Instituto de Ciências Matemáticas e de Computação

ISSN - 0103-2577

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Nº 83

NOTAS DO ICMC
Série Matemática

São Carlos
Ago./1999

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To G. Reeb, in memory

ABSTRACT. We define a concept of holonomy for a circle leaf of a smooth d-dimensional foliation of an m-manifold M, $1 \leq d \leq m$, which coincides with the Poincaré map when $d = 1$. Then, we use it to prove a theorem on the $C^{(1,1)}$ structural stability of a two dimensional foliation, defined by a smooth integrable (m-2)-form, in the neighborhood of a circle leaf. The $C^{(r,r)}$ topology measures the C^r proximity between two forms and also between their exterior derivatives.

1. INTRODUCTION

We shall consider foliations in the sense of P. Molino [9]: a C^r-*foliation* $\mathcal{F}$, $r \geq 1$, is a partition of an m-manifold M in connected immersed C^r submanifolds, called *leaves*, such that the module $\mathfrak{X}^r(\mathcal{F})$ of the C^r vector fields of M tangent to the leaves is transitive, i.e., given $v \in T_p(L)$, where L is any leaf, there exists $X \in \mathfrak{X}^r(\mathcal{F})$ such that $X_p = v$. There may be leaves of different dimensions, the *dimension* d of the foliation is the greatest of these numbers, and $c = m - d$ its *codimension*. If $\dim L < d$, then L is called a *singular leaf* and $\text{sing}(\mathcal{F}) = \{p \in L \mid L \text{ is singular}\}$ the *singular set* of $\mathcal{F}$. A leaf of dimension d is called a *regular leaf* and the subset $\text{reg}(\mathcal{F})$ of points of M which belong to regular leaves is open. A foliation with only regular leaves is called a *regular foliation.* The main results in this paper refer to foliations originated from integrable C^∞ forms.

Here we present a concept of holonomy for a circle leaf of a smooth d-foliation $\mathcal{F}$ of an m-manifold M, $1 \leq d \leq m$, which coincides with the Poincaré map when $d = 1$. Then, we use it to prove a theorem on the $C^{(1,1)}$ local structural stability of a 2-foliation, defined by a smooth

Date: September 1, 1999.

1991 *Mathematics Subject Classification.* Primary 57R .

Key words and phrases. Foliation, Holonomy, Singular leaf, Structural Stability.

This paper is in final form and no version of it will be submitted for publication elsewhere.

integrable $(m-2)$-form, in the neighborhood of one such leaf. The $C^{(r,r)}$ topology measures the C^r proximity between two forms and also between their exterior derivatives. A circle leaf $L \in \mathcal{F}$ is given by an embedding $\gamma : S^1 \to M$ with $\gamma(S^1) = L$. Fix a pair (L,p) with $p \in L$, and a cross section to $\mathcal{F}$ at $p \in M$, i.e., a smooth embedding $\sigma : D^{m-1} \to \Sigma \subset M$, transversal to $\mathcal{F}$, with $\sigma(0) = p$, where D^n denotes the open unit disk in $\mathbb{R}^n$. If $\mathcal{U}$ is a disk neighborhood of p in Σ, we denote by $\mathcal{F} \cap \mathcal{U}$ the $(d-1)$-foliation of $\mathcal{U}$ obtained by intersecting leaves of $\mathcal{F}$ with $\mathcal{U}$ and by $\frac{\mathcal{U}}{\mathcal{F}}$ the space of leaves of this foliation with the quotient topology. The space $\frac{\mathcal{U}}{\mathcal{F}}$ is in general topologically complicated, but it contains the subspace $\frac{\mathcal{U}_\bullet}{\mathcal{F}}$, where $\mathcal{U}_\bullet = \mathcal{U} \backslash \operatorname{sing}(\mathcal{F} \cap \mathcal{U})$, which in many cases admits a natural structure of a $(m-d)$-dimensional smooth manifold which may not be Hausdorff. We shall only consider situations in which $\frac{\mathcal{U}_\bullet}{\mathcal{F}}$ is a smooth manifold.

Denote by $\operatorname{Diff}^r_p(M,\mathcal{F})$, $r \geq 1$, the set of C^r-diffeomorphisms $f : \mathcal{U} \to M$, with $f(p) = p$, which take leaves of $\mathcal{F} \cap \mathcal{U}$ into leaves of $\mathcal{F}$, where $\mathcal{U}$ is an open disk of M containing p, and by $\operatorname{Diff}^r(M,\mathcal{F})$ if $\mathcal{U} = M$.

Now, assume that $\mathcal{G}$ is a smooth foliation of D^n having $\{0\}$ as the unique 0-dimensional leaf. Each $f \in \operatorname{Diff}^r_0(D^n,\mathcal{G})$ induces a continuous map $\bar{f} : \frac{\mathcal{U}}{\mathcal{G}} \to \frac{D^n}{\mathcal{G}}$ which restricts to a C^r-map from $\frac{\mathcal{U}_\bullet}{\mathcal{G}}$ into $\frac{D^n_\bullet}{\mathcal{G}}$, also denoted by $\bar{f}$.Two elements $f : \mathcal{U} \to D^n$ and $g : \mathcal{V} \to D^n$ of $\operatorname{Diff}^r_0(D^n,\mathcal{G})$ are said to be *equivalent* $f \sim g$ if there exists a disk neighborhood $\mathcal{W}$ of p in D^n and a continuous path f_t, $0 \leq t \leq 1$, in $\operatorname{Diff}^r_0(D^n,\mathcal{G})$, each f_t defined on $\mathcal{W}$, such that $f_0 = f$, $f_1 = g$, and $\bar{f}_t : \frac{\mathcal{W}}{\mathcal{G}} \to \frac{D^n}{\mathcal{G}}$ does not depend on t. The equivalence class $\widehat{f}$ is called the *class of* f, and we shall denote by $G^r(\mathcal{G})$ the group of these classes with the multiplication $\widehat{f} \circ \widehat{g} = \widehat{f \circ g}$.

The *holonomy* of a circle leaf L, defined at the end of section 2, is a homomorphism $H : \pi_1(L,p) = \mathbb{Z} \to G^\infty(\sigma^*\mathcal{F})$, and the subgroup $\operatorname{Hol}(L,p) = H(\pi_1(L,p))$ of $G^\infty(\sigma^*\mathcal{F})$ is called the *holonomy group* of L at p. If we use another point $q \in L$, then $\operatorname{Hol}(L,q)$ is isomorphic to $\operatorname{Hol}(L,p)$. Now assume that $\mathcal{F}$ is a two dimensional foliation of a m-manifold defined by a smooth integrable $(m-2)$-form ω and L a circle leaf of $\mathcal{F}$. If $p \in L$, then there exists a cross section $\sigma : D^{m-1} \to \Sigma \subset M$ to $\mathcal{F}$, with $\sigma(0) = p$, such that $\sigma^*\mathcal{F}$ is parametrized by a vector field X with a singularity at the origin and the generator of $\operatorname{Hol}(L,p)$ is the class of an element $f \in \operatorname{Diff}^r_0(D^n,X)$, i.e., a local diffeomorphism of D^{m-1} that takes orbits of X in its domain into orbits of X in D^{m-1}. Thus the holonomy is determined by the pair

(f, X). In section 3 we define a concept of hyperbolicity for a pair and prove theorems 3.5 and 3.6 that characterize the hyperbolicity of a pair (f, X) in terms of the structurally stability of the leaf map $\overline{f}$. In section 4 we prove Theorem 4.5. on the $C^{(1,1)}$ structural stability of a circle leaf L which is the main result of this paper.

It is possible to define a concept of holonomy for a leaf of any smooth foliation, which may be singular or not, and this concept coincides with the usual one on regular leaves, but we shall do it elsewhere. There are several papers that study a foliation, defined by an integrable form, in the neighborhood of a singular leaf. The first result on the C^1 structural stability of a foliation defined by a 1-form in the neighborhood of an isolated singular point is due to Reeb [11]. He considered the case of a central point. Medeiros in [8] extended it to the other posible cases. Lins in [6] considers foliations of $\mathbb{R}^3$ defined by C^2 1-forms whose 1-jet is null in the neighborhood of an isolated singular point. He proves the C^2 local structural stability at the singular points that satisfy a certain ' hyperbolicity condition '. In the references we also include other papers with related results.

2. THE HOLONOMY OF A CIRCLE LEAF

In this section we define a notion of holonomy for an S^1-leaf of a foliation. We start by proving some lemmas. Helpful references here are [4] and [10]. Let

$$\mathbf{C}^m = \{x \in \mathbb{R}^m \mid |x_i| < 1\} \quad \text{and} \quad \mathbf{C}^{m-l} = \{x \in \mathbf{C}^m \mid x_i = 0,\, 1 \leq i \leq l\}$$

Lemma 2.1. *(foliation-box) Let $\mathcal{F}$ be a codimension c smooth foliation of M, L a leaf of dimension l and $p \in L$. There exists a codimension c smooth foliation $\mathcal{G}_l$ of $\mathbf{C}^{m-l}$ with $\{0\}$ as a singular leaf, a neighborhood V of p in M and a smooth diffeomorphism $\mathbf{x} : V \to \mathbf{C}^m$ which takes the leaves of $\mathcal{F} \cap V$ onto the leaves of the foliation $(-1,1)^l \times \mathcal{G}_l$ of $\mathbf{C}^m$.*

Proof. Take a non-zero vector $v_1 \in T_p(L)$ and let $X_1 \in \mathfrak{X}^\infty(\mathcal{F})$ be an extension of v_1. By the flow-box theorem, there exists a neighborhood V_0 of p and a smooth diffeomorphism $h_0 : V_0 \to \mathbf{C}^m$ with $h_0(p) = 0$ such that h_0 takes the orbits of X_1 onto the orbits of $\frac{\partial}{\partial x_1}$. Call $\mathcal{F}_1$ the image foliation of $\mathcal{F} \cap V_0$ under h_0. Since $\frac{\partial}{\partial x_1}$ is tangent to $\mathcal{F}_1$, it follows that $\mathcal{F}_1$ is transversal to the foliation of $\mathbf{C}^m$ whose leaves are $\{x_1 = a,\ -1 < a < 1\}$. Let $\mathcal{G}_1 = \mathcal{F}_1 \cap \mathbf{C}^{m-1}$, then $\mathcal{F}_1 = (-1,1) \times \mathcal{G}_1$ and $\mathrm{cod}(\mathcal{G}_1) = c$. If $l = 1$, make $V = V_0$, $\mathbf{x} = h_0$ and the proof is complete but, if $l > 1$, then take a non-zero vector $v_2 \in T_0(L_0)$, where L_0 is the leaf of $\mathcal{G}_1$ by the point $0 \in \mathbf{C}^{m-1}$, and let $X_2 \in \mathfrak{X}^\infty(\mathcal{G}_1)$

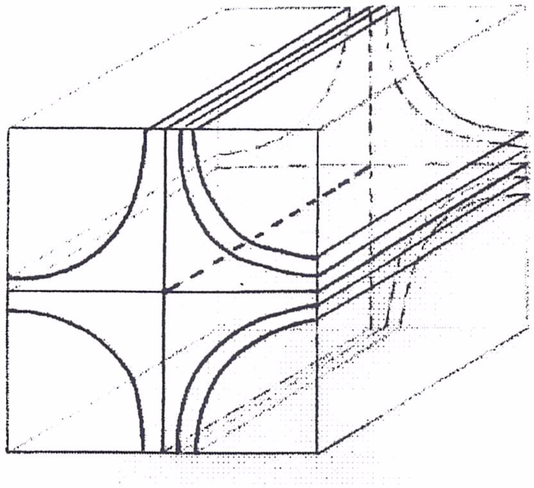

FIGURE 1

be an extension of v_2. Again, by the flow-box theorem, there exists a nbhd V_1 of 0 in $\mathbf{C}^{m-1}$ and a smooth diffeomorphism $g_1 : V_1 \to \mathbf{C}^{m-1}$, with $g_1(0) = 0$, such that g_1 takes the orbits of X_2 in V_1 onto the orbits of $\frac{\partial}{\partial x_2}$ in $\mathbf{C}^{m-1}$. Call $\mathcal{F}_{12}$ the foliation of $\mathbf{C}^{m-1}$ which is the image of $\mathcal{G}_1 \cap V_1$ under g_1 and put $\mathcal{F}_2 = (-1,1) \times \mathcal{F}_{12}$. Notice that $\mathcal{F}_2$ is the image of $(-1,1) \times \mathcal{G}_1$ under the diffeomorphism

$$h_1 = id \times g_1 : (-1,1) \times V_1 \to (-1,1) \times \mathbf{C}^{m-1} = \mathbf{C}^m$$

Now, since $\frac{\partial}{\partial x_2}$ is tangent to $\mathcal{F}_{12}$, it follows that $\mathcal{F}_{12}$ is transversal to the foliation of $\mathbf{C}^{m-1}$ whose leaves are $\{x_1 = 0, x_2 = a, \ -1 < a < 1\}$. Let $\mathcal{G}_2 = \mathcal{F}_{12} \cap \mathbf{C}^{m-2}$, it is clear that $\mathcal{F}_{12} = (-1,1) \times \mathcal{G}_2$ and thus $\mathcal{F}_2 = (-1,1)^2 \times \mathcal{G}_2$ with $\text{cod}(\mathcal{G}_2) = c$. If $l = 2$, make $V = (h_1 \circ h_0)^{-1}\mathbf{C}^m$ and $\mathbf{x} = h_1 \circ h_0$ and the lemma is proved. If $l > 2$, we continue the process until we arrive to a foliation $\mathcal{F}_l = (-1,1)^l \times \mathcal{G}_l$ of $\mathbf{C}^m$, where $\mathcal{G}_l$ is a smooth cod. c foliation of $\mathbf{C}^{m-l}$ whose leaf through 0 reduces to a point. Figure 1 shows an example of a foliation box with $m = 3$, $l = 1$ and $c = 1$. ∎

Definition 2.1. *A map $\mathbf{x} : U \to \mathbf{C}^m$, as in lemma 2.1, will be called a chart of $\mathcal{F}$ at $p \in L$ or simply, a chart of $\mathcal{F}$ at L.*

Plaques and cross sections. If $\mathbf{x} : V \to \mathbf{C}^m$ is a chart of $\mathcal{F}$ at $p \in L$, the smooth sub-manifolds of V of the form $\mathbf{x}^{-1}((-1,1)^l \times \{x\})$, with $x \in \mathbf{C}^{m-l}$, define an l-dimensional smooth regular foliation of V that we denote by $\mathcal{P}$. The leaves of $\mathcal{P}$ are called *plaques* and each plaque is contained in a leaf of the foliation $\mathcal{F} \cap V$. Notice that if $\mathcal{F}$ is a regular foliation or if L is a regular leaf and V is small enough, then

$\mathcal{P} = \mathcal{F} \cap V$. Let $\sigma : D^{m-l} \to M$ be a smooth embedding, with $\sigma(0) \in L$, which is transversal to $\mathcal{F}$. The map σ and also the submanifold $\Sigma = \sigma(D^{m-l})$ are called a *cross section* to $\mathcal{F}$ at $\sigma(0) \in L$. A natural way to obtain cross sections is by restricting $\mathbf{x}^{-1}$ to disks contained in $\mathbf{C}^{m-l}$. Since we are dealing with local sections we may always assume that σ itself is transversal to $\mathcal{P}$. The saturated $\mathcal{P}\Sigma$ of Σ by the foliation $\mathcal{P}$ is an open subset of V and there is a smooth projection $\Pi : \mathcal{P}\Sigma \to \Sigma$ defined by $\Pi(q) = P_q \cap \Sigma$, where P_q is the plaque through q. Since σ is transversal to $\mathcal{F}$, it follows that $\mathcal{F} \cap \Sigma$ is a cod. c foliation of Σ, and it is clear that Π sends leaves of $\mathcal{F} \cap \mathcal{P}\Sigma$ into leaves of $\mathcal{F} \cap \Sigma$. When Σ is contained in the intersection of the domains of two charts $(\mathbf{x}_j, V_j)$, $j = 0, 1$, both Π_0 and Π_1 are defined on $\mathcal{P}_0\Sigma \cap \mathcal{P}_1\Sigma$, but, in general, they are different. If L is a regular leaf, then $\Pi_0 = \Pi_1$.

