,1} \subset \mathcal{H}_{d,1} \subset \mathcal{T}_{d,1}.$$

By Theorem 2.1, any $h \in \mathcal{T}_d$ is real analytically conjugate to a map $\hat{h} \in \mathcal{M}_d$, and so by Proposition 4.1 we also have $h \in \mathcal{P}_d$. So we obtain

$$\mathcal{P}_d = \mathcal{T}_d, \quad \mathcal{P}_{d,*} = \mathcal{T}_{d,*}, \quad \text{and} \quad \pi_d(\mathcal{M}_d) = \pi_d(\mathcal{P}_d) = \pi_d(\mathcal{T}_d).$$

4.3.1. Proof of Lemma 4.1. This subsection is completely devoted to proving Lemma 4.1. Let us start by noticing that it follows from the definition of \mathcal{M}_d that h only has finitely many parabolic points. The proof of Lemma 4.1 uses the idea of the proof of Theorem 3.2 to recursively construct conjugacies to maps which full-fills the requirements of \check{H} to higher and higher orders. It turns out that after two steps of the recursion we arrive at the desired map \check{H} and obtain the conjugacy as the composition of the pair of conjugacies from the recursion.

The recursion is given by the following procedure:

Let $h \in \mathcal{M}_d$ be arbitrary, let $N = N_h$ denote the least common multiple of the periods of parabolic orbits for h and let $L := (d^N - 1)/(d - 1)$. Define a real analytic diffeomorphism $\phi : \mathbb{R} \rightarrow \mathbb{R}$ and a new real analytic diffeomorphism \hat{H} (lift of degree d covering \hat{h}) as follows:

$$\phi(x) := \frac{1}{L} \sum_{k=0}^{N-1} (H^k)(x) \quad \text{and} \quad \hat{H} := \phi \circ H \circ \phi^{-1}.$$

Then $\phi(x+1) = 1 + \phi(x)$, $\text{Par}(\widehat{H}) = \phi(\text{Par}(H))$, $N_{\widehat{h}} = N_h$ and

$$\begin{aligned}\widehat{H}'(\phi(x)) &= \frac{\phi'(H(x)) \cdot H'(x)}{\phi'(x)} = \frac{H'(x) \sum_{k=0}^{N-1} (H^k)'(H(x))}{\sum_{k=0}^{N-1} (H^k)'(x)} \\ &= \frac{\sum_{k=1}^N (H^k)'(x)}{\sum_{k=0}^{N-1} (H^k)'(x)} = \frac{(H^N)'(x) + \sum_{k=1}^{N-1} (H^k)'(x)}{1 + \sum_{k=1}^{N-1} (H^k)'(x)} \geq 1\end{aligned}\quad (3)$$

with equality if and only if $x \in \text{Par}(H)$, thus $\widehat{H} \in \mathcal{M}_d$.

For $p \in \text{Par}(H)$ with period s , set $\widehat{p} := \phi(p) \in \text{Par}(\widehat{H})$, $p_k := H^k(p)$, $\widehat{p}_k = \phi(p_k) = \widehat{H}^k(\widehat{p})$, then $p_{s+k} - p_k = \widehat{p}_{s+k} - \widehat{p}_k \in \mathbb{Z}$ for each $k \geq 0$. Let $2n > 0$ denote the common parabolic degeneracy. A priori the power series developments (Taylor expansions) of H around the points p_k could have non-linear terms of order less than $2n+1$. However, since $h \in \mathcal{M}_d$, the leading non-linear term must be of odd order, say $2m+1$ (and have positive coefficient), and Claim 4.1 (statement and proof of which are below) implies $m = n$.

Write $h_0 := h$, $H_0 := H$ and $\phi_0 := \phi$. Set $H_1 := \widehat{H}$, and define

$$\phi_1(x) := \frac{1}{L} \sum_{k=0}^{N-1} (H_1^k)(x) \quad \text{and} \quad H_2 := \phi_1 \circ H_1 \circ \phi_1^{-1}.$$

Then $\phi := \phi_1 \circ \phi_0$ and $\tilde{H} := H_2$ satisfy the Lemma, with $\tilde{P} := \widehat{P} + x^{2n} \widehat{R}$, where \widehat{P} and \widehat{R} are given by Claim 4.2 applied to H_1 (statement and proof of Claim 4.2 are below, after the proof of Claim 4.1).

