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ON A FUNCTION MODULE WITH APPROXIMATE HYPERPLANE SERIES PROPERTY

T. GRANDO✉ and M. L. LOURENÇO

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Abstract

We present a sufficient and necessary condition for a function module space X to have the approximate hyperplane series property (AHSP). As a consequence, we have that the space $C_0(L, E)$ of bounded and continuous E-valued mappings defined on the locally compact Hausdorff space L has AHSP if and only if E has AHSP.

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1. Introduction

Throughout the paper, E and F will be complex Banach spaces. As usual, S_E, B_E and E^* will denote the unit sphere, the closed unit ball, and the (topological) dual of E, respectively. Given two Banach spaces E and F, $\mathcal{L}(E, F)$ denotes the space of all bounded linear operators from E into F.

The Bishop–Phelps theorem states that the set of norm-attaining functionals on E is dense in E^* [8]. It has been usefully extended in many directions and in the study of optimization. After the celebrated Bishop–Phelps theorem, it was a natural question as to whether the set of norm-attaining linear operators is dense in $\mathcal{L}(E, F)$, for all E and F.

In 1963, J. Lindenstrauss [18] gave a counterexample showing that it does not hold in general and he also showed that, if E is reflexive, then the set of all norm-attaining operators is always dense in the space of $\mathcal{L}(E, F)$.

Motivated by the study of numerical ranges of operators, B. Bollobás in [9] proved a refinement of the Bishop–Phelps theorem, nowadays known as the Bishop–Phelps–Bollobás theorem [9, Theorem 1].

Carrying Bollobás's ideas to the vector-valued case in 2008, Acosta, Aron, García and Maestre [1] introduced the notion of the Bishop–Phelps–Bollobás property for operators (BPBP for operators, for short) (see the Definition 2.1). BPBP for operators is a stronger property than the denseness of norm-attaining operators. It had been known that the set of norm-attaining operators from ℓ_1 to any Banach space F is dense, but the pair (ℓ_1, F) has BPBP for operators if F has a special property. This property was introduced in [1], called the approximate hyperplane series property (AHSP, for short), with the purpose of characterizing those Banach spaces F such that (ℓ_1, F) has BPBP for operators. These two properties have attracted the attention of many researchers. For more details and recent results about BPBP for operators or AHSP, see [2–6, 10–12, 14–17].

In this note we study when a function module space X has AHSP and we obtain that the space $C(K, E)$ has AHSP if, and only if, a Banach space E has AHSP. In this sense, we have generalized a result of Choi and Kim [13]. We also obtain as a consequence, the space $C_0(L, E)$ of bounded and continuous E-valued mappings defined on the locally compact Hausdorff space L has AHSP if and only if E has AHSP.

2. Results

For our purposes, it will be useful to recall the definition of BPBP for operators.

Definition 2.1. Let E and F be Banach spaces. We say that the pair (E, F) has the *Bishop–Phelps–Bollobás property for operators* (*BPBP for operators*, for short) if given $\varepsilon > 0$, there is $\eta(\varepsilon) > 0$ such that whenever $T \in S_{\mathcal{L}(E,F)}$ and $x_0 \in S_E$ satisfy that $\|T x_0\| > 1 - \eta(\varepsilon)$, then there exist a point $u_0 \in S_E$ and an operator $S \in S_{\mathcal{L}(E,F)}$ satisfying the following conditions.

$$\|S u_0\| = 1, \qquad \|u_0 - x_0\| < \epsilon, \quad \text{and} \quad \|S - T\| < \epsilon.$$

Now we will give the definition of AHSP introduced in [1]. Recall that if $(x_k)_{k\in\mathbb{N}} \subset E$ and $(\lambda_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ such that $\lambda_k \geq 0$ for all $k \in \mathbb{N}$, we say that the series given by $\sum_{k=1}^{\infty} \lambda_k x_k$ is a *convex series* if $\sum_{k=1}^{\infty} \lambda_k = 1$.