A map $\gamma : [a, b] \to M$ is called an *arc of flow* if there is a vector field $X \in \mathfrak{X}^\infty(M)$ such that $\gamma(t) = X^t(p)$, $a \le t \le b$, where p is a non-singular point of X. Notice that we can also parametrize $\gamma[a, b]$ by $\gamma(a + s) = X^s(\gamma(a))$ with $0 \le s \le b - a$. If γ is an arc of flow, then for every $t \in [a, b]$ we have $\gamma'(t) = X(\gamma(t)) \ne 0$, i.e., γ is an immersion and also X is an *extension* of γ'. A smooth parametrization $\gamma : S^1 \to L$ of a circle leaf can be regarded as a smooth periodic map $\gamma : \mathbb{R} \to L$ of period one.

Lemma 2.2. *Let L be a circle leaf of a smooth d-foliation $\mathcal{F}$. Then, there exists $Y \in \mathfrak{X}^\infty(\mathcal{F})$ such that L is a periodic orbit of Y.*

Proof. Let $\gamma : S^1 \to L$ be a smooth parametrization. For each $s \in S^1$, by lemma 2.1, there is a chart $\mathbf{x}_s : U_s \to \mathbf{C}^m$ of $\mathcal{F}$ at $\gamma(s)$ such that $L \cap U_s = \gamma(I_s)$, for some open interval $I_s \ni s$. We can assume that the vector fields γ' and $(\mathbf{x}_s^{-1}|_{(-1,1)\times\{0\}})'$ define the same orientation on $\gamma(I_s)$. Let $V_s = \mathbf{x}_s^{-1}(-\frac{2}{3}, \frac{2}{3})^m$, $W_s = \mathbf{x}_s^{-1}(-\frac{1}{3}, \frac{1}{3})^m$, $J_s = \gamma^{-1}(W_s)$ and let $\lambda_s : M \to \mathbb{R}^+$ be a C^∞-function such that $\lambda_s(p) = 1$ if $p \in W_s$ and $\lambda_s(p) = 0$ if $p \in M - V_s$.

The family $\{J_s\}_{s\in S^1}$ is an open covering of S^1, thus there is a finite subcovering $\{J_i = J_{s_i}\}_{i=0}^n$. We can assume w.l.o.g. that the finite covering is a closed chain i.e., no interval J_i is contained in the union of two other intervals of the collection. For each $0 \le i \le n$ put $U_i = U_{s_i}$, $V_i = V_{s_i}$, $W_i = W_{s_i}$, $\mathbf{x}_i = \mathbf{x}_{s_i}$, $\lambda_i = \lambda_{s_i}$ and define $X_i = \lambda_i \cdot (\mathbf{x}_i^{-1})_*(\frac{\partial}{\partial x_1})$. It is not difficult to verify that $Y = X_0 + ... + X_n$ has L as a periodic orbit with the orientation induced by γ' ∎

Fix a circle leaf $L \in \mathcal{F}$ and a parametrization $\gamma : \mathbb{R} \to L$ which is an arc of flow for some $X \in \mathfrak{X}^\infty(\mathcal{F})$. Let $\sigma : D^{m-1} \to \Sigma$ be a smooth cross section to $\mathcal{F}$ at $p = \gamma(0)$ and $P_X : \sigma(\mathcal{U}) \to \Sigma$ the Poincaré transformation of L with respect to X, where $\mathcal{U}$ is a disk nbhd of 0 in D^{m-1}. Let $\gamma_X = \sigma^{-1} \circ P_X \circ \sigma : \mathcal{U} \to D^{m-1}$ and $\mathcal{G} = \sigma^* \mathcal{F}$. Since X is tangent to $\mathcal{F}$, it is clear that P_X takes leaves of $\mathcal{F} \cap \sigma(\mathcal{U})$ into leaves of $\mathcal{F} \cap \Sigma$ or equivalently that γ_X takes leaves of $\mathcal{G} \cap \mathcal{U}$ into leaves of $\mathcal{G}$, i.e., $\gamma_X \in \mathrm{Diff}_0^\infty(D^{m-1}, \mathcal{G})$. The maps P_X and γ_X are called *holonomy maps* associated to γ. The equivalence class $\widehat{\gamma_X}$ is called the *holonomy class* associated to the generator $[\gamma] \in \pi_1(L,p)$.

Define $H : \pi_1(L,p) \to G^\infty(\mathcal{G})$ by $H([\gamma]^n) = (\widehat{\gamma_X})^n$ and $\mathrm{Hol}(L,p)$ as the subgroup of $G^\infty(\mathcal{G})$ generated by $\widehat{\gamma_X}$. To see that $\widehat{\gamma_X}$ does not depend on X, take another extension $Y \in \mathfrak{X}^\infty(\mathcal{F})$ of γ' and define $X_t = (1-t)X + tY$. It is clear that $\gamma_X \sim \gamma_Y$.

3. ORBIT PRESERVING DIFFEOMORPHISMS

Let $X(x) = (a_1(x), ..., a_n(x))$ be a smooth vector field defined on a neighborhood of $0 \in \mathbb{R}^n$ where $a_i(x) = \Sigma_{j=1}^n a_{ij}\, x_j + Q_i(x)$, $1 \le i \le n$, with $a_{ij} = \frac{\partial a_i}{\partial x_j}(0)$ and $\lim_{x \to 0} \frac{Q_i(x)}{\|x\|} = 0$. If we write points of $\mathbb{R}^n$ as column vectors, then $X(x) = Ax + Q(x)$, with $A = [a_{ij}]$. The set of eigenvalues $\{\mu_1, \mu_2, ..., \mu_l\}$ of A are said to be *non-resonant* if

$$\mu_i \neq \Sigma_{j=1}^l m_j \cdot \mu_j$$

for all $1 \le i \le l$ and for all non-negative integers $m_1, ..., m_l$ with $\Sigma_{j=1}^l m_j \ge 2$. X is said to be *non-resonant at* 0 if the set of eigenvalues of A are non-resonant.

Definition 3.1. *We say that* 0 *is a singularity of* type (s,u) *or that* X *is of* type (s,u) *at* 0 *if the eigenvalues of* A *are all real, satisfy*

$$\alpha_s < \alpha_{s-1} < .... < \alpha_1 < 0 < \beta_1 < \beta_2 < .... < \beta_u$$

with $s+u = n$ *for some* $0 \le s \le n$, *and there is at least one eigenvalue* μ *such that* $-\mu$ *is not an eigenvalue.*

Assume now that X is linear and of type (s,u) at 0. Let E_i, $1 \le i \le s$, and F_i, $1 \le i \le u$, be the eigenspaces associated to α_i and β_i, respectively. Put $E = E_1 \oplus .... \oplus E_s$ and $F = F_1 \oplus .... \oplus F_u$, then $\mathbb{R}^n = E \oplus F$. Define $E^i = E_{i+1} \oplus ... \oplus E_s$ for $0 \le i \le s-1$, and $F^i = F_{i+1} \oplus ... \oplus F_u$, for $0 \le i \le u-1$.

The manifolds $\mathbf{S}^{n-1}_{(u,v)}$. Fix a basis of $\mathbb{R}^n$ by choosing a vector $\overrightarrow{e_i}$ in each E_i and a vector $\overrightarrow{f}_i$ in each F_i and define an inner product on $\mathbb{R}^n$ by imposing that this basis be orthonormal. In this eigenbasis a point of $\mathbb{R}^n$ has coordinates $(x,y) = (x_1, ..., x_s, y_1, ..., y_u)$.

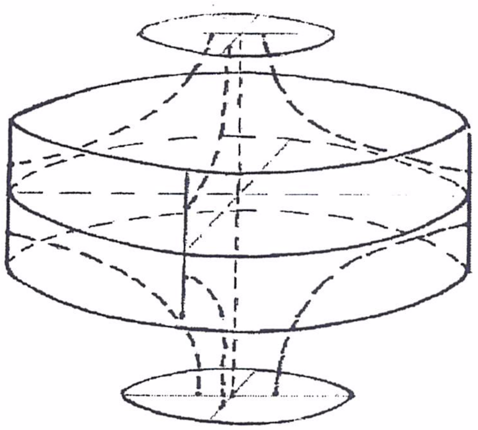

FIGURE 2

The orbit $\mathfrak{o}(x,y)$ of X through (x,y) is parametrized by $X^t(x,y) = (e^{\alpha_1 t}x_1, ..., e^{\alpha_s t}x_s, e^{\beta_1 t}y_1, ..., e^{\beta_u t}y_u)$, and we denote by $\mathcal{F}_X$ the foliation of $\mathbb{R}^n$ given by the orbits of X. Put $D^s_\delta = \{(x,0) \mid |x| < \delta\}$, $S^{s-1}_\delta = \partial\overline{D^s_\delta}$, $D^u_\delta = \{(0,y) \mid |y| < \delta\}$, $S^{u-1}_\delta = \partial\overline{D^u_\delta}$, and consider the $(n-1)$-manifold $\mathbf{S}^{n-1}_{(s,u)}$ obtained from $S^{s-1}_\delta \times F \cup E \times S^{u-1}_\delta$ by identifying a point $(x,y) \in S^{s-1}_\delta \times F$, $y \neq 0$, with the point of intersection of $\mathfrak{o}(x,y)$ and $E \times S^{u-1}_\delta$. The differentiable structure on $S^{n-1}_{(s,u)}$ is such that S^{s-1}_δ and S^{u-1}_δ are submanifolds and $\mathbf{S}^{n-1}_{(s,u)} \backslash S^{u-1}_\delta$ ($\mathbf{S}^{n-1}_{(s,u)} \backslash S^{s-1}_\delta$) is diffeomorphic to $S^{s-1}_\delta \times F$ ($E \times S^{u-1}_\delta$). The positive number δ will be chosen according to convenience.

Lemma 3.1. *Let X be a linear vector field of $\mathbb{R}^n$ of type (s,u) at 0 and $\mathcal{U}$ a disk neighborhood of the origin. Then, the space $\frac{\mathcal{U}^n_\bullet}{\mathcal{F}_X}$ of non-singular orbits of X in $\mathcal{U}$ is diffeomorphic to $\mathbf{S}^{n-1}_{(s,u)}$, and the following properties are satisfied:*

1) $\mathbf{S}^{n-1}_{(j,k)}$ is diffeomorphic to $\mathbf{S}^{n-1}_{(k,j)}$ and $\mathbf{S}^{n-1}_{(0,n)}$ is diffeomorphic to the sphere S^{n-1}

2) If $0 < s < n$, then any two points $(x,0)$, $(0,y)$ of $\mathbf{S}^{n-1}_{(s,u)}$ are inseparable *i.e., any neighborhood of $(x,0)$ intersects any neighborhood of $(0,y)$, in particular $\mathbf{S}^{n-1}_{(s,u)}$ is non-Hausdorff; however every point is closed.*

The manifolds $\mathbf{K}_\varepsilon$. When $0 < s < n$, we let $S^{s-1} \subset E$ ($S^{u-1} \subset F$) denote any small sphere S^{s-1}_ρ (S^{u-1}_ρ) contained in a given disk neighborhood $\mathcal{U}$ of 0. This lack of precision is not serious because after making a choice, we keep it all the way through. Take $\varepsilon > 0$ so small that the *fence* $S^{s-1} \times \overline{D^u_\varepsilon}$ be contained in $\mathcal{U}$ and define

$$\psi : S^{s-1} \times (\overline{D^u_\varepsilon})_\bullet \to E \times S^{u-1}$$

by $\psi(x,y) = \mathfrak{o}(x,y) \cap E \times S^{u-1}$. Denote by $\mathbf{K}_\varepsilon = \mathbf{K}_\varepsilon(X)$ the topological subspace of $\frac{\mathcal{U}_\bullet}{\mathcal{F}_X}$ or of $\mathbf{S}^{n-1}_{(s,u)}$ defined by the orbits of X through the points of $(S^{s-1} \times \overline{D^u_\varepsilon}) \cup S^{u-1}$. This space is in fact a compact non-Hausdorff $(n-1)$-manifold whose boundary $\partial\mathbf{K}_\varepsilon$ is diffeomorphic to $S^{s-1} \times S^{u-1}$. Notice that each orbit in $\mathbf{K}_\varepsilon$ intersects the set $(S^{s-1} \times \overline{D^u_\varepsilon}) \cup S^{u-1}$ in a unique point. Thus, one can transfer the differentiable structure of $\mathbf{K}_\varepsilon$ to this set and in this form render concrete a manifold whose points are orbits. This differentiable manifold will also be denoted by $\mathbf{K}_\varepsilon$. The fences $\mathbf{F}^s_\varepsilon = S^{s-1} \times \overline{D^u_\varepsilon}$ and $\mathbf{F}^u_\varepsilon = \psi(S^{s-1} \times (\overline{D^u_\varepsilon})_\bullet) \cup S^{u-1}$ are compact Hausdorff submanifolds of $\mathbf{K}_\varepsilon$. Figure 2 illustrates the definition of $\mathbf{K}_\varepsilon \subset \mathbf{S}^2_{(2,1)}$

We consider pairs (f, X) where X is linear of type (s,u) at 0 and $f \in \mathrm{Diff}^r_0(\mathbb{R}^n, X)$, i.e., $f : \mathcal{U} \to \mathbb{R}^n$ is a C^r-diffeomorphism, $r \geq 1$, of $\mathcal{U}$ into $\mathbb{R}^n$ fixing the origin and taking orbits of $X|_\mathcal{U}$ into orbits of X. Each f induces a leaf map $\overline{f} : \frac{\mathcal{U}_\bullet}{\mathcal{F}_X} \to \frac{D^n_\bullet}{\mathcal{F}_X}$. Our objective in this section is to find out under which conditions $\overline{f}$ is structurally stable. It is proven in Proposition 1.1 of [3] that $DX(0) \circ Df(0) = \frac{1}{\lambda} Df(0) \circ DX(0)$, with $\lambda \in \{-1, 1\}$. Let μ be an eigenvalue of $DX(0)$ such that $-\mu$ is not an eigenvalue and $\overrightarrow{u}$ an eigenvector associated to it, then

$$DX(0) \circ Df(0)(\overrightarrow{\mathrm{u}}) = \tfrac{1}{\lambda} Df(0) \circ DX(0)(\overrightarrow{\mathrm{u}}) = \tfrac{\mu}{\lambda} Df(0)(\overrightarrow{\mathrm{u}})$$

i.e., $\frac{\mu}{\lambda}$ is an eigenvalue of $DX(0)$, and therefore $\lambda = 1$. This means that $Df(0)$ commutes with $DX(0)$ and also that f preserves the orientation of the orbits. Therefore $Df(0)\, E_i = E_i$, for $1 \leq i \leq s$, and $Df(0)\, F_i = F_i$, for $1 \leq i \leq u$, or in other words

$$f(x,y) = (a_1 x_1 + R_1, ..., a_s x_s + R_s, b_1 y_1 + S_1, ..., b_u y_u + S_u)$$

for some non-zero real numbers $a_1, ..., a_s, b_1, ..., b_u$, with

$$lim_{(x,y)\to(0,0)} \tfrac{R_i}{|(x,y)|} = lim_{(x,y)\to(0,0)} \tfrac{S_i}{|(x,y)|} = 0\,.$$

Notice that $Df(0)$ is a diffeomorphism of $\mathbb{R}^n$ which takes orbits of X onto orbits of X and in particular $Df(0) \in \mathrm{Diff}^r_0(\mathbb{R}^n, X)$. For each $1 \leq i \leq s$ denote by $x_i^\pm$ the point of E whose only non-zero coordinate is its i-coordinate which is ± 1, and for each $1 \leq i \leq u$ by $y_i^\pm$ the point of F whose only non-zero coordinate is its i-coordinate which is ± 1. It is clear that either $\mathfrak{o}(x_i^+)$ and $\mathfrak{o}(x_i^-)$ are fixed points of $\overline{Df(0)}$ or of $\overline{Df^2(0)}$. The same statement is true for $\mathfrak{o}(y_i^+)$ and

$\mathfrak{o}(y_i^-)$. In the next lemma and whenever convenient, we shall write x_{s+i}, α_{s+i}, a_{s+i} and R_{s+i} instead of y_i, β_i, b_i and S_i, respectively, for each $1 \leq i \leq u$.