Claim 4.1. *Suppose that for some $m > 0$ the Taylor expansions of H around the points p_k take the form*

$$H(x) = p_{k+1} + (x - p_k)(1 + (x - p_k)^{2m} \cdot P_k(x - p_k) + \mathcal{O}(x - p_k)^{4m}),$$

where P_k is a polynomial of degree at most $(2m-1)$, $P_{s+k} = P_k$ for $k \geq 0$ and where $P_k(0) > 0$ for at least one k , $0 \leq k < s$. Then for each k the Taylor approximation to order $4m$ of \widehat{H} at \widehat{p}_k takes the form

$$\widehat{H}(\widehat{x}) = \widehat{p}_{k+1} + (\widehat{x} - \widehat{p}_k)(1 + (\widehat{x} - \widehat{p}_k)^{2m} \cdot \widehat{P}(\widehat{x} - \widehat{p}_k) + \mathcal{O}(\widehat{x} - \widehat{p}_k)^{4m}), \quad (4)$$

where

$$\widehat{P}(x) := \frac{L^{2m}}{s} \cdot \sum_{k=0}^{s-1} P_k(Lx) \quad (5)$$

is independent of $k \geq 0$ and moreover for \widehat{x} close to \widehat{p}_k and $j \geq 1$:

$$\widehat{H}^j(\widehat{x}) = \widehat{p}_{j+k} + (\widehat{x} - \widehat{p}_k)(1 + j \cdot (\widehat{x} - \widehat{p}_k)^{2m} \cdot \widehat{P}(\widehat{x} - \widehat{p}_k) + \mathcal{O}(\widehat{x} - \widehat{p}_k)^{4m}). \quad (6)$$

Let us first see that the Claim implies $m = n$. Since H and \widehat{H} are analytically conjugate, the parabolic degeneracy of \widehat{H} at \widehat{p} is also $2n$. However, since the coefficient of the leading terms in (4) are non-negative and at least one of them is positive, it follows from (5) that the constant term of \widehat{P} is positive, and then from (6) that the degeneracy is $2m$. Therefore $m = n$.

Proof. Towards a proof of the Claim a routine computation and induction shows that for all $j \geq 0$ the Taylor series of H^j to order $4m$ at p_k is given by:

$$H^j(x) = H^j(p_k) + (x - p_k)(1 + (x - p_k)^{2m} \cdot \sum_{l=0}^{j-1} P_{l+k}(x - p_k) + \mathcal{O}(x - p_k)^{4m})$$

and thus with $Q_k := (2m+1)P_k + x \cdot P'_k = Q_{s+k}$

$$(H^j)'(x) = 1 + (x - p_k)^{2m} \cdot \sum_{l=0}^{j-1} Q_{l+k}(x - p_k) + \mathcal{O}(x - p_k)^{4m}.$$

Continuing to compute $\widehat{H}'(\phi(x))$ for x near p_k starting from the first term of (3) and using $(H^0)' \equiv 1$, $(H^j)' - 1 = \mathcal{O}((x - p_k)^{2m})$ we find

$$\begin{aligned} \widehat{H}'(\phi(x)) &= \frac{\sum_{j=1}^N (H^j)'(x)}{\sum_{j=0}^{N-1} (H^j)'(x)} = \frac{N + \sum_{j=1}^N ((H^j)'(x) - 1)}{N + \sum_{j=1}^{N-1} ((H^j)'(x) - 1)} \quad (7) \\ &= \left(1 + \frac{1}{N} \sum_{j=1}^N ((H^j)'(x) - 1) \right) \left(1 - \frac{1}{N} \sum_{j=1}^{N-1} ((H^j)'(x) - 1) \right) + \mathcal{O}(x - p_k)^{4m} \\ &= 1 + \frac{1}{N} ((H^N)'(x) - 1) + \mathcal{O}(x - p_k)^{4m} \\ &= 1 + \frac{(x - p_k)^{2m}}{N} \cdot \sum_{l=0}^{N-1} Q_{l+k}(x - p_k) + \mathcal{O}(x - p_k)^{4m} \end{aligned}$$

From the formula for ϕ we find the expansion of ϕ to order $2m$ at p_k :

$$\phi(x) = \widehat{p}_k + \frac{1}{L} (x - p_k) (1 + \mathcal{O}(x - p_k)^{2m})$$

so that the expansion for ϕ^{-1} to order $2m$ at \widehat{p}_k is:

$$\phi^{-1}(\widehat{x}) = p_k + L(\widehat{x} - \widehat{p}_k) (1 + \mathcal{O}(\widehat{x} - \widehat{p}_k)^{2m})$$

and thus the expansion for \widehat{H}' to order $(4m-1)$ at \widehat{p}_k is:

$$\widehat{H}'(\widehat{x}) = 1 + \frac{(L(\widehat{x} - \widehat{p}_k))^{2m}}{N} \cdot \sum_{l=0}^{N-1} Q_{l+k}(L(\widehat{x} - \widehat{p}_k)) + \mathcal{O}(\widehat{x} - \widehat{p}_k)^{4m}.$$