Definition 2.2. A Banach space E is said to have AHSP (approximate hyperplane series property) if for every $\epsilon > 0$ there exists $0 < \eta < \epsilon$ such that for every sequence $(x_k)_k \subset S_E$ and convex series $\sum_{k=1}^{\infty} \alpha_k x_k$ with

$$\left\|\sum_{k=1}^{\infty} \alpha_k x_k\right\| > 1 - \eta,$$

there exist a subset $A \subset \mathbb{N}$ and a subset $\{z_k : k \in A\}$ satisfying

(i) $\sum_{k\in A} \alpha_k > 1 - \epsilon$;
(ii) $\|z_k - x_k\| < \epsilon$ for all $k \in A$;
(iii) $x^*(z_k) = 1$ for a certain $x^* \in S_{X^*}$ and for all $k \in A$.

We observe that the above property holds if it is satisfied just for a finite convex combination (instead of convex series). The very useful comment in [1] is:

'Geometrically, E has AHSP if whenever we have a convex series of vectors in B_E whose norm is very close to 1, then a preponderance of these vectors are uniformly close to unit vectors that lie in the same hyperplane $(x^*)^{-1}(1)$, where $\|x^*\| = 1$.'

Among the spaces with AHSP, we may cite finite dimensional spaces, uniformly convex spaces and $C_0(L)$ spaces, as representative examples [1]. On the other hand, there are spaces failing this property: every strictly convex space which is not uniformly convex [1]. We refer the reader to the paper [15] for more examples of spaces with AHSP.

It was verified in [1] that in the Definition 2.2, we can consider sequences $(x_k)_k$ of vectors in the unit ball of E.

Proposition 2.3. *Let E be a Banach space. E has AHSP if and only if for all $\epsilon > 0$ there exist $0 < \gamma(\epsilon) < \epsilon$ and $\eta(\epsilon) > 0$ with $\lim_{\epsilon \to 0^+} \gamma(\epsilon) = 0$ such that for every sequence, $(x_k)_k \subset B_E$ and every convex series $\sum_{k=1}^{\infty} \alpha_k x_k$ satisfying*

$$\left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\| > 1 - \eta(\epsilon),$$

there exist a subset $A \subset \mathbb{N}$, $\{z_k : k \in A\} \subset S_E$ and $x^* \in S_{E^*}$ such that

(i) $\sum_{k \in A} \alpha_k > 1 - \gamma(\epsilon)$;
(ii) $\|z_k - x_k\| < \epsilon$ *for all* $k \in A$;
(iii) $x^*(z_k) = 1$ *for all* $k \in A$.

Our objective is to study when a function module space has AHSP. For this, we define a function module space. Recall that a space X is a $C(K)$-module space if for all $x \in X$ and for all $h \in C(K)$, we have that $hx \in X$, where $(hx)(t) := h(t)x(t)$.

Definition 2.4. *Function Module* is (the third coordinate of) a triple $(K, (X_t)_{t\in K}, X)$, where K is a nonempty compact Hausdorff topological space, $(X_t)_{t\in K}$ a family of Banach spaces, and X a closed $C(K)$-submodule of the $C(K)$-module $\prod_{t\in K}^{\infty} X_t$ (the ℓ_∞-sum of the spaces X_t) such that the following conditions are satisfied:

(1) for every $x \in X$, the function $t \mapsto \|x(t)\|$ from K to $\mathbb{R}$ is upper semicontinuous;
(2) for every $t \in K$, we have $X_t = \{x(t) : x \in X\}$;
(3) the set $\{t \in K : X_t \neq 0\}$ is dense in K.

Remark 2.5. In the Definition 2.4, K is called the base space and the family $(X_t)_{t\in K}$ is called the component spaces.

For function modules we follow the notation of [7], where the basic results of such theory can be found.

Examples 2.6.

(a) Let K be a nonempty compact Hausdorff space and $E \neq \{0\}$. The space $C(K, E)$ can be viwed as a function module space when $X_t = E$ for all $t \in K$ and $X = C(K, E)$.

(b) Let L be a nonempty locally compact Hausdorff space. The space $C_0(L, E)$ is the space of all continuous function $f : L \to E$ such that for all $\epsilon > 0$ there exists a compact set $C \subset L$ such that $\|f(t)\| \leq \epsilon$, for all $t \in L \setminus C$. It can be regarded in a natural way as a function module with base space $K = \beta L$ (the Stone–Cech compactification of L) and the component spaces $(X_t)_{t\in K}$ given by $X_t = E$ if $t \in L$ and $X_t = \{0\}$ if $t \in K \setminus L$.