Put $T_{c_i} = \{x \in \mathbb{R}^n \mid x_i = c_i\}$ and let $\pi : T_{c_i} \to \mathbb{R}^{n-1}$ be given by $\pi(x_1, ..., x_{i-1}, c_i, x_{i+1}, ..., x_n) = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$. Define

$$\psi_{c_i} : \{x \in \mathbb{R}^n \mid x_i \cdot c_i > 0\} \to \mathbb{R}^{n-1}$$

by $\psi_{c_i}(x) = \pi(\mathfrak{o}(x) \cap T_{c_i})$. Then

$$(3.1)\quad \psi_{c_i}(x) = ((\frac{c_i}{x_i})^{\frac{\alpha_1}{\alpha_i}} x_1, ..., (\frac{c_i}{x_i})^{\frac{\alpha_{i-1}}{\alpha_i}} x_{i-1}, (\frac{c_i}{x_i})^{\frac{\alpha_{i+1}}{\alpha_i}} x_{i+1}, ..., (\frac{c_i}{x_i})^{\frac{\alpha_n}{\alpha_i}} x_n)$$

The map ψ_{c_i} induces a chart $\mathbf{c}_i : V_i^{\pm} \subset \frac{U_\bullet}{\mathcal{F}_X} \to \mathbb{R}^{n-1}$, given by $\mathbf{c}_i(\mathfrak{o}(x)) = \psi_{c_i}(x)$, where V_i^+ (V_i^-) is a neighborhood of $\mathfrak{o}(x_i^+)$ ($\mathfrak{o}(x_i^-)$) if $c_i > 0$ ($c_i < 0$), and $\mathbf{c}_i(\mathfrak{o}(x_i^{\pm})) = 0$. We call $(\mathbf{c}_i, V_i^{\pm})$ *a canonical chart of* $\frac{U_\bullet}{\mathcal{F}_X}$ *at* $\mathfrak{o}(x_i^+)$ ($\mathfrak{o}(x_i^-)$)

Lemma 3.2. *Let (f, X) be a pair where X is a linear vector field of $\mathbb{R}^n$, $n \geq 2$, of type (s, u) at 0 and $f \in \mathit{Diff}_0^r(\mathbb{R}^n, X)$ with $r \geq 1$. Let $\overline{f} : \frac{U_\bullet}{\mathcal{F}_X} \to \frac{D_\bullet^n}{\mathcal{F}_X}$ be its leaf map. Assume that for some $1 \leq i \leq n$ the orbits $\mathfrak{o}(x_i^+)$ and $\mathfrak{o}(x_i^-)$ are fixed points of $\overline{f}$. Then, in any canonical chart $(\mathbf{c}_i, V_i^{\pm})$ if $1 \leq j \leq n$ and $j \neq i$*

$$(3.2)\qquad D\overline{f}(\mathfrak{o}(x_i^+))\frac{\partial}{\partial x_j} = D\overline{f}(\mathfrak{o}(x_i^-))\frac{\partial}{\partial x_j} = a_j / a_i^{\frac{\alpha_j}{\alpha_i}} \frac{\partial}{\partial x_j}$$

If instead of fixed points they are periodic of period two, then

$$(3.3)\qquad D\overline{f}^2(\mathfrak{o}(x_i^+))\frac{\partial}{\partial x_j} = D\overline{f}^2(\mathfrak{o}(x_i^-))\frac{\partial}{\partial x_j} = a_j^2 / a_i^{2\frac{\alpha_j}{\alpha_i}} \frac{\partial}{\partial x_j}$$

Proof. Assume that $\mathfrak{o}(x_i^+)$ and $\mathfrak{o}(x_i^-)$ are fixed points of $\overline{f}$, or equivalently that $a_i > 0$. We will compute now $D\overline{f}(\mathfrak{o}(x_i^+))$ and $D\overline{f}(\mathfrak{o}(x_i^-))$. The leaf map in a canonical chart is $\mathbf{c}_i \circ \overline{f} \circ \mathbf{c}_i^{-1} = (\overline{f}_1, ..., \overline{f}_{i-1}, \overline{f}_{i+1}, ..., \overline{f}_n)$, and given by

$$\mathbf{c}_i \circ \overline{f} \circ \mathbf{c}_i^{-1}(x_1, ..., x_{i-1}, x_{i+1}, ...x_n) = \psi_{c_i} \circ f(x_1, ..., x_{i-1}, c_i, x_{i+1}, ...x_n)$$
$$=((\tfrac{f_i}{c_i})^{-\frac{\alpha_1}{\alpha_i}} f_1, ..., (\tfrac{f_i}{c_i})^{-\frac{\alpha_{i-1}}{\alpha_i}} f_{i-1}, (\tfrac{f_i}{c_i})^{-\frac{\alpha_{i+1}}{\alpha_i}} f_{i+1}, ..., (\tfrac{f_i}{c_i})^{-\frac{\alpha_n}{\alpha_i}} f_n)$$

Now we compute all partial derivatives $\frac{\partial \overline{f}_j}{\partial x_k}$, where $j, k \in \{1, ..., i-1, i+1, ..., n\}$. Assume, for example, that $k < i$, then

$$\tfrac{\partial \overline{f}_j}{\partial x_k}(0) = \lim_{x_k \to 0}(\tfrac{f_i(0,...,x_k,...,c_i,...,0)}{c_i})^{-\frac{\alpha_j}{\alpha_i}} \tfrac{f_j(0,...,x_k,...,c_i,...,0)}{x_k}$$
$$=(\tfrac{f_i(0,...,c_i,...,0)}{c_i})^{-\frac{\alpha_j}{\alpha_i}} \tfrac{\partial f_j}{\partial x_k}(0, ..., c_i, ..., 0)$$

But $\frac{\partial f_j}{\partial x_k}(0)$ does not depend on c_i. In fact, let d_i be a real number with $c_i \cdot d_i > 0$ and $(\mathbf{d}_i, V_i^{\pm})$ the corresponding chart. Compute the Jacobian matrix of $\mathbf{d}_i \circ \overline{f} \circ \mathbf{d}_i^{-1}$ in two ways using each side of the equality

$$\mathbf{d}_i \circ \overline{f} \circ \mathbf{d}_i^{-1} = (\mathbf{c}_i \circ \mathbf{d}_i^{-1})^{-1} \circ \mathbf{c}_i \circ \overline{f} \circ \mathbf{c}_i^{-1} \circ (\mathbf{c}_i \circ \mathbf{d}_i^{-1})$$

and compare the results. Therefore, letting $c_i \to 0$ one obtains

$$\frac{\partial \overline{f}_j}{\partial x_k}(0) = \left(\frac{\partial f_i}{\partial x_i}(0)\right)^{-\frac{\alpha_j}{\alpha_i}} \frac{\partial f_j}{\partial x_k}(0) = a_i^{-\frac{\alpha_j}{\alpha_i}} a_j \delta_{jk}$$

This proves 3.2. The other case is analogous ∎

Remark 3.1. *If (f, X) is a pair satisfying the hypothesis of Lemma 3.2, then $(Df(0), X)$ is also a pair with that property. The map $\overline{Df(0)} : K_\varepsilon \to S^{n-1}_{(s,u)}$ leaves invariant the submanifold S^{s-1} (S^{u-1}) and for each $1 \le i \le s$ ($1 \le i \le u$) it is always true that $\mathfrak{o}(x_i^+)$ and $\mathfrak{o}(x_i^-)$ ($\mathfrak{o}(y_i^+)$ and $\mathfrak{o}(y_i^-)$) are either fixed or periodic points of period two. Since the conclusions of the lemma depend only on the first jet of X and f at 0, they are also valid for $\overline{Df(0)}$, i.e., if they are fixed points, then in any canonical chart $(\mathbf{c}_i, V_i^{\pm})$*

$$D\overline{Df(0)}(\mathfrak{o}(x_i^{\pm}))\left(\frac{\partial}{\partial x_j}\right) = a_j / a_i^{\frac{\alpha_j}{\alpha_i}} \frac{\partial}{\partial x_j} \qquad D\overline{Df(0)}(\mathfrak{o}(x_i^{\pm}))\left(\frac{\partial}{\partial y_j}\right) = b_j / a_i^{\frac{\beta_j}{\alpha_i}} \frac{\partial}{\partial x_j}$$

$$D\overline{Df(0)}(\mathfrak{o}(y_i^{\pm}))\left(\frac{\partial}{\partial x_j}\right) = a_j / b_i^{\frac{\alpha_j}{\beta_i}} \frac{\partial}{\partial x_j} \qquad D\overline{Df(0)}(\mathfrak{o}(y_i^{\pm}))\left(\frac{\partial}{\partial y_j}\right) = b_j / b_i^{\frac{\beta_j}{\beta_i}} \frac{\partial}{\partial x_j}$$

There are also the corresponding formulas if the points are periodic.

The next theorem, which is a particular case of theorem 2 in [3], will be very useful in this section.

Theorem 3.3. *(Camacho-Lins)- Let (f, X) be a pair where X is a linear vector field of $\mathbb{R}^n$, $n \ge 2$, non-resonant at 0 and of type $(n, 0)$ or $(0, n)$, and $f \in Diff_0^r(\mathbb{R}^n, X)$ with $r > \left|\frac{\alpha_n}{\alpha_1}\right|$. Then $\mathbf{c}_1 \circ \overline{f} \circ \mathbf{c}_1^{-1} : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is a linear map.*

Lemma 3.4. *Let (f, X) be a pair where X is a linear vector field of $\mathbb{R}^n$, $n \ge 2$, non-resonant at 0 and of type (s, u), and $f \in Diff_0^r(\mathbb{R}^n, X)$ with $r > max\{\left|\frac{\alpha_s}{\alpha_1}\right|, \left|\frac{\beta_u}{\beta_1}\right|\}$. Then*

1) $\overline{f}\,|_{S^{s-1}} = \overline{Df(0)}\,|_{S^{s-1}}$ and $\overline{f}\,|_{S^{u-1}} = \overline{Df(0)}\,|_{S^{u-1}}$,

2) $D\overline{f}(x, 0) = D\overline{Df(0)}(x, 0)$ and $D\overline{f}(0, y) = D\overline{Df(0)}(0, y)$.

Proof. It follows from the commutativity of $Df(0)$ with $DX(0)$ that $Df(0)E_i = E_i$ and $Df(0)F_i = F_i$, in particular $Df(0)E = E$ and $Df(0)F = F$. Since E (F) is the stable (unstable) manifold of X at 0 and f preserves the orientation of the orbits, it follows from the continuity of f at 0 that $f(E \cap \mathcal{U}) \subset E$ ($f(F \cap \mathcal{U}) \subset F$). Thus, the submanifolds S^{s-1} and S^{u-1} of $K_\varepsilon \subset S^{n-1}_{(s,u)}$ are left invariant under both $\overline{Df(0)}$ and $\overline{f}$. One knows that $Df(0)E^1 = E^1$ and it can be verified that the orbit $\mathfrak{o}(p)$ of any $p \in E \backslash E^1$ enters the origin with a tangent vector parallel to E_1 when $t \to \infty$. Therefore $f(E^1 \cap \mathcal{U})$ has to be contained in E^1. Denote by (f_{E^1}, X_{E^1}) the restriction of the pair (f, X) to E^1 and notice that X_{E^1} is non-resonant at 0, of type $(s-1, 0)$, with eigenvalues $\{\alpha_2, ..., \alpha_s\}$, and that $f_{E^1} \in Diff^r(D^{s-1}, \mathcal{F}_{X_{E^1}})$, with $r > \left|\frac{\alpha_s}{\alpha_1}\right| > \left|\frac{\alpha_s}{\alpha_2}\right|$. Let $\overline{f_{E^1}}, \overline{Df_{E^1}(0)} : S^{s-2} \to S^{s-2}$, with $S^{s-2} = S^{s-1} \cap E^1$, be the leaf maps of f_{E^1} and Df_{E^1}. By the induction hypothesis $\overline{f_{E^1}} = \overline{Df_{E^1}(0)}$. This implies that either $\overline{f}\mathfrak{o}(x_i^{\pm}) = \mathfrak{o}(x_i^{\pm})$ or $\overline{f}\mathfrak{o}(x_i^{\pm}) = \mathfrak{o}(x_i^{\mp})$, for each $2 \le i \le s$.