So by integration from \widehat{p}_k we find

$$\widehat{H}(\widehat{x}) = \widehat{p}_{k+1} + (\widehat{x} - \widehat{p}_k) (1 + (\widehat{x} - \widehat{p}_k)^{2m} \frac{L^{2m}}{N} \cdot \sum_{l=0}^{N-1} P_{l+k}(L(\widehat{x} - \widehat{p}_k)) + \mathcal{O}(\widehat{x} - \widehat{p}_k)^{4m}),$$

from which the Claim follows, since N is a multiple of s and the terms of the sum are repeated N/s times. \square

Claim 4.2. *Suppose the Taylor expansions of H around the points p_k take the form*

$$H(x) = p_{k+1} + (x - p_k) (1 + (x - p_k)^{2n} \cdot P(x - p_k) + (x - p_k)^{4n} \cdot R_k(x - p_k) + \mathcal{O}(x - p_k)^{6n}),$$

where P and R_k are polynomials of degree at most $(2n-1)$, P with $P(0) > 0$ is independent of k and $R_{s+k} = R_k$ for $k \geq 0$. Then for each k the Taylor expansion of \widehat{H} to order $6n$ at \widehat{p}_k takes the form

$$\widehat{H}(\widehat{x}) = \widehat{p}_{k+1} + (\widehat{x} - \widehat{p}_k) (1 + (\widehat{x} - \widehat{p}_k)^{2n} \cdot \widehat{P}(\widehat{x} - \widehat{p}_k) + (\widehat{x} - \widehat{p}_k)^{4n} \cdot \widehat{R}(\widehat{x} - \widehat{p}_k) + \mathcal{O}(\widehat{x} - \widehat{p}_k)^{6n}), \quad (8)$$

where \widehat{R} and $\widehat{P}(x) = L^{2n}P(Lx)$ with $\widehat{P}(0) > 0$ are polynomials of degree at most $2n-1$ and are independent of the point in the orbit of $\widehat{p} = \phi(p)$.

Proof. The proof of this Claim is similar to the proof of the first Claim, and we only indicate the differences.

For proving a formula for the j -th iterate the following formula is simple and useful

$$P(x(1 + x^{2n}P(x))) = P(x) + x^{2n} \cdot x \cdot P'(x) \cdot P(x) + \mathcal{O}(x)^{4n} \quad (9)$$

(Note that the term $x^{2n} \cdot x \cdot P'(x) \cdot P(x)$ contains terms of order larger than or equal to $4n$, but taking them out only complicates the formula.)

By induction, for each $j \geq 1$ and x close to p_k we find

$$\begin{aligned} H^j(x) &= p_{j+k} + (x - p_k)(1 + (x - p_k)^{2n} \cdot j \cdot P(x - p_k)) \\ &\quad + (x - p_k)^{4n} \cdot \frac{j(j-1)}{2} ((2n+1)(P(x - p_k))^2 \\ &\quad + (x - p_k)P'(x - p_k)P(x - p_k)) \\ &\quad + (x - p_k)^{4n} \cdot \sum_{l=0}^{j-1} R_{l+k}(x - p_k) + \mathcal{O}(x - p_k)^{6n} \\ &= p_{j+k} + F_j(x - p_k) + (x - p_k)^{4n} \cdot \sum_{l=0}^{j-1} R_{l+k}(x - p_k) + \mathcal{O}(x - p_k)^{6n} \end{aligned}$$

where

$$\begin{aligned} F_j(x) &:= x(1 + x^{2n} \cdot j \cdot P(x)) \\ &\quad + x^{4n} \cdot \frac{j(j-1)}{2} ((2n+1)(P(x))^2 + xP'(x)P(x)) \end{aligned}$$

is independent of k , i.e. independent of the starting point in the orbit of p . As above, define Q by the formula $x^{2n}Q(x) := \frac{d}{dx}(x^{2n+1}P(x))$, and thus $Q(x) = (2n+1)P + x \cdot P'$, and S_k by the formula $x^{4n}S_k(x) := \frac{d}{dx}(x^{4n+1}R_k(x))$, and thus $S_k(x) = (4n+1)R_k + x \cdot R'_k = S_{s+k}(x)$. Then

$$\begin{aligned} (H^j)'(x) &= F'_j(x - p_k) + (x - p_k)^{4n} \cdot \sum_{l=0}^{j-1} S_{l+k}(x - p_k) + \mathcal{O}(x - p_k)^{6n} \\ &= 1 + j(x - p_k)^{2n}Q(x - p_k) + \mathcal{O}(x - p_k)^{4n} \end{aligned}$$