Theorem 2.7. *Let $(K, (X_t)_{t\in K}, X)$ be a complex function module and $\epsilon > 0$. Suppose that for all $t \in K$, $(X_t)_{t\in K}$ has AHSP with the same function $\eta(\epsilon)$ given by Proposition 2.3, and for every $x_t \in X_t$ there exists $f \in X$ such that $f(t) = x_t$ and $\|f\| \leq \|x_t\|$ then X has AHSP.*

Proof. Let $0 < \epsilon < 1$. We consider a finite convex series $\sum_{k=1}^{n} \alpha_k x_k$ for $(x_k)_{k=1}^{n} \subset B_X$ such that

$$\Big\|\sum_{k=1}^{n} \alpha_k x_k\Big\| > 1 - \eta\Big(\frac{\epsilon}{2}\Big).$$

Since

$$\Big\|\sum_{k=1}^{n} \alpha_k x_k\Big\| = \sup\Big\{\Big\|\sum_{k=1}^{n} \alpha_k x_k(t)\Big\|_{X_t} : t \in K\Big\} > 1 - \eta\Big(\frac{\epsilon}{2}\Big),$$

there exists $t_0 \in K$ such that,

$$\Big\|\sum_{k=1}^{n} \alpha_k x_k(t_0)\Big\|_{X_{t_0}} > 1 - \eta\Big(\frac{\epsilon}{2}\Big).$$

And for all $k \in \{1, \ldots, n\}$ we have,

$$\|x_k(t_0)\|_{X_{t_0}} \leq \|x_k\| \leq 1.$$

Then, the sequence $(x_k(t_0))_{k=1}^{n} \subset B_{X_{t_0}}$. By hypothesis X_{t_0} has AHSP and by Proposition 2.3 there exist $A \subset \{1, \ldots, n\}$, $\{z_k : k \in A\} \subset S_{X_{t_0}}$ and $z^* \in S_{X^*_{t_0}}$ such that:

(1) $\sum_{k\in A} \alpha_k > 1 - \gamma(\epsilon/2)$;

(2) $\|z_k - x_k(t_0)\|_{X_{t_0}} < \epsilon/2$ for all $k \in A$;

(3) $z^*(z_k) = 1$ for all $k \in A$.

By hypothesis, for all $k \in A$ there exist $f_k \in X$ such that $f_k(t_0) = z_k$ and $\|f_k\| \leq \|z_k\| = 1$. Now, we define the following subset in K:

$$U = \bigcap_{k\in A}\Big\{t \in K : \|f_k(t) - x_k(t)\|_{X_t} \leq \frac{\epsilon}{2}\Big\}.$$

It is clear that $U \neq \emptyset$ and U is an open set of K, since the function $t \in K \mapsto \|x(t)\| \in \mathbb{R}$ is upper semicontinuous for all $x \in X$. By Urysohn's lemma there exists a function $\varphi : K \to [0, 1]$ such that $\varphi(t_0) = 1$ and $\varphi(t) = 0$ for all $t \in K \setminus U$.

Now, for each $k \in A$ let $g_k : K \to \bigcup_{t \in K} X_t$ defined by

$$g_k(t) = \varphi(t) f_k(t) + \left(1 - \frac{\epsilon}{2}\right)(1 - \varphi(t)) x_k(t).$$

It is clear that $g_k \in X$ for all $k \in A$. We claim that $(g_k)_{k \in A} \subset S_X$. Indeed,

$$\begin{aligned} \|g_k(t)\|_{X_t} &= \left\| \varphi(t) f_k(t) + \left(1 - \frac{\epsilon}{2}\right)(1 - \varphi(t)) x_k(t) \right\|_{X_t} \\ &\leq \varphi(t) + 1 - \varphi(t) = 1, \end{aligned}$$

and $\|g_k(t_0)\|_{X_{t_0}} = \|z_k\|_{X_{t_0}} = 1$. So , for all $k \in A$

$$\|g_k\| = \sup\{\|g_k(t)\|_{X_t} : t \in K\} = 1,$$

which means $g_k \in S_X$ for all $k \in A$.