Now, assume that $a_1 > 0$, then f transforms $\{x \mid x_1 > 0\} \cap \mathcal{U}$ into $\{x \mid x_1 > 0\}$, or equivalently, $\overline{f}$ leaves invariant $V^+ = \{\mathfrak{o}(x) \mid x_1 > 0\}$ and also $V^- = \{\mathfrak{o}(x) \mid x_1 < 0\}$. If $a_1 < 0$, one would work with f^2 instead of f. By Theorem 3.3 $\mathfrak{c}_1 \circ \overline{f} \circ \mathfrak{c}_1^{-1}$ and $\mathfrak{c}_1 \circ \overline{Df(0)} \circ \mathfrak{c}_1^{-1}$ are linear and in particular $\overline{f}\mathfrak{o}(x_1^{\pm}) = \mathfrak{o}(x_1^{\pm})$, and by Lemma 3.2 they are given, in the base $\{\overrightarrow{e_2}, ..., \overrightarrow{e_s}\}$, by the same matrix

$$\mathrm{diag}(a_2/a_1^{\frac{\alpha_2}{\alpha_1}}, ..., a_s/a_1^{\frac{\alpha_s}{\alpha_1}}) \ .$$

Thus, $\overline{f}\mid_{V^\pm} = \overline{Df(0)}\mid_{V^\pm}$. Finally, since $S^{s-1} = V^- \cup S^{s-2} \cup V^+$, we conclude that $\overline{f}\mid_{S^{s-1}} = \overline{Df(0)}\mid_{S^{s-1}}$. This proves the first part of 1). Notice that either $\overline{f}\mathfrak{o}(x_i^{\pm}) = \mathfrak{o}(x_i^{\pm})$ or $\overline{f}\mathfrak{o}(x_i^{\pm}) = \mathfrak{o}(x_i^{\mp})$, for each $1 \le i \le s$. The second part is analogous. We can prove 4) by elementary computations ∎

A triangulation of S^{n-1} invariant under $\overline{f}$. Let (f, X) be a pair as in lemma 3.4 with $s = n$. Choose a small sphere $S^{n-1}_\varepsilon \subset \mathcal{U} \subset D^n$ such that S^{n-1}_ε renders concrete both $\frac{\mathcal{U}_\bullet}{\mathcal{F}_X}$ and $\frac{D^n_\bullet}{\mathcal{F}_X}$. Define $g : S^{n-1} \to S^{n-1}_\varepsilon$ by $g(x) = \mathfrak{o}(x) \cap S^{n-1}_\varepsilon$. The leaf map, after conjugation by g, becomes a diffeomorphism of S^{n-1} that we keep denoting by $\overline{f}$. It follows from Lemma 3.4 that $\overline{f}(x_i^{\pm}) = x_i^{\pm}$ if $a_i > 0$ and that $\overline{f}(x_i^{\pm}) = x_i^{\mp}$ if $a_i < 0$. Let Δ_{n-1} be the triangulation of S^{n-1} defined inductively by:

Δ_0 is the triangulation of $S^0 = E^{n-1} \cap S^{n-1}$ with vertices $\{x_n^+, x_n^-\}$. Assuming defined Δ_{i-1} of $S^{i-1} = E^{n-i} \cap S^{n-1}$, with $1 \le i$, define the triangulation Δ_i of $S^i = E^{n-i-1} \cap S^{n-1}$ by $\Delta_i = \mathrm{susp}(\Delta_{i-1})$. The vertices of Δ_i are $\{x^+_{n-i}, x^-_{n-i}, \dots, x^+_n, x^-_n\}$. The k-simplexes, $1 \le k \le i$ are those of Δ_{i-1} plus the ones defined in the following way: for every sequence $n-i < j_1 < \dots < j_k \le n$ consider the $\binom{i}{k}2^{k+1}$ $(k+1)$-tuples obtained from $(x^+_{n-i}, x^-_{j_1}, \dots, x^-_{j_k})$ by changing the superscripts in all possible ways. Each of these tuples determines a k-simplex of Δ_i, for example $[x^+_{n-i}, x^-_{j_1}, \dots, x^-_{j_k}]$ is the set

$$\{x \in (E_{n-i} \oplus E_{j_1} \oplus \dots \oplus E_{j_k}) \cap S^{n-1} \mid x_{n-1} \ge 0,\, x_{j_1} \le 0, \dots, x_{j_k} \le 0\}$$

The triangulation Δ_{n-1} has the property that its restriction to S^i coincides with Δ_i, and it is a consequence of Lemma 3.4 that $\overline{f}\Delta_i = \overline{Df(0)}\Delta_i = \Delta_i$ for each $0 \le i \le n-1$. Notice that $\mathrm{star}(x^+_{n-i}, \Delta_i)$, the star of the vertex x_{n-i} in Δ_i, is the hemisphere of S^i which contains x^+_{n-i}, thus $S^i = \mathrm{star}(x^+_{n-i}, \Delta_i) \cup S^{i-1} \cup \mathrm{star}(x^-_{n-i}, \Delta_i)$.

Definition 3.2. *Let (f, X) be a pair where X is a smooth vector field of $\mathbb{R}^n$, $n \ge 2$, 0 is a singularity of type (s,u) with $s+u=n$, and $f \in \mathrm{Diff}^\infty_0(\mathbb{R}^n, X)$. Let $\{\alpha_1, \dots, \alpha_s, \beta_1, \dots, \beta_u\}$ be the eigenvalues of $DX(0)$, $\{a_1, \dots, a_s, b_1, \dots, b_u\}$ the corresponding eigenvalues of $Df(0)$. If*

1) $|a_k| / |a_j|^{\frac{\alpha_k}{\alpha_j}} \ne 1$ for all $1 \le j < k \le s$ and $|b_k| / |b_j|^{\frac{\beta_k}{\beta_j}} \ne 1$ for each $1 \le j < k \le u$,

2) $|b_j| / |a_i|^{\frac{\beta_j}{\alpha_i}} < \min_{1 \le k \le s,\, k \ne i}\{|a_k| / |a_i|^{\frac{\alpha_k}{\alpha_i}}\}$, for each $1 \le j \le u$ and $1 \le i \le s$, and

3) $|a_i| / |b_j|^{\frac{\alpha_i}{\beta_j}} < \min_{1 \le k \le u,\, k \ne j}\{|b_k| / |b_j|^{\frac{\beta_k}{\beta_j}}\}$ for each $1 \le i \le s$ and $1 \le j \le u$,

then (f, X) is called a hyperbolic pair at 0.

Let (f, X) be a hyperbolic pair at 0 with X of type (s,u). Fix a compact disk $\overline{\mathcal{D}} \subset \mathbb{R}^n$ such that $\mathrm{sing}(X) \cap \overline{\mathcal{D}} = \{0\}$. Consider pairs (g, Y), where Y is a smooth vector field on a neighborhood of $\overline{\mathcal{D}}$ and g is an element of $\mathrm{Diff}^\infty_0(\mathbb{R}^n, Y)$ whose domain of definition contains a closed disk $\overline{D^n_\rho} \subset \mathcal{D}$. Use $\overline{\mathcal{D}}$ to measure the C^1-proximity between Y and X and $\overline{D^n_\rho}$ the C^1-proximity between g and f. There exists $\xi > 0$ and $\zeta > 0$ such that $d_1(Y, X) < \xi$ and $d_1(g, f) < \zeta$ implies that Y has a unique singularity $y \in \overline{\mathcal{D}}$ with type (s,u) and the pair (g, Y) is hyperbolic at y. Since each of the properties in the definition of a hyperbolic pair are open, it follows that there exists $\delta > 0$ such that: if $d_1((g,Y), (f,X)) < \delta$, then (g, Y) is also a hyperbolic pair at 0 of the same type as (f, X).

Theorem 3.5. *Let (f, X) be pair where X is a linear vector field of $\mathbb{R}^n$, $n \geq 2$, non-resonant at 0 and of type $(n,0)$ or type $(0,n)$. The following statements are equivalent:*

1) (f, X) is hyperbolic at 0.

2) $\Omega(\overline{f}) = \{o(x_1^+), o(x_1^-), \ldots.., o(x_n^+), o(x_n^-)\}$ is the non-wandering set of $\overline{f}$, and for each $1 \leq i \leq n$ there exists a point $p \in \Omega(\overline{f})$ such that either $D\overline{f}(p)$ or $D\overline{f}^2(p)$ is $(i-1, n-i)$-hyperbolic.

3) The leaf map $\overline{f} : S^{n-1} \to S^{n-1}$ is a Morse-Smale diffeomorphism.

Proof. Assume that X is of type $(n,0)$ and that $\overline{f}$ satisfies *3)*, then every fixed and every periodic point of $\overline{f}$ must be hyperbolic. Since, by Lemma 3.4, every $o(x_i^{\pm})$ is fixed or periodic, one obtains from Lemma 3.2 that condition *1)* is satisfied.

The proof that *1)* implies *2)* is by induction on n. Assume that X is of type $(2,0)$ and decompose $\frac{\mathcal{U}_\bullet}{\mathcal{F}_X} = S^1$ as $V_1^- \cup \{o(x_2^+), o(x_2^-)\} \cup V_1^+$, where $V_1^- = \{o(x) \mid x_1 < 0\}$ and $V_1^+ = \{o(x) \mid x_1 > 0\}$. Let (c_1, V_1) be a canonical chart and assume $a_1 > 0$. It follows from theorem 3.3 that $c_1 \circ \overline{f} \circ c_1^{-1} : \mathbb{R} \to \mathbb{R}$ is a linear map and therefore $o(x_1^+)$ and $o(x_1^-)$ are fixed points of $\overline{f}$. From lemma 3.2 and the inequality $|a_2| \neq |a_1|^{\frac{\alpha_2}{\alpha_1}}$ one obtains that $c_1 \circ \overline{f} \circ c_1^{-1}(x_2) = a_2/a_1^{\frac{\alpha_2}{\alpha_1}} \cdot x_2$ is either a contraction or an expansion, which in turn implies that $o(x_1^+)$ and $o(x_1^-)$ are the only non-wandering points of $\overline{f}$ in $V_1^- \cup V_1^+$. We know that $o(x_2^+)$ and $o(x_2^-)$ are fixed points of $\overline{f}$ if $a_2 > 0$ and periodic if $a_2 < 0$. Thus

$$\Omega(\overline{f}) = \{o(x_1^+), o(x_1^-), o(x_2^+), o(x_2^-)\}.$$

Again, from lemma 3.2 $D\overline{f}(o(x_2^{\pm})) = a_1/a_2^{\frac{\alpha_1}{\alpha_2}}$ and it is clear that $a_2/a_1^{\frac{\alpha_2}{\alpha_1}} < 0 (> 0) \Leftrightarrow a_1/a_2^{\frac{\alpha_1}{\alpha_2}} > 0 (< 0)$ i.e., $D\overline{f}(o(x_1^{\pm}))$ is $(1,0)$-hyperbolic if and only if $D\overline{f}(o(x_2^{\pm}))$ is $(0,1)$-hyperbolic and vice versa. The cases $a_1 < 0$ and $a_2 > 0$ and $a_1 < 0$ and $a_2 < 0$ are analogous.

Next assume that X is of type $(n,0)$ with $n > 2$ and decompose $\frac{\mathcal{U}_\bullet}{\mathcal{F}_X} = S^{n-1}$ as $V_1^- \cup S^{n-2} \cup V_1^+$. We consider the case $a_i > 0$ for $1 \leq i \leq n$ in which every point in

$$\{o(x_1^+), o(x_1^-), \ldots.., o(x_n^+), o(x_n^-)\}$$

is a fixed point of $\overline{f}$. One knows from lemma 3.4 that $f(E^1 \cap \mathcal{U}) \subset E^1$. Denote by (f_{E^1}, X_{E^1}) the restriction of the pair (f, X) to E^1, and notice that X_{E^1} is linear, non-resonant at 0, of type $(n-1, 0)$ with eigenvalues $\{\alpha_2, \ldots, \alpha_n\}$, and $f \in \mathrm{Diff}_0^r(D^{n-1}, X_{E^1})$ with $r > \left|\frac{\alpha_n}{\alpha_1}\right| >$

$\left|\frac{\alpha_n}{\alpha_2}\right|$. Notice also that the eigenvalues of $Df_{E^1}(0)$ are $\{a_2, ..., a_n\}$ and of course $|a_k| / |a_j|^{\frac{\alpha_k}{\alpha_j}} \neq 1$ for all $2 \leq j < k \leq n$. By the induction hypothesis

$$\Omega(\overline{f}_{E^1}) = \{\mathfrak{o}(x_2^+), \mathfrak{o}(x_2^-), ..., \mathfrak{o}(x_n^+), \mathfrak{o}(x_n^-)\}$$

and one can suppose, permuting the subscripts if necessary, that $D\overline{f_{E^1}}(x_i^{\pm})$ is $(i-2, n-i)$-hyperbolic for $2 \leq i \leq n$. The decomposition $S^{n-1} = V_1^- \cup S^{n-2} \cup V_1^+$ is invariant under $\overline{f}$. Let (c_1, V_1) be a canonical chart. By theorem 3.3 $\mathrm{c}_1 \circ \overline{f} \circ \mathrm{c}_1^{-1} : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ is linear and from condition *1)* one obtains that it is hyperbolic. Therefore $\mathfrak{o}(x_1^-)$ $(\mathfrak{o}(x_1^+))$ is the only non-wandering point of $\overline{f}$ in V_1^- (V_1^+). This implies that $\Omega(\overline{f}) = \{\mathfrak{o}(x_1^+), \mathfrak{o}(x_1^-), ..., \mathfrak{o}(x_n^+), \mathfrak{o}(x_n^-)\}$.

In order to prove the second part of *2)*, we construct a $n \times n$-matrix $B = [b_{ij}]$ which contains the information on $D\overline{f}$ at all points of $\Omega(\overline{f})$. Define

$$b_{ij} = a_j / a_i^{\frac{\alpha_j}{\alpha_i}} \text{ if } i \neq j \text{ and } b_{ii} = 0$$

Since we assumed that each $a_i > 0$, then $b_{ij} > 0$ if $i \neq j$. It follows from the hypothesis that each $b_{ij} \neq 1$ and also that $b_{ij} < 1 \, (> 1)$ if and only if $b_{ji} > 1 (< 1)$. Recall that $D\overline{f}(\mathfrak{o}(x_i^{\pm}))\frac{\partial}{\partial x_j} = b_{ij}\frac{\partial}{\partial x_j}$, which says that $\frac{\partial}{\partial x_j}$ is tangent to the stable (unstable) manifold of $\overline{f}$ at $\mathfrak{o}(x_i^{\pm})$ if and only if $b_{ij} < 1$ (> 1). Call C the matrix obtained from B by deleting the first line and the first column. The assumption that $D\overline{f_{E^1}}(\mathfrak{o}(x_i^{\pm}))$ is $(i-2, n-i)$-hyperbolic translates into: every element of C below the main diagonal is less than 1 and above is greater than 1. A simple manipulation with inequalities gives the following sequence of implications:

$$a_{n-1}/a_n^{\frac{\alpha_{n-1}}{\alpha_n}} < 1 \Rightarrow a_{n-2}/a_n^{\frac{\alpha_{n-2}}{\alpha_n}} < a_{n-2}/a_{n-1}^{\frac{\alpha_{n-2}}{\alpha_{n-1}}} < 1 \Rightarrow ...$$

$$... \Rightarrow a_2/a_n^{\frac{\alpha_2}{\alpha_n}} < a_2/a_{n-1}^{\frac{\alpha_2}{\alpha_{n-1}}} < .... < a_2/a_3^{\frac{\alpha_2}{\alpha_3}} < 1 \Rightarrow$$

$$a_1/a_n^{\frac{\alpha_1}{\alpha_n}} < a_1/a_{n-1}^{\frac{\alpha_1}{\alpha_{n-1}}} < .... < a_1/a_2^{\frac{\alpha_1}{\alpha_2}} \tag{3.4}$$

Notice that the terms which appear in 3.4 are precisely those of B^1, the first column of B. Now we shall show that, after a permutation of subscripts, for each $1 \leq i \leq n$ the linear transformation $D\overline{f}(\mathfrak{o}(x_i^{\pm}))$ is $(i-1, n-i)$-hyperbolic. This amounts to say that every element of B below the diagonal is less than 1. There are two cases:

If for every $2 \leq j \leq n$ the term $b_{j1} > 1$, then use the permutation

$$\begin{pmatrix} 1 & 2 & . & . & . & (n-1) & n \\ n & 1 & . & . & . & (n-2) & (n-1) \end{pmatrix}$$

If not, there is a minimum $2 \leq j_0 \leq n$ such that $b_{j_0 1} < 1$ and it follows from 3.4 that also $b_{k1} < 1$ if $j_0 \leq k \leq n$. In this case, use the permutation

$$\begin{pmatrix} 1 & 2 & . & . & . & (j_0-1) & j_0 & . & . & . & n \\ (j_0-1) & 1 & . & . & . & (j_0-2) & j_0 & . & . & . & n \end{pmatrix}$$

To prove that 2) implies 3) one has to show that for any two points $p, q \in \Omega(\overline{f})$ the unstable manifold $W^u(p)$ (stable manifold $W^s(p)$) is transversal to the stable manifold $W^s(q)$ (unstable manifold $W^u(q)$). We assume, after a permutation of subscripts, that for each $1 \leq i \leq n$ the linear map $D\overline{f}(\mathfrak{o}(x_i^{\pm}))$ is $(i-1, n-i)$-hyperbolic. This means that $\dim W^u(\mathfrak{o}(x_i^{\pm})) = n-i$ and that $\dim W^s(\mathfrak{o}(x_i^{\pm})) = i-1$. With the identification of both $\frac{U_\bullet}{\mathcal{F}_X}$ and $\frac{D^n_\bullet}{\mathcal{F}_X}$ with S^{n-1}, we have in fact that $W^u(x_i^{\pm})$ =star$(x_i^{\pm}, \Delta_i)$ and that $W^s(x_i^{\pm})$ is equal to $W^u(x_i^{\pm}, \overline{f}^{-1})$, the unstable manifold of $x_i^{\pm}$ with respect to $\overline{f}^{-1}$. The transversality is now clear. The other cases are similar. Notice that we assumed $n \geq 2$. For $n = 1$, if (f, X) is a pair such that X is a vector field of $\mathbb{R}$ that has an isolated singularity at 0, and f is any local diffeomorphism of $\mathbb{R}$ fixing 0, then $\overline{f} : S^0 \to S^0$ is C^0 structurally stable ∎

In order to prove an analogous result when X is a linear vector field of type(s, u) with $0 < s < n$, we shall make use of the compact non-Hausdorff manifolds $\mathbf{K}_\varepsilon = \mathbf{K}_\varepsilon(X)$ defined at the beginning of this section. Denote by $\mathcal{E}^r(\mathbf{K}_\varepsilon)$ the space of C^r-embeddings of $\mathbf{K}_\varepsilon$ into $S^{n-1}_{(s,u)}$ that leave invariant S^{s-1} and S^{u-1}. We consider in $\mathcal{E}^r(\mathbf{K}_\varepsilon)$ the C^1-topology defined through a finite collection of charts, see [12] section 2.

Definition 3.3. *An element $\overline{f} \in \mathcal{E}^r(\mathcal{K}_\varepsilon)$ such that $\overline{f}(\mathbf{K}_\varepsilon) \subset int(\mathbf{K}_\varepsilon)$ is said to be C^1-structurally stable if there exists a neighborhood N of $\overline{f}$ in the C^1-topology such that for each $\overline{g} \in N$ there is a homeomorphism $h : \mathbf{K}_\varepsilon \to \mathbf{K}_\varepsilon$ that satisfies $\overline{g} = h \circ \overline{f} \circ h^{-1}$.*

Theorem 3.6. *Let (f, X) be a pair where X is a linear vector field of $\mathbb{R}^n, n \geq 2$, non-resonant at 0 and of type(s, u), $0 < s < n$, and $f \in Diff_0^\infty(\mathbb{R}^n, X)$. If (f, X) is hyperbolic, then for ε small enough the leaf map $\overline{f}$, restricted to $\mathbf{K}_\varepsilon$, is C^1 structurally stable.*

Proof. Given $f \in \mathrm{Diff}_0^r(D^n, X)$ we first choose $\delta > 0$ so small that $\overline{D^n_{2\delta}}$ be contained in the domain $\mathcal{U}$ of f. If $G : V \supset \overline{D^n_{2\delta}} \to \mathbb{R}^n$ is a C^r-map, $r \geq 1$, put

$$\|G\|_1 = \sup_{x \in \overline{D^n_{2\delta}}} \{\|G(x)\|, \|DG(x)\|\}$$

Therefore, if $g \in \mathrm{Diff}_0^r(D^n, X)$ is defined in a neighborhood of $\overline{D^n_{2\delta}}$, then $\|g - f\|_1$ measures the C^1-proximity between g and f. Next, put $S^{s-1} = \partial \overline{D^s_\delta} \subset E$, $S^{u-1} = \partial \overline{D^u_\delta} \subset F$, and choose $\varepsilon < \delta$ so small that $S^{s-1} \times \overline{D^u_\varepsilon} \subset \overline{D^n_{2\delta}}$ and that every solution of X with initial condition $(x, y) \in S^{s-1} \times (\overline{D^u_\varepsilon})_\bullet$ intersects $E \times S^{u-1}$ before leaving $\overline{D^n_{2\delta}}$. Then, g induces an embedding $\overline{g} : \mathbf{K}_\varepsilon \to \mathbf{S}^{n-1}_{(s,u)}$ which leaves invariant S^{s-1} and S^{u-1}, and the map $g \to \overline{g}$ is continuous at f.

Notice that the vector field X_E (X_F) is linear, non-resonant at 0, of type $(s, 0)$ (type $(0, u)$), with eigenvalues $\{\alpha_1, ..., \alpha_s\}$ $(\{\beta_1, ..., \beta_u\})$ and $f_E \in \mathrm{Diff}_0^\infty(D^s, X|_E)$ $(f_F \in \mathrm{Diff}_0^\infty(D^u, X|_F))$, i.e., the pair (f_E, X_E) $((f_F, X_F))$ satisfies the hypothesis of Lemma 3.4. Furthermore, since the pair (f, X) is hyperbolic, so is the pair (f_E, X_E) $((f_F, X_F))$. Thus, by Theorem 3.5, $\overline{f_E} = \overline{Df(0)_E} : S^{s-1} \to S^{s-1}$ $(\overline{f_F} = \overline{Df(0)_F} : S^{u-1} \to S^{u-1})$ is a Morse-Smale diffeomorphism and in particular it is C^1 structurally stable.

It follows from part 2) (part 3)) of Definition 3.2 that $\overline{f}$ and $\overline{Df(0)}$ are 1-normally attracting at S^{s-1} (S^{u-1}). The reader may consult section 14 of [12] for the definition of r-normally attracting, stable manifolds and its persistence properties.

Since $\overline{f}$ is 1-normally attracting, there exists $\varepsilon > 0$ such that $\overline{f}(\mathbf{K}_\varepsilon) \subset \mathrm{int}(\mathbf{K}_\varepsilon)$, and C^0-fibrations

$$\pi^s_f : \mathbf{F}^s_\varepsilon \to S^{s-1} \qquad\qquad \pi^u_f : \mathbf{F}^u_\varepsilon \to S^{u-1}$$

where the stable manifolds $\mathcal{V}^f_x = (\pi^s_f)^{-1}(x)$ and $\mathcal{V}^f_y = (\pi^u_f)^{-1}(y)$ are C^1–submanifolds diffeomorphic to $\overline{D^u}$ and $\overline{D^s}$, respectively. These two fibrations are $\overline{f}$-invariant, i.e., $\overline{f}\mathcal{V}^f_x \subset \mathcal{V}^f_{\overline{f}x}$ and $\overline{f}\mathcal{V}^f_y \subset \mathcal{V}^f_{\overline{f}y}$. There are the corresponding statements for $\overline{Df(0)}$. If we denote the invariant stable submanifolds for $\overline{Df(0)}$ by $\mathcal{V}_x$ and $\mathcal{V}_y$, then, using elementary computations, we obtain that $\mathcal{V}_x = \{x\} \times \overline{D^u_\varepsilon}$ and that $\mathcal{V}_y = \psi(B_y) \cup S^{u-1}$, where

$$B_y = \{(x, e^{\beta_1(t_0 - \tau)}y_1, ..., e^{\beta_u(t_0 - \tau)}y_u) \mid |x| = \delta, \tau \geq 0, |X^{t_0}y| = \varepsilon\}$$

The submanifolds $\mathcal{V}_x$ and $\mathcal{V}_y$ intersect transversely along the curve $\gamma_{(x,y)}$ parametrized by:

$$\gamma_{(x,y)}(\tau) = (x, e^{\beta_1(t_0-\tau)}y_1, ..., e^{\beta_u(t_0-\tau)}y_u)\,,\ 0 \le \tau\,,\ \text{if viewed in } \mathbf{F}^s_\varepsilon$$
$$\gamma_{(x,y)}(\tau) = (e^{\alpha_1(\tau-t_0)}x_1, ..., e^{\alpha_s(\tau-t_0)}x_s\,, y)\,,\ 0 \le \tau\,,\ \text{if viewed in } \mathbf{F}^u_\varepsilon$$

Notice that $\gamma_{(x,y)}$ is transversal to $\{x\} \times S_t^{u-1}$ for each $0 < t \le \varepsilon$ and thus, its closure contains the point $(x,0)$. By a similar argument, it also contains $(0,y)$. The one dimensional foliation Γ defined by the curves $\{\gamma_{(x,y)}\,,\ (x,y) \in S^{s-1} \times S^{u-1}\}$ is $\overline{Df(0)}$-invariant. Taking ε smaller we may assume that $\mathcal{V}_x^f$ $(\mathcal{V}_y^f)$ is as C^1 close to $\mathcal{V}_x$ $(\mathcal{V}_y)$ as necessary and can write $\mathcal{V}_y^f = \psi(B_y^f) \cup S^{u-1}$, where B_y^f is an open submanifold of $\mathbf{K}^s_\varepsilon$ which is C^1 close to B_y. Therefore $\mathcal{V}_x^f$ and $\mathcal{V}_y^f$ intersect transversely along a curve $\gamma^f_{(x,y)}$ whose closure contains the points $(x,0)$ and $(0,y)$. We denote this curve by $\gamma_{(x,y)^f}$, where $(x,y)^f$ stands for the unique point of the curve which belongs to $\partial(\mathbf{F}^s_\varepsilon) = S_\delta^{s-1} \times S_\varepsilon^{u-1}$.

Fix ε. If $\overline{g} \in \mathcal{E}^r(\mathbf{K}_\varepsilon)$ is close enough to $\overline{f}$ for the C^1 topology, then $\overline{g}$ is also 1-normally attracting. Thus $\overline{g}(K_\varepsilon) \subset \mathring{\mathbf{K}}_\varepsilon$ and $\overline{g}_{S^{s-1}}$ $(\overline{g}_{S^{u-1}})$ is topologically equivalent to $\overline{f}_{S^{s-1}}$ $(\overline{f}_{S^{u-1}})$, i.e., there exists a homeomorphism $h_s : S^{s-1} \to S^{s-1}$ $(h_u : S^{u-1} \to S^{u-1})$ such that $\overline{g} = h_s \circ \overline{f} \circ h_s^{-1}$ $(\overline{g} = h_u \circ \overline{f} \circ h_u^{-1})$. Furthermore, by the persistence theorem, the stable manifolds $\mathcal{V}_x^g$ and $\mathcal{V}_y^g$ for $\overline{g}$ are C^1 close to the corresponding manifolds for $\overline{f}$, and thus they intersect transversely along a curve $\gamma_{(x,y)^g}$ whose closure contains $(x,0)$ and $(0,y)$. Now, we are going to construct a topological equivalence $h : \mathbf{K}_\varepsilon \to \mathbf{K}_\varepsilon$ between $\overline{f}$ and $\overline{g}$ whose restriction to S^{s-1} (S^{u-1}) is h_s (h_u). Since $\overline{f}$ and $\overline{g}$ are 1-normally attracting on S^{s-1}, then $\overline{f}^{i+1}\mathbf{F}^s_\varepsilon \subset \mathrm{int}(\overline{f}^i\mathbf{F}^s_\varepsilon)$ $(\overline{g}^{i+1}\mathbf{F}^s_\varepsilon \subset \mathrm{int}(\overline{g}^i\mathbf{F}^s_\varepsilon)$, for each $i \ge 0$. Put $A_i^f = \overline{f}^i(\mathbf{F}^s_\varepsilon) \setminus \overline{f}^{i+1}\mathbf{F}^s_\varepsilon$, then $\mathbf{F}^s_\varepsilon \setminus S^{s-1} = \cup_{i=1}^\infty A_i^f$ and $A_i^f \cap A_j^f = \emptyset$ if $i \ne j$. There are also sets A_i^g satisfying the corresponding properties.

First, define $h : \partial\mathbf{F}^s_\varepsilon \cup \overline{f}(\partial\mathbf{F}^s_\varepsilon) \to \partial\mathbf{F}^s_\varepsilon \cup \overline{g}(\partial\mathbf{F}^s_\varepsilon)$ by $h \circ \overline{f}^i(x,y)^f = \overline{g}^i(h_s x, h_u y)^g$, $i = 0,1$. Next, extend it to a homeomorphism $h : A_0^f \to A_0^g$ sending $\gamma_{(x,y)^f} \cap \mathcal{V}_x^f$ onto $\gamma_{(h_s x, h_u y)^g} \cap \mathcal{V}^g_{h_s x}$. Finally, extend it to $h : \mathbf{F}^s_\varepsilon \to \mathbf{F}^s_\varepsilon$ by the following rule, if $p \in \mathbf{F}^s_\varepsilon \setminus S^{s-1}$, then there exists a unique $i \ge 0$ such that $p \in A_i^f$. Define $h(p) = \overline{g}^i \circ h \circ \overline{f}^{-i}(p)$. The map h has the required properties ∎

4. LOCAL STRUCTURAL STABILITY

Let $\Lambda^q(M)$, $0 \le q \le m$, denote the set of smooth q-forms on M. If $\omega \in \Lambda^q(M)$ and $x \in \mathrm{reg}(\omega) = M \setminus \mathrm{sing}(\omega)$, then $\ker \omega_x = \{v \in T_x M \mid$

$i_v\omega_x = 0\}$ is a codimension q linear subspace of T_xM. A q-form ω is said to be *integrable* if the distribution defined by $\ker\omega$ is involutive, or equivalently, if for every $x \in M\backslash\mathrm{sing}(\omega)$ there exists a neighborhood U of x and q 1-forms $\omega_1, ..., \omega_q$ on U such that:

1) $\omega\,|_U = \omega_1 \wedge .... \wedge \omega_q$
2) $\omega_1 \wedge .... \wedge \omega_q \wedge d\omega_j = 0$ for each $j = 1, ..., q$

Denote by $\mathfrak{I}^q(M)$ (resp. $\mathfrak{I}^q(0)$) the set of smooth integrable q-forms defined on M (resp. in a neighborhood of $0 \in \mathbb{R}^m$). By a theorem of Frobenius, an element $\omega \in \mathfrak{I}^q(M)$ defines a cod. q regular foliation of $M\backslash\mathrm{sing}(\omega)$. Notice that a 0-form, i.e., a smooth function $f : M \to \mathbb{R}$, is always integrable, and the leaves of the regular foliation defined by f are the connected components of $M \setminus f^{-1}(0)$.