Thus

$$\begin{aligned} \sum_{j=1}^{N-1} ((H^j)'(x) - 1) &= \sum_{j=1}^{N-1} j(x - p_k)^{2n}Q(x - p_k) + \mathcal{O}(x - p_k)^{4n} \\ &= \frac{N(N-1)}{2}(x - p_k)^{2n}Q(x - p_k) + \mathcal{O}(x - p_k)^{4n} \end{aligned} \quad (10)$$

Computing $\widehat{H}'(\phi(x))$ from the second formula in (7) we obtain

$$\begin{aligned} \widehat{H}'(\phi(x)) &= \frac{N + \sum_{j=1}^N ((H^j)'(x) - 1)}{N + \sum_{j=1}^{N-1} ((H^j)'(x) - 1)} \\ &= \left(1 + \frac{1}{N} \sum_{j=1}^N ((H^j)'(x) - 1) \right). \end{aligned}$$

$$\begin{aligned}
& \left(1 - \frac{1}{N} \sum_{j=1}^{N-1} ((H^j)'(x) - 1) + \frac{(N-1)^2}{4} (x - p_k)^{4n} (Q(x - p_k))^2 \right) \\
& \quad + \mathcal{O}(x - p_k)^{6n} \\
& = 1 + \frac{1}{N} ((H^N)'(x) - 1) + \frac{(N-1)^2 - (N^2 - 1)}{4} (x - p_k)^{4n} (Q(x - p_k))^2 \\
& \quad + \mathcal{O}(x - p_k)^{6n} \\
& = 1 + \frac{1}{N} (F'_N(x - p_k) - 1) + \frac{(x - p_k)^{4n}}{N} \cdot \sum_{l=0}^{N-1} S_{l+k}(x - p_k) \\
& \quad - \frac{1}{2} (x - p_k)^{4n} (Q(x - p_k))^2 + \mathcal{O}(x - p_k)^{6n}
\end{aligned}$$

That is, the terms of $\widehat{H}'(\phi(x))$ depending on k are the terms

$$\frac{(x - p_k)^{4n}}{N} \cdot \sum_{l=0}^{N-1} S_{l+k}(x - p_k) + \mathcal{O}(x - p_k)^{6n}$$

of order at least $4n$.

From the definition of ϕ and (10) we see that ϕ is independent of k to order $4n$ and thus the same holds for ϕ^{-1} . Combining this with the above shows that \widehat{H}' is independent of k to order $6n - 1$ and thus \widehat{H} is independent of k up to and including order $6n$, as promised by the Claim. \square

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REFERENCES

- [1] B. Branner and N. Fagella, *Quasiconformal Surgery in Holomorphic Dynamics*, Cambridge University Press, 2014.
- [2] G. Cui, *Circle expanding maps and symmetric structures*, *Ergodic Theory and Dynamical Systems*, **18** (1998), 831–842.
- [3] A. Douady and J. H. Hubbard, *On the dynamics of polynomial-like mappings*, *Annales scientifiques de l’École normale supérieure*, **18** (1985), 287–343.
- [4] L. Lomonaco, *Parabolic-like maps*, *Ergodic Theory and Dynamical Systems*, **35** (2015), 2171–2197.
- [5] J. Ma., *On Evolution of a Class of Markov Maps*, Undergraduate thesis (in Chinese), University of Science and Technology of China, 2007.
- [6] R. Mañé, *Hyperbolicity, sinks and measure in one-dimensional dynamics*, *Communications in Mathematical Physics*, **100** (1985), 495–524.
- [7] M. Martens, W. de Melo and S. van Strien, *Julia-Fatou-Sullivan theory for real one-dimensional dynamics*, *Acta Mathematica*, **168** (1992), 273–318.
- [8] W. de Melo and S. van Strien, *One-Dimensional Dynamics*, Springer-Verlag, 1993.
- [9] W. Rudin, *Real and Complex Analysis*, New York-Toronto, Ont.-London, 1966.
- [10] M. Shishikura, Bifurcation of parabolic fixed points, in *The Mandelbrot set, theme and variations*, London Mathematical Society Lecture Note Series, Cambridge University Press, **274** (2000), 325–363.

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