Now, we will show that $\|g_k - x_k\| < \epsilon$ for all $k \in A$. In fact, if $t \in U$, then

$$\begin{aligned} \|g_k(t) - x_k(t)\|_{X_t} &= \left\| \varphi(t) f_k(t) + \left(1 - \frac{\epsilon}{2}\right)(1 - \varphi(t)) x_k(t) - x_k(t) \right\|_{X_t} \\ &= \left\| \varphi(t)(f_k(t) - x_k(t)) - \frac{\epsilon}{2}(1 - \varphi(t)) x_k(t) \right\|_{X_t} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

If $t \in K \setminus U$, then $\|g_k(t) - x_k(t)\|_{X_t} = \|(1 - \epsilon/2) x_k(t) - x_k(t)\|_{X_t} < \epsilon$.

Thus,

$$\|g_k - x_k\| = \sup\{\|(g_k - x_k)(t)\| : t \in K\} < \epsilon, \quad \forall k \in A.$$

Now we consider the valuation mapping $\delta_{t_0} : X \longrightarrow X_{t_0}$ and define the linear function $x^* := z^* \circ \delta_{t_0}$. If $x \in S_X$, then $|x^*(x)| = |z^*(x(t_0))| \leq \|z^*\| \|x(t_0)\| \leq 1$.

Besides that, for all $k \in A$, $|x^*(g_k)| = |z^*(g_k(t_o))| = |z^*(z_k)| = 1$. So $\|x^*\| = 1$.

Finally, $x^* \in S_{X^*}$ and $x^*(g_k) = 1$, for all $k \in A$. Then X has AHSP. □

In the next theorem we will show that it is possible to get the reciprocal of Theorem 2.7. We need to add the additional hypothesis that the mapping $t \in K \mapsto \|x(t)\|$ is continuous for all $x \in X$, when $X_t = E$, for all $t \in K$ and for some E.

Theorem 2.8. *Let $(K, (X_t)_{t \in K}, X)$ be a complex function module where $X_t = E$, for all $t \in K$ for some Banach space E. Suppose that the mapping $t \in K \mapsto \|x(t)\|$ is continuous for all $x \in X$. If X has AHSP, then X_t has AHSP for all $t \in K$.*

Proof. Let $\epsilon > 0$. Since X has AHSP, there exist $\eta(\epsilon), \gamma(\epsilon) > 0$ that satisfy the Proposition 2.3. Consider $(x_k)_k \subset B_E$ and the convex series $\sum_{k=1}^{\infty} \alpha_k x_k$ such that

$$\left\| \sum_{k=1}^{\infty} \alpha_k x_k \right\|_E > 1 - \eta(\epsilon).$$

For all $k \in \mathbb{N}$, we define $f_k : K \to \bigcup_{t\in K} X_t$ by $f_k(t) = x_k$. So $(f_k)_k \subset B_X$ and

$$\Big\|\sum_{k=1}^{\infty} \alpha_k f_k\Big\| = \sup\Big\{\Big\|\sum_{k=1}^{\infty} \alpha_k f_k(t)\Big\| : t \in K\Big\} = \Big\|\sum_{k=1}^{\infty} \alpha_k x_k\Big\|_E > 1 - \eta(\epsilon).$$

Since X has AHSP by Proposition 2.3, there are $A \subset \mathbb{N}$, $\{z_k : k \in A\} \subset S_X$ and $\Phi \in S_{X^*}$ such that $\sum_{k\in A} \alpha_k > 1 - \gamma(\epsilon)$, $\|z_k - f_k\| < \epsilon$ and $\Phi(z_k) = 1$, for all $k \in A$. Now, we claim that $\|\sum_{k\in A} \alpha_k z_k\| = \sum_{k\in A} \alpha_k$. Indeed,

$$\Big\|\sum_{k\in A} \alpha_k z_k\Big\| \le \sum_{k\in A} \alpha_k \|z_k\| = \sum_{k\in A} \alpha_k.$$