Lemma 4.1. *Let $\omega \in \mathfrak{I}^q(M)$, $0 \leq q \leq m-1$. Then*

1) *$d\omega \in \mathfrak{I}^{q+1}(M)$.*
2) *If $\omega_x \neq 0$ and $d\omega_x \neq 0$, then the leaf of $d\omega$ through x is contained in the leaf of ω through x.*
3) *If $\omega_x = 0$ and $d\omega_x \neq 0$, then the leaf of $d\omega$ through x is contained in sing(ω).*

Proof. Since the integrability of a form is a local property, one can assume that $\omega \in \mathfrak{I}^q(0)$ and that $d\omega_0 \neq 0$. Write $d\omega$ in coordinates

$$d\omega = \textstyle\sum_{1\leq i_1 < .... < i_{(q+1)} \leq m} a_{i_1 ... i_{q+1}}\, dx_{i_1} \wedge .... \wedge dx_{i_{(q+1)}}$$

and assume that $a_{1...(q+1)}(0) \neq 0$. Define the 1-forms

$$\xi_j = i_{\partial/\partial x_{q+1}} \circ ... \circ \widehat{i_{\partial/\partial x_j}} \circ ... \circ i_{\partial/\partial x_1}\, d\omega \qquad 1 \leq j \leq q+1$$

It is laborious but not difficult to verify that

$$\xi_1 \wedge .... \wedge \xi_{q+1} = (-1)^{q+1} a^q_{1....(q+1)}\, d\omega$$

Put $\omega_1 = a^{-q}_{1....(q+1)}\,\xi_1$, $\omega_2 = \xi_2$, ..., $\omega_{(q+1)} = \xi_{(q+1)}$, then

$$d\omega = \omega_1 \wedge .... \wedge \omega_{(q+1)}$$

It follows from $d^2\omega = 0$ that $\omega_1 \wedge .... \wedge \omega_{(q+1)} \wedge d\omega_j = 0$ for every $1 \leq j \leq (q+1)$. Therefore $d\omega$ is integrable and 1) is proved.

To prove 2) and 3), we first prove that: if $d\omega_x \neq 0$ and $i_v\, d\omega_x = 0$, with $v \in T_xM$, then $i_v\omega_x = 0$. In fact, one can assume that $\omega_x \neq 0$ and thus locally $\omega = \omega_1 \wedge .... \wedge \omega_q$ with $\omega_1 \wedge .... \wedge \omega_q \wedge d\omega_j = 0$ for every $1 \leq j \leq q$. Now, $\omega_j \wedge d\omega = 0$ and thus

$$0 = i_v(\omega_j \wedge d\omega) = i_v\omega_j \wedge d\omega_x - \omega_j \wedge i_v\, d\omega = i_v\omega_j \wedge d\omega_x$$

But, since $d\omega_x \neq 0$, it follows that $\omega_j(v) = 0$, $1 \leq j \leq q$, which implies that $i_v\omega_x = 0$ and proves 2). Finally, let X be a smooth vector field tangent to the leaves of $d\omega$, i.e., $i_X\, d\omega = 0$ at every $x \in \mathrm{reg}(d\omega)$. Then, $i_X\,\omega = 0$ at these points too, and therefore $L_X\,\omega = 0$, which in turn implies 3) ∎

Decompose $\mathrm{sing}(\omega)$ as the union of an open subset $\mathrm{sing}_1(\omega) = \{x \mid d\omega_x \neq 0\}$ and a closed subset $\mathrm{sing}_0(\omega) = \{x \mid d\omega_x = 0\}$. Part 3) of lemma 4.1 says that $\mathrm{sing}_1(\omega)$ is saturated by the leaves of $d\omega$.

Definition 4.1. *If $\omega \in \mathfrak{I}^q(M)$, we let $\mathcal{F}_\omega$ denote the partition of M by the leaves of the regular foliation defined by ω, plus the leaves of the regular foliation defined by $d\omega$ which are contained in $sing_1(\omega)$, plus the points in $sing_0(\omega)$.*

It can be readily proved that:

Proposition 4.2. *If ω is an element of $\mathfrak{I}^q(M)$, $0 \leq q \leq m-1$, then $\mathcal{F}_\omega$ is a smooth cod$q$ foliation of M.*

A form $\omega \in \mathfrak{I}^q(M)$ may define more than one foliation. If we write

$$M = \mathrm{reg}(\omega) \cup \mathrm{sing}_1(\omega) \cup \mathrm{sing}_0(\omega)$$

then the leaves of $\mathcal{F}_\omega$ in $\mathrm{reg}(\omega)$ are $(m-q)$-dimensional, those in $\mathrm{sing}_1(\omega)$ are $(m-q-1)$-dimensional and those in $\mathrm{sing}_0(\omega)$ are 0-dimensional. Sometimes it is possible to decompose $\mathrm{sing}_0(\omega)$ as a union of submanifolds, other than points, in such a way that the new decomposition $\mathcal{F}'_\omega$ is still a foliation. For example: consider the 1-form $\xi = y\,dx + x\,dy$ on $\mathbb{R}^2$ and the projection $\pi : \mathbb{R}^2 \times S^1 \to \mathbb{R}^2$. The form $\omega = \pi^*\xi$ is closed and $\mathrm{sing}(\omega) = \mathrm{sing}_0(\omega) = \{0\} \times S^1$. We obtain two different foliations of $\mathbb{R}^2 \times S^1$ out of ω. The foliation $\mathcal{F}'_\omega$ whose leaves are $L \times S^1$, for every leaf L of ξ, and the foliation $\mathcal{F}_\omega$ defined above. In $\mathcal{F}_\omega$ each point of $\{0\} \times S^1$ is a leaf while in $\mathcal{F}'_\omega$ the whole set $\{0\} \times S^1$ is a leaf. Next, we recall some familiar definitions:

Let $\omega_i \in \mathfrak{I}^q(M_i)$, $i = 0, 1$, and $\mathcal{F}_i = \mathcal{F}_{\omega_i}$ be the corresponding cod. q foliation of the m-manifold M_i. The pair $(M_0, \mathcal{F}_0)$ is said to be *topologically equivalent* to $(M_1, \mathcal{F}_1)$ if there exists a homeomorphism $g : M_0 \to M_1$ which takes leaves of $\mathcal{F}_0$ onto leaves of $\mathcal{F}_1$. This implies that necessarily $g(\mathrm{sing}(\mathcal{F}_0)) = \mathrm{sing}(\mathcal{F}_1)$.

Definition 4.2. *Let $\omega \in \mathfrak{I}^q(M)$ and L a compact leaf of $\mathcal{F}_\omega$. Fix a finite covering $\{V_i\}_{i=1}^k$ of L by open sets of M such that each $\bar{V}_i$ be contained in the domain of a chart $(\mathbf{x}_i, U_i)$ at L. For $\eta \in \mathfrak{I}^q(M)$ denote by $\eta^i = (\eta^i_1, \ldots, \eta^i_s)$, $s = \binom{n}{q}$, the coordinates of η in the chart $(\mathbf{x}_i, U_i)$ and define*

$$\|\eta\|_r = \max_i \sup_{x\in \mathbf{x}_i(\bar{V}_i)}\{\|\eta^i(x)\|, \|d\eta^i(x)\|, ..., \|d^r\eta^i(x)\|\}$$
$$\|\eta\|_{(r,r)} = \max\{\|\eta\|_r, \|d\eta\|_r\}$$

The positive numbers $d_r(\eta,\omega) = \|\eta-\omega\|_r$ *and* $d_{(r,r)}(\eta,\omega) = \|\eta-\omega\|_{r,r}$ *are called the* C^r distance between η and ω *and the* $C^{(r,r)}$ distance between η and ω, *respectively. It is possible for* η *to be* $C^{(r-1,r-1)}$*-close to* ω *without been* C^r*-close.*

A compact leaf L of $\mathcal{F}_\omega$ is said to be $C^{(r,r)}$ *locally structurally stable* if for any neighborhood W of L, there exists $\delta > 0$ such that if $\eta \in \mathfrak{I}^q(M)$ satisfies $\|\eta-\omega\|_{(r,r)} < \delta$, then

1) $\mathcal{F}_\eta$ has a compact leaf $L_\eta \subset W$, and

2) There exist neighborhoods U of L and V of L_η and a topological equivalence $g: V \to U$ between the pairs $(V, \mathcal{F}_\eta \cap V)$, $(U, \mathcal{F}_\omega \cap U)$ such that $g(L_\eta) = L$.

From now on we shall assume that $\omega \in \mathfrak{I}^{m-2}(M)$. Then $\mathcal{F}_\omega$ is a 2-foliation and all singular one dimensional leaves are contained in $\text{sing}_1(\omega)$. We shall now translate Lemma 2.1 into the language of forms giving what was called the Fundamental Lemma in [8].

Lemma (2.1)$'$. *Let* L *be a 1-dimensional leaf of* $\mathcal{F}_\omega$ *and* $p \in L$. *There exists a smooth* $\omega^{\mathbf{x}} \in \mathfrak{I}^{m-2}(\mathbf{C}^{m-1})$

$$\omega^{\mathbf{x}} = \textstyle\sum_{j=2}^{m} (-1)^j a_j(x_2, ..., x_m)\, dx_2 \wedge ... \wedge \widehat{dx_j} \wedge ... \wedge dx_m$$

with $\xi(0) = 0$, *a neighborhood* U *of* p *in* M *and a smooth diffeomorphism* $\mathbf{x}: U \to \mathbf{C}^m$, *with* $\mathbf{x}(x) = 0$, *such that* $\omega\,|_U = \mathbf{x}^*\omega^{\mathbf{x}}$. *Furthermore,* $\mathbf{x}$ *takes leaves of* $\mathcal{F}_{d\omega} \cap U$ *onto segments* $\{(t, x_2, ..., x_m) \mid -1 < t < 1\}$.

Proof. Since $d\omega_p \neq 0$, it follows that $d\omega \neq 0$ in a neighborhood W of p. Choose any vector field X tangent to $\ker(d\omega)$ with $X_p \neq 0$ and apply Lemma 2.1 to $\mathcal{F} = \mathcal{F}_\omega$, and X ∎

The map $\mathbf{x} : U \to \mathbf{C}^m$ is called a *chart* of $\mathcal{F}_\omega$ *at* $p \in L$ *along* $\mathcal{F}_{d\omega}$. One can obtain cross sections to $\mathcal{F}_{d\omega}$ at p by restricting the map $\sigma : \mathbf{C}^{m-1} \to \Sigma \subset M$, $\sigma = \mathbf{x}^{-1}\,|_{\mathbf{C}^{m-1}}$, to disks D_ρ^{m-1}, $0 < \rho < 1$. The smooth vector field

$$k(\omega^{\mathbf{x}}) = \sum_{j=2}^{m} a_j(x_2, ..., x_m)\, \partial/\partial x_j \qquad (4.1)$$

generates $\ker(\omega^{\mathbf{x}})$ and the orbits of $\sigma_* k(\omega^{\mathbf{x}})$ on Σ are the leaves of $\mathcal{F}_\omega \cap \Sigma$. For $\eta \in \mathfrak{I}^{m-2}(M)$ we let $\eta^{\mathbf{x}}$ denote the form on $\mathbf{C}^m$ such that $\eta\,|_U = \mathbf{x}^* \eta^{\mathbf{x}}$, and by $k(\eta^{\mathbf{x}})$ the vector field on $\mathbf{C}^m$ defined by

$$\mathrm{i}_{k(\eta^{\mathbf{x}})}\, dx_2 \wedge \dots \wedge dx_m = j_{x_1}^*\, \eta^{\mathbf{x}}$$

where $j_{x_1} : \{(x_1, 0, 0)\} \times \mathbf{C}^{m-1} \to \mathbf{C}^m$ is the natural injection. For $m = 3$ one obtains the following expressions:

$$\begin{aligned} \omega^{\mathbf{x}} &= -a_3(x_2, x_3)\, dx_2 + a_2(x_2, x_3)\, dx_3 \\ k(\omega^{\mathbf{x}}) &= a_2(x_2, x_3)\, dx_2 + a_3(x_2, x_3)\, dx_3 \\ \eta^{\mathbf{x}} &= b_1(x)\, dx_1 - b_3(x)\, dx_2 + b_2(x)\, dx_3 \\ k(\eta^{\mathbf{x}}) &= b_2(x)\, \partial/\partial x_2 + b_3(x)\, \partial/\partial x_3 \end{aligned}$$

When there is no danger of misunderstanding, we shall also denote by $k(\eta^{\mathbf{x}})$ the restriction of $k(\eta^{\mathbf{x}})$ to $\mathbf{C}^{m-1}$. Let $\omega \in \mathfrak{I}^{m-2}(M)$ and L a 1-dimensional leaf of $\mathcal{F}_\omega$. The leaf L is called *normally hyperbolic* if there is a point $p \in L$ and a chart $(\mathbf{x}, U)$ of $\mathcal{F}_\omega$ at $p \in L$ along $\mathcal{F}_{d\omega}$ such that $\mathbf{x}(p) = 0$ is a hyperbolic singularity of $k(\omega^{\mathbf{x}})$. This definition does not depend on the chart. The independence on the point $p \in L$ follows from the fact that $L_X\, \omega = 0$ if X is tangent to $\mathcal{F}_{d\omega}$.

From now on we suppose that L is a circle leaf of $\mathcal{F}_\omega$ and thus also of $\mathcal{F}_{d\omega}$. Fix a Riemannian metric $\langle\, ,\rangle$ on M and a normal tubular neighborhood $\pi_0 : T_0(L) \to L$ such that each fiber $\pi_0^{-1}(x)$ be transversal to $\mathcal{F}_{d\omega}$. Denote by $\langle\, ,\rangle_x$ the Riemannian metric induced on $\pi_0^{-1}(x)$. If $\tilde{u}, u \in \pi_0^{-1}(x)$, the *fiber distance* $d_f(\tilde{u}, u)$ is by definition their distance as points of $\pi_0^{-1}(x)$. For any sufficiently small $\gamma > 0$ the subset

$$T_\gamma(L) = \{u \in T_0(L) \mid d_f(u, \pi_0(u)) < \gamma\}$$

is called a *γ-tubular neighborhood* of L. Let $\tilde{L}$ be another circle embedded in $T_0(L)$ which intersects each fiber $\pi_0^{-1}(x)$ in a unique point $\tilde{x}$. Define the C^0-*distance* between $\tilde{L}$ and L by

$$d_0(\tilde{L}, L) = \sup_{x \in L}\{\, d_f(\tilde{x}, x)\}$$

Choices. In order to facilitate the proof of the main results, we are going to make some choices and construct some vector fields. First, we choose the finite covering $\{V_i\}_{i=1}^k$ of L, mentioned in definition 4.2, so that $\overline{V}_i \subset U_i \subset T_0(L)$, where $\mathbf{x}_i : U_i \to \mathbf{C}^m$ is a chart along $\mathcal{F}_{d\omega}$ at some point $y_i \in L$. Thus, each $\mathbf{x}_i$ takes leaves of $\mathcal{F}_{d\omega} \cap U_i$ onto orbits of $\frac{\partial}{\partial x_1}\,|_{\mathbf{C}^m}$ and in particular $L \cap U_i$ onto $(-1, 1) \times \{0\}$. Therefore, $\omega\,|_{U_i} = \mathbf{x}_i^*(\omega^{\mathbf{x}_i})$ with

$$\omega^{\mathbf{x}_i} = \sum_{j=2}^{m}(-1)^j\, a_j^i(x_2,...,x_m)\, dx_2 \wedge ... \wedge \widehat{dx_j} \wedge ... \wedge dx_m$$

We may assume, and shall do it from now on, that $\mathbf{x}_i^{-1}$ takes the fibers of the first projection $P : \mathbf{C}^m \to (-1,1)$ into the fibers of π_0. We choose γ so small that $T_\gamma(L) \subset \cup_{i=1}^k V_i$, and write simply $T(L)$ when it is not important to specify the size of the tubular neighborhood and π instead of $\pi_0\,|_{T(L)}$. Let $\eta \in \mathfrak{I}^{m-2}(M)$ with $d_1(\eta,\omega)$ so small that $\mathcal{F}_{d\eta}$ be transversal to the fibers $\pi^{-1}(x)$. We are going to define two vector fields $X_{d\eta}$ and V_η tangent to $\ker(\eta)$ on $T(L)$.

The vector field $X_{d\eta}$. Fix a smooth periodic parametrization $\gamma : \mathbb{R} \to L$ of period one of L and denote by $X_{d\eta}$ the vector field which is the lifting of the vector field γ' to the subbundle $\ker(d\eta)$ of the tangent bundle to $T(L)$, i.e., $X_{d\eta}(u) \in \ker(d\eta)$ and $d\pi_u(X_{d\eta}(u)) = \gamma'(\pi(u))$. Notice that $X_{d\eta}^t \circ \pi = \pi \circ X_{d\eta}^t$, whenever both sides are defined, in particular $X_{d\eta}^1$ leaves invariant each fiber $\pi^{-1}(x)$.

Take $p \in L$ and a chart $(\mathbf{x},U)$ of $\mathcal{F}_\omega$ at p along $\mathcal{F}_{d\omega}$. Let $\Sigma = \pi^{-1}(p) \cap T(L)$, then $X_{d\omega}^1$, the Poincaré map of $X_{d\omega}$ at L, is a local diffeomorphism of Σ. It follows from 2) and 3) of Lemma 4.1 that $X_{d\omega}^1 \in \mathrm{Diff}_p^\infty(\Sigma, \mathcal{F}_\omega \cap \Sigma)$ and therefore $\widehat{X_{d\omega}^1}$ is a generator of $Hol(L,p)$.

Definition 4.3. *A circle leaf L is said to have a* hyperbolic holonomy *if there is a chart $(\mathbf{x}, U)$ at some point $p \in L$ along $X_{d\omega}$ such that $(\mathbf{x} \circ X_{d\omega}^1 \circ \mathbf{x}^{-1}, k(\omega^{\mathbf{x}}))$ is a hyperbolic pair at 0, in the sense of Definition 3.2. It is said that L has a* non-resonant holonomy *if $k(\omega^{\mathbf{x}})$ is non-resonant at 0.*

Notice that these concepts neither depend on the chart nor on the point $p \in L$, and also that if L has a hyperbolic holonomy, then it is normally hyperbolic.

The vector field V_η. Choose a smooth volume element ϑ_x on $\pi^{-1}(x)$ that varies smoothly with x and which coincides with $\mathbf{x}_i^*(dx_2 \wedge dx_3 \wedge .... \wedge dx_m)$ on a neighborhood of y_i, for each $1 \leq i \leq k$. Let $j_x : \pi^{-1}(x) \to T(L)$ be the inclusion map and define V_η by

$$i_{V_\eta}\vartheta_x = j_x^*\,\eta$$

It follows from this definition that V_η is tangent to the leaves of $\mathcal{F}_\eta$ and also to the fibers of π, and that $V_\eta(u) = 0$ if and only if $\eta_u = 0$. Notice that $(\mathbf{x}_i)_* V_\eta = \phi_i k(\eta^{\mathbf{x}_i})$, where $\phi_i : \mathbf{x}_i(U_i) \to \mathbb{R}$ is a non-vanishing smooth function such that $\phi_i = 1$ in a neighborhood of y_i, and also that L is normally hyperbolic if and only if x is an hyperbolic singularity of $V_\omega\,|_{\pi^{-1}(x)}$ for each $x \in L$.

Proposition 4.3. *Let* $\omega \in \mathfrak{I}^{m-2}(M)$ *and* L *a normally hyperbolic circle leaf of* $\mathcal{F}_\omega$. *Then* L *is* C^1*-locally stable, i.e., given* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any* $\eta \in \mathfrak{I}^{m-2}(M)$ *satisfying* $\|\eta - \omega\|_1 < \delta$ *the foliation* $\mathcal{F}_\eta$ *has a circle leaf* L_η *with* $d_0(L_\eta, L) < \varepsilon$.

Proof. Choose a γ-tubular neighborhood $\pi : T(L) \to L$ and a covering $\{V_i\}_{i=1}^k$ of L, as above. Since L is normally hyperbolic, we can assume that $\operatorname{sing}(\omega) \cap T(L) = L$. If $\eta \in \mathfrak{I}^{m-2}(M)$ is C^1 close enough to ω, then each $\pi^{-1}(x)$ is transversal to $\mathcal{F}_{d\eta}$ and we have available the vector fields $X_{d\eta}$ and V_η on an open neighborhood of $\overline{T(L)}$. We shall use the covering $\{\overline{V_i'}\}_{i=1}^k$, with $V_i' = V_i \cap T(L)$, to measure the C^r-distance between $X_{d\eta}$ and $X_{d\omega}$ and between V_η and V_ω. Notice that $d_{r-1}(X_{d\eta}, X_{d\omega}) \leq A_{r-1} \cdot d_r(\eta,\omega)$ and that $d_r(V_\eta, V_\omega) \leq B_r \cdot d_r(\eta,\omega)$ for some positive constants A_{r-1} and B_r. Now, since L is normally hyperbolic, given $\varepsilon > 0$, there exists δ_1 such that $d_1(\eta,\omega) < \delta_1$ implies that for each $x \in L$ there is a unique singularity $x_\eta \in \pi^{-1}(x)$ of V_η with $d_f(x_\eta, x) < \varepsilon$. We know that $V_\eta(x_\eta) = 0$ if and only if $\eta_{x_\eta} = 0$, therefore, it follows from part 3) of Lemma 4.1. that the set $L_\eta = \{x_\eta \mid x \in L\}$ is a periodic orbit of $X_{d\eta}$, i.e., a circle leaf of $\mathcal{F}_\eta$ satisfying $d_0(L_\eta, L) < \varepsilon$. To finish the proof choose $\delta = \frac{\delta_1}{C}$ ∎

In the proof of the above proposition we saw that if $d_1(\eta,\omega) < \delta$, then L_η is a periodic orbit of period one of $X_{d\eta}$ and that $x_\eta \in L_\eta$ is the unique singular point of V_η in $\pi^{-1}(x)$.

Lemma 4.4. *Let* $\omega \in \mathfrak{I}^{m-2}(M)$ *and* L *a circle leaf of* $\mathcal{F}_\omega$ *with non-resonant hyperbolic holonomy. Given* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that: for every* $\eta \in \mathfrak{I}^{m-2}(M)$, *with* $d_1(\eta,\omega) < \delta$, *there is a vector field* Y_η *tangent to* $\mathcal{F}_\eta$ *and transversal to the fibers of* π *such that*

1) The circle leaf $L_\eta \in \mathcal{F}_\eta$ *with* $d_0(L_\eta, L) < \varepsilon$, *given by the proposition above, is a periodic orbit of period one of* Y_η,

2) $Y_\eta^t \circ \pi = \pi \circ Y_\eta^t$ *whenever both sides are defined,*

3) Fix $p \in L$. *There is a compact disk neighborhood* $K \subset \pi^{-1}(p)$ *of* $\widetilde{p} \in L_\eta$ *such that* $Y_\eta^1 : K \to K$ *is a contraction with fix point* $\widetilde{p}$, *and*

4) Y_η^1 *and* $X_{d\eta}^1$ *give the same germ in* $Hol(L_\eta, \widetilde{p})$.

Proof. Since L has hyperbolic holonomy, the pair $(\mathrm{x}_1 \circ X_{d\omega} \circ \mathrm{x}_1^{-1}, k(\omega^{\mathrm{x}_1}))$ is hyperbolic at 0, where (x_1, U_1) is one of the charts mentioned in the paragraph choices above. $k(\omega^{\mathrm{x}_1})$ does not depend on the first coordinate and as a vector field on $\mathbb{C}^{m-1}$ is of type (s,u) for some $0 \leq s \leq m-1$. Furthermore, since the holonomy of L is non-resonant, by a theorem of Stenberg on linearization, see [5], there exists $g \in \mathrm{Diff}_0^\infty(\mathbb{C}^{m-1})$ such that $g_* k(\omega^{\mathrm{x}})$ is a linear vector field. Put

$p = y_1$. Then, there is a chart $\mathbf{x} : \mathbf{U} \to \mathbf{C}^m$ of L at p along $\mathcal{F}_{d\omega}$, with $U \subset U_1$, such that

$$k(\omega^{\mathbf{x}}) = \sum_{i=1}^{s} \alpha_i x_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{u} \beta_i y_i \frac{\partial}{\partial y_i}$$

where the coordinates (x, y) of $\mathbf{C}^{m-1} \subset \mathbb{R}^{m-1}$ are chosen as they were at the beginning of section 3. Fix a closed disk $\overline{\mathcal{D}} \subset \mathbf{C}^{m-1}$ and use the compact subset $[-\frac{2}{3}, \frac{2}{3}] \times \overline{\mathcal{D}}$ to measure the C^r-distance between $k(\eta^{\mathbf{x}})$ and $k(\omega^{\mathbf{x}})$. There exists $\xi > 0$ such that if $d_1(k(\eta^{\mathbf{x}}), k(\omega^{\mathbf{x}})) < \xi$, then $k(\eta^{\mathbf{x}}) \mid_{\overline{\mathcal{D}}(x_1)}$ has a unique singularity and of the same type than $k(\omega^{\mathbf{x}})$, where $\overline{\mathcal{D}}(x_1) = \{(x_1, 0, 0)\} \times \overline{\mathcal{D}}$. Furtheremore, there is a diffeomorphism $G = G(\eta)$, isotopic to the identity, of an open neighborhood of $[-\frac{2}{3}, \frac{2}{3}] \times \overline{\mathcal{D}}$ and preserving the first coordinate of each point such that $(x_1, 0, 0)$ is the unique singularity of $G_* k(\eta^{\mathbf{x}})$ in $\overline{\mathcal{D}}(x_1)$ and the local stable and unstable manifolds of $G_* k(\eta^{\mathbf{x}}) \mid_{\overline{\mathcal{D}}(x_1)}$ at the singularity $(x_1, 0, 0)$ are contained in the corresponding stable and unstable manifolds of $k(\omega^{\mathbf{x}}) \mid_{\overline{\mathcal{D}}(x_1)}$. Let $\mu : \mathbf{C}^m \to \mathbb{R}$ be the quadratic function $-\frac{1}{2}(\alpha_1 x_1^2 + \ldots + \alpha_s x_s^2 + \beta_1 y_1^2 + \ldots + \beta_u y_u^2)$ and $\lambda : M \to \mathbb{R}$ be a C^∞ non-negative real function such that $\lambda = 1$ on $(G \circ \mathbf{x})^{-1}([-\frac{1}{2}, \frac{1}{2}] \times \overline{\mathcal{D}})$ and $\lambda = 0$ outside of $(G \circ \mathbf{x})^{-1}([-\frac{2}{3}, \frac{2}{3}] \times \overline{\mathcal{D}})$. For each $j \in \mathbb{Z}^+$ define $Y_j = X_{d\eta} + j \cdot \lambda (G \circ \mathbf{x})_*^{-1}(\mu G_* k(\eta^{\mathbf{x}}))$. The vector field $Y_\eta = Y_j$ for any j sufficiently large has the desired properties ∎

Theorem 4.5. *Let L be a circle leaf of $\mathcal{F}_\omega$, where $\omega \in \mathfrak{I}^{m-2}(M)$. If $L \subset sing_1(\omega)$ has a non-resonant hyperbolic holonomy, then L is $C^{(1,1)}$ locally structurally stable.*

Proof. Fix a Riemannian metric $\langle , \rangle$ on M and a normal tubular neighborhood $\pi_0 : T_0(L) \to L$ such that each fiber $\pi_0^{-1}(q)$ be transversal to $\mathcal{F}_{d\omega}$, and also that $\text{sing}(\omega) \cap T_0(L) = L$. Choose a covering $\{V_i\}_{i=1}^k$ of L and a γ-tubular neighborhood $\pi : T(L) \to L$, as indicated above. If $\eta \in \mathfrak{I}^{m-2}(M)$ and $d_1(\eta, \omega)$ is small enough, then $\mathcal{F}_{d\eta}$, on the open set $\cup_{i=1}^k V_i$, is transversal to the fibers of π_0. Thus, one has available the vector fields $X_{d\eta}$ and V_η on an open neighborhood of $\overline{T(L)}$. One shall use the covering $\{\overline{V_i'}\}_{i=1}^k$, where $V_i' = V_i \cap T(L)$, to measure the C^r-distance between $X_{d\eta}$ and $X_{d\omega}$. Then, it becomes clear that $d_r(X_{d\eta}, X_{d\omega}) \leq A_r \cdot d_r(d\eta, d\omega)$, for some positive constant A_r.

Now, given an open neighborhood W of L take $\gamma > 0$ so small that $T_\gamma(L) \subset T(L) \cap W$. By Proposition 4.3, there exists $\delta_1 > 0$ such that $d_1(\eta, \omega) < \delta_1$ implies that $\mathcal{F}_\eta$ has a circle leaf L_η with $d_0(L_\eta, L) < \gamma$. Therefore $L_\eta \subset T_\gamma(L) \subset W$. By reducing the size of δ_1, if necessary, one also obtains that $\text{sing}(\eta) \cap T_\gamma(L) = L_\eta$.

Let $p = y_1$ and $p_\eta = L_\eta \cap \pi^{-1}(p)$. Since L has a non-resonant hyperbolic holonomy, as we saw in the lemma above, there is a chart $(\mathbf{x},U)$ of L at p along $\mathcal{F}_{d\omega}$ with $U \subset U_1$ such that

$$k(\omega^{\mathbf{x}}) = \sum_{i=1}^{s} \alpha_i x_i \tfrac{\partial}{\partial x_i} + \sum_{i=1}^{u} \beta_i y_i \tfrac{\partial}{\partial y_i}$$

where the first coordinate of the chart does not appear in the expression above. At this point it is convenient to recall that $X^1_{d\eta}$ leaves invariant each fiber of $\pi : T(L) \to L$ and also that $\mathbf{x}^{-1}$ takes the fibers of the first projection $P : \mathbf{C}^m \to (-1,1)$ into the fibers of π. Fix a disk $\overline{\mathcal{D}} \subset \mathbf{C}^{m-1}$ such that $\Sigma = \mathbf{x}^{-1}(\mathcal{D}) \subset \pi^{-1}(p)$. Notice that $X^1_{d\omega} \in \mathrm{Diff}^\infty_p(\Sigma,V_\omega)$ or equivalently $\mathbf{x} \circ X^1_{d\omega} \circ \mathbf{x}^{-1} \in \mathrm{Diff}^\infty_0(\mathcal{D},k(\omega^{\mathbf{x}}))$. The closed disk $\overline{\mathcal{D}}$ will be used to measure the C^r-distance between $k(\eta^{\mathbf{x}})$ and $k(\omega^{\mathbf{x}})$. It is then clear that $d_r(k(\eta^{\mathbf{x}}),k(\omega^{\mathbf{x}})) < B_r \cdot d_r(\eta,\omega)$, for some positive constant B_r.

There exists $\delta_2 > 0$ and a neighborhood $N \subset \Sigma$ of p such that $d_1(X_{d\eta}, X_{d\omega}) < \delta_2$ implies $X^1_{d\eta}(N) \subset \Sigma$. Fix a disk $D^{m-1}_\rho \subset \mathcal{D}$ with $\mathbf{x}^{-1}(\overline{D^{m-1}_\rho}) \subset N$ and use $\overline{D^{m-1}_\rho}$ to measure the C^1-proximity between the Poincaré diffeomorphisms $X^1_{d\eta}$ and $X^1_{d\omega}$. In fact define

$$d_1(X^1_{d\eta}, X^1_{d\omega}) = d_1(\mathbf{x} \circ X^1_{d\eta} \circ \mathbf{x}^{-1}, \mathbf{x} \circ X^1_{d\omega} \circ \mathbf{x}^{-1})$$

It is well known that the function $X_{d\eta} \to X^1_{d\eta}$ is continuous at $X_{d\omega}$. We divide the rest of the proof in two cases: $s = m-1$ or $s = 0$ and $0 < s < m-1$.

First suppose that $s = m-1$. In order to simplify the notation put $f_\eta = \mathbf{x} \circ X^1_{d\eta} \circ \mathbf{x}^{-1}$ and recall that $f_\eta \in \mathrm{Diff}^\infty_{\mathbf{x}(p_\eta)}(\mathcal{D}, k(\eta^{\mathbf{x}}))$ and that $\widehat{f_\eta}$ is a generator of $\mathrm{Hol}(L_\eta, p_\eta)$. Since the pair $(f_\omega, k(\omega^{\mathbf{x}}))$ is hyperbolic at 0 there exist $\xi_1 > 0$ and $\zeta_1 > 0$ such that $d_1(k(\eta^{\mathbf{x}}), k(\omega^{\mathbf{x}})) < \xi_1$ and $d_1(f_\eta, f_\omega) < \zeta_1$ implies that $(f_\eta, k(\eta^{\mathbf{x}}))$ is a hyperbolic pair at $\mathbf{x}(p_\eta)$ with $k(\eta^{\mathbf{x}})$ also of type (s,u). Put $\delta_3 = \frac{\xi_1}{B_1}$ and let $\delta_4 > 0$ be such that $d_1(X_{d\eta}, X_{d\omega}) < \delta_4$ implies that $d_1(f_\eta, f_\omega) < \zeta_1$.

Recapitulaiting, if $\eta \in \mathfrak{I}^{m-2}(M)$ and $d_{(1,1)}(\eta,\omega) < \delta_i$, $1 \le i \le 4$, then $\mathcal{F}_\eta$ has a circle leaf $L_\eta \subset T_\gamma(L) \subset W$, $\mathrm{sing}(\eta) \cap T_\gamma(L) = L_\eta$, the pair $(f_\eta, k(\eta^{\mathbf{x}}))$ is hyperbolic at $\mathbf{x}(p_\eta)$ with $k(\eta^{\mathbf{x}})$ also of type $(m-1,0)$, and $f_\eta(\overline{D^{m-1}_\rho}) \subset \mathcal{D}$. To facilitate the utilization of the results of section 3 put $n = m-1$ and $\mathcal{U} = D^{m-1}_\rho$.

Fix an $\varepsilon < \rho$ so that $\overline{D^n_\varepsilon} \subset \mathcal{U}$. Notice that $k(\omega^{\mathbf{x}})$ is transversal to the sphere S^{n-1}_ε. Thus, there exists $\xi_2 > 0$ such that $d_1(k(\eta^{\mathbf{x}}), k(\omega^{\mathbf{x}})) < \xi_2$ implies that $k(\eta^{\mathbf{x}})$ is also transversal to S^{n-1}_ε. Put $\delta_5 = \frac{\xi_2}{B_1}$. Let $\mathcal{U}_\bullet = \mathcal{U} \backslash \{0\}$ and denote by $\frac{\mathcal{U}_\bullet}{V_\eta}$ the space of orbits of the restriction of V_η to $\mathcal{U}_\bullet$. It is clear that S_ε^{n-1} renders concrete both orbit spaces $\frac{\mathcal{U}_\bullet}{V_\omega}$ and $\frac{\mathcal{D}_\bullet}{V_\eta}$. Therefore $f_\eta : \mathcal{U} \to \mathcal{D}$ induces a C^∞ diffeomorphism $\overline{f_\eta} : S_\varepsilon^{n-1} \to S_\varepsilon^{n-1}$. It is not difficult to verify that the map $f_\eta \to \overline{f_\eta}$ is continuous at f_ω. Now, since the pair $(f_\omega, k(\omega^\mathrm{x}))$ is hyperbolic at 0, it follows from Theorem 3.5 that $\overline{f_\omega} : S_\varepsilon^{n-1} \to S_\varepsilon^{n-1}$ is a Morse-Smale diffeomorphism. Thus, there exists $\theta > 0$ such that $d_1(\overline{f_\omega}, \overline{f_\eta}) < \theta$ implies that $\overline{f_\eta}$ and $\overline{f_\omega}$ are topologically conjugated. It follows from the continuity of the map $f_\eta \to \overline{f_\eta}$ at f_ω that there exists $\zeta_2 > 0$ such that if $d_1(f_\eta, f_\omega) < \zeta_2$, then $d_1(\overline{f_\omega}, \overline{f_\eta}) < \theta$, and from the continuity of the map $\eta \to f_\eta$ at ω in the $C^{(1,1)}$-topology that there is δ_6 such that $d_{(1,1)}(\eta, \omega) < \delta_6$ implies $d_1(f_\eta, f_\omega) < \zeta_2$.

Summarizing, if $d_{(1,1)}(\eta, \omega) < \delta_i$, $1 \le i \le 6$, then there exists a homeomorphism $h_0 : S_\varepsilon^{n-1} \to S_\varepsilon^{n-1}$ such that $\overline{f_\eta} = h_0 \circ \overline{f_\omega} \circ h_0^{-1}$. If one looks at the cross section Σ, then $\overline{X^1_{d\omega}}$ and $\overline{X^1_{d\eta}}$ are smooth diffeomorphisms of $\mathcal{C}_0 = \mathrm{x}^{-1}(S_\varepsilon^{n-1})$ topologically conjugated by $h = \mathrm{x}^{-1} \circ h_0 \circ \mathrm{x}$, i.e., $\overline{X^1_{d\eta}} = h \circ \overline{X^1_{d\omega}} \circ h^{-1}$. The next step is to extend h to $\mathrm{x}^{-1}(\overline{D_\varepsilon^{n-1}})$ taking orbits of V_ω to orbits of V_η. At this moment we substitute $X^1_{d\eta}$ by Y^1_η because, according to Lemma 4.4, they are equivalent, but Y^1_η has the advantage of being a contraction. For each integer $i \ge 0$ let $\mathcal{C}_i^\eta = Y_\eta^i(\mathcal{C}_0)$ and $\mathcal{A}_i^\eta$ the closed region with boundary $\partial\mathcal{A}_i^\eta = \mathcal{C}_i^\eta \cup \mathcal{C}_{i+1}^\eta$. Notice that $\mathcal{C}_0^\eta = \mathcal{C}_0^\omega = \mathcal{C}_0$ and that $\mathcal{A}_i^\eta$ is foliated by the orbits of V_η. Now, we extend h to a homeomorphism of $\mathrm{x}^{-1}(\overline{D_\varepsilon^{n-1}})$ taking orbits of V_ω to orbits of V_η. Choose an extension of $h : \mathcal{C}_0 \to \mathcal{C}_0$ to a homeomorphism $h : \mathcal{A}_0^\omega \to \mathcal{A}_0^\eta$ taking orbits of V_ω to orbits of V_η and notice that if $q \in \mathrm{x}^{-1}(\overline{D_\varepsilon^{n-1}})$ and $q \ne 0$, then there exists a unique integer k such that $Y_\omega^k(q) \in \mathcal{A}_0^\omega \backslash \mathcal{C}_0$. Define $h(q) = Y_\eta^{-k}(q) \circ h \circ Y_\omega^k(q)$ and $h(p) = p_\eta$. It is clear that h satisfies $Y_\eta^1 \circ h = h \circ Y_\omega^1$. Finally, extend h to a neighborhood of L by defining $h(Y_\omega^t(q)) = Y_\eta^t(h(q))$. For $s = 0$ the proof is completely analogous.

Next, suppose that $0 < s < m-1$. Given $\lambda_1 > 0$ there exists $\xi_1 > 0$ such that for each $\eta \in \mathfrak{I}^{m-2}(M)$ with $d_1(k(\eta^\mathrm{x}), k(\omega^\mathrm{x})) < \xi_1$ there is a diffeomorphism $g = g(\eta)$ between neighborhoods of $\overline{\mathcal{D}}$ such that the unique singularity of $g_* k(\eta^\mathrm{x})$ in $\overline{\mathcal{D}}$ is $(x, y) = (0, 0)$, that the local stable and unstable manifolds of $g_* k(\eta^\mathrm{x})$ at $(0,0)$ are contained in those of $k(\omega^\mathrm{x})$, i.e., $W^s \cap \overline{\mathcal{D}} \subset E = \{(x,y) \mid y = 0\}$ and $W^u \cap \overline{\mathcal{D}} \subset F = \{(x,y) \mid x = 0\}$, and that $d_1(g_* k(\eta^\mathrm{x}), k(\omega^\mathrm{x})) < \lambda_1$. Notice that $(g \circ \mathrm{x}) \circ X^1_{d\eta} \circ (\mathrm{x} \circ g)^{-1}$ take orbits of $g_* k(\eta^\mathrm{x})$ in $\mathcal{U} = D_\rho^{m-1}$ into orbits of $g_* k(\eta^\mathrm{x})$ in $\mathcal{D}$. In order to simplify the notation put $f_\eta = (g \circ \mathrm{x}) \circ X^1_{d\eta} \circ (g \circ \mathrm{x})^{-1}$ and $f_\omega = \mathrm{x} \circ X^1_{d\omega} \circ \mathrm{x}^{-1}$. Since the pair

$(f_\omega, k(\omega^\times))$ is hyperbolic at $(0,0)$ there exists $\lambda_2 > 0$ and $\zeta_1 > 0$ such that $d_1(g_* k(\eta^\times), k(\omega^\times)) < \lambda_2$ and $d_1(f_\eta, f_\omega) < \zeta_1$ implies that the pair $(f_\eta, g_* k(\eta^\times))$ is hyperbolic at $(0,0)$ with $g_* k(\eta^\times)$ of the same type than $k(\omega^\times)$. Put $\delta_3 = \frac{\xi_1}{B_1}$ and choose $\delta_4 > 0$ so small that $d_{(1,1)}(\eta, \omega) < \delta_4$ implies that $d_1(g_* k(\eta^\times), k(\omega^\times)) < \lambda_2$ and $d_1(f_\eta, f_\omega) < \zeta_1$.

Recapitulating, if $d_{(1,1)}(\eta, \omega) < \delta_i$, $1 \leq i \leq 4$, then $\mathcal{F}_\eta$ has a circle leaf $L_\eta \subset T_\gamma(L) \subset W$, $\operatorname{sing}(\eta) \cap T_\gamma(L) = L_\eta$, the pair $(f_\eta, g_* k(\eta^\times))$ is hyperbolic at $(0,0)$ with $g_* k(\eta^\times)$ also of type (s, u).

Let $S^{s-1} = \{(x, 0) \mid |x| = \frac{\rho}{2}\}$ and $S^{u-1} = \{(0, y) \mid |y| = \frac{\rho}{2}\}$ and consider the manifold with boundary $\mathbf{K}_\varepsilon = \mathbf{K}_\varepsilon(k(\omega^\times))$ defined in section 3. Notice that each orbit in $\mathbf{K}_\varepsilon$ intersects the set $(S^{s-1} \times \overline{D^u_\varepsilon}) \cup S^{u-1}$ in a unique point. Thus, one can transfer the differentiable structure of $\mathbf{K}_\varepsilon$ to this set and in this way render concrete a manifold whose points are orbits. This differentiable manifold will be denoted by $\mathbf{K}^*_\varepsilon$. The vector field $k(\omega^\times)$ is transversal to the fence $\mathbf{F}_\varepsilon$, therefore there exists $\xi_2 > 0$ such that $d_1(g_* k(\eta^\times), k(\omega^\times)) < \xi_2$ implies that $g_* k(\eta^\times)$ is also transversal to $\mathbf{F}_\varepsilon$. Let δ_5 be a positive number such that if $d_1(\eta, \omega) < \delta_5$ then $d_1(g_* k(\eta^\times), k(\omega^\times)) < \xi_2$. Thus, if $d_1(\eta, \omega) < \delta_5$, then the space $\mathbf{K}^\eta_\varepsilon$ of orbits of $g_* k(\eta^\times)$ in $\mathcal{U}_\bullet$ which cut $(S^{s-1} \times \overline{D^u_\varepsilon}) \cup S^{u-1}$ is a manifold that can also be identified with $\mathbf{K}^*_\varepsilon$. Since the pair $(f_\omega, k(\omega^\times))$ is hyperbolic at $(0,0)$ and $k(\omega^\times)$ has type (s, u), with $s > 0$, by Theorem 3.6 the map $\overline{f_\omega}$ restricted to $\mathbf{K}^*_\varepsilon$ is structurally stable. Furthermore, since the map $f_\eta \to \overline{f_\eta}$ is continuous at f_ω, there exists $\delta_6 > 0$ such that $d_1(d\eta, d\omega) < \delta_6$ implies that $\overline{f_\eta}(\mathbf{K}^*_\varepsilon) \subset \operatorname{int} \mathbf{K}^*_\varepsilon$ and also that $\overline{f_\eta}$ and $\overline{f_\omega}$ are topologically conjugated.

Summarizing, if $d_{(1,1)}(\eta, \omega) < \delta_i$, $1 \leq i \leq 6$, then there exists a homeomorphism $h_0 : \mathbf{K}^*_\varepsilon \to \mathbf{K}^*_\varepsilon$ such that $\overline{f_\eta} = h_0 \circ \overline{f_\omega} \circ h_0^{-1}$. The map h_0 leaves invariant the subspaces S^{s-1}, S^{u-1} and the fence $\mathbf{F}^s_\varepsilon$. Consider in $C^\omega_0 = \mathbf{x}^{-1}\mathbf{K}^*_\varepsilon$ ($C^\eta_0 = (g \circ \mathbf{x})^{-1}\mathbf{K}^*_\varepsilon$) the differentiable structure that