Since for all $k \in A$, $\Phi(z_k) = 1$, then

$$\Phi\Big(\sum_{k\in A} \alpha_k z_k\Big) = \sum_{k\in A} \alpha_k,$$

so

$$\Big\|\sum_{k\in A} \alpha_k z_k\Big\| = \sup\Big\{\Big|\varphi\Big(\sum_{k\in A} \alpha_k z_k\Big)\Big| : \varphi \in S_{X^*}\Big\} = \sum_{k\in A} \alpha_k.$$

Now,

$$\sum_{k\in A} \alpha_k = \Big\|\sum_{k\in A} \alpha_k z_k\Big\| = \sup\Big\{\Big\|\sum_{k\in A} \alpha_k z_k(t)\Big\| : t \in K\Big\}.$$

By hypothesis, the function $t \in K \mapsto \|\sum_{k\in A} \alpha_k z_k(t)\|_E$ is continuous. Then there is $t_0 \in K$, such that $\|\sum_{k\in A} \alpha_k z_k(t_0)\| = \sum_{k\in A} \alpha_k$. Thus, $\sum_{k\in A} \alpha_k z_k(t_0) \ne \mathbf{0}$. So, there is a function $x^* \in S_{E^*}$ such that

$$x^*\Big(\sum_{k\in A} \alpha_k z_k(t_0)\Big) = \Big\|\sum_{k\in A} \alpha_k z_k(t_0)\Big\|_E = \sum_{k\in A} \alpha_k.$$

Now we consider $g_k := z_k(t_0)$ and observe that $x^*(g_k) = 1$ for all $k \in A$. That is, for all $k \in A$, $(g_k)_k \subset S_E$ and

$$\|g_k - x_k\|_E = \|z_k(t_0) - f_k(t_0)\|_E \le \|z_k - f_k\| < \epsilon, \quad \forall k \in A.$$

The theorem follows. □

Corollary 2.9. *Let X be a dual complex Banach space such that X can be regarded as a function module space, where $X_t = E$, for all $t \in K$ and E a Banach space. Then X has AHSP if and only if X_t has AHSP for all $t \in K$.*

Proof. Now for X as a dual space which can be represented as a complex function module with a base space K (see [7, Proposition 3.10]), then for every $x \in X$, the function $t \in K \mapsto \|x(t)\|$ is continuous [7, Theorem 5.13]. Therefore the assumptions in both Theorems 2.7 and 2.8 hold. □

Corollary 2.10. *Let L be a locally compact Hausdorff space and X a Banach space. Then $C_0(L, X)$ has AHSP if, and only if X has AHSP.*

Proof. Consider $K = \beta L$ the Stone–Cech compactification of L; the Theorems 2.7 and 2.8 imply the result. □

As a consequence of Corollary 2.10 and Theorem 4.1 in [1] we have that $(\ell_1, C_0(L, E))$ has BPBP if, and only if E has AHSP. Generally, if $(K, (X_t)_{t\in K}, X)$ is a function module space with AHSP, then (ℓ_1, X) has BPBP.

Corollary 2.11. *Let $K \neq \emptyset$ be a compact Hausdorff topological space and E be a Banach space. Then E has AHSP if, and only if $C(K, E)$ has AHSP.*

Proof. Since $C(K, E)$ is a function module, with K base space and $X_t = E$ for all $t \in K$ (see Examples 2.6(a)) and the mapping $t \in K \mapsto \|f(t)\|$ is continuous for all $f \in C(K, E)$, the result follows straight away by Theorems 2.7 and 2.8. □

S. Y. Choi and S. Kim in [13, Theorem 11] showed that if $C(K, E)$ has AHSP, then E has AHSP. Here the Corollary 2.11 generalizes, in a sense, the result of Choi and Kim and we have the reciprocal.

Open problem: We do not know if Theorem 2.8 is true if there are two distinct component spaces. That means, there are $t_1, t_2 \in K$ such that $X_{t_1} \neq X_{t_2}$.

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T. GRANDO, Department of Mathematics, University of São Paulo,
P.O. Box 66281, 05315-970,
São Paulo, Brazil
e-mail: tgrando@ime.usp.br

M. L. LOURENÇO, Department of Mathematics, University of São Paulo,
P.O. Box 66281, 05315-970,
São Paulo, Brazil
e-mail: mllouren@ime.usp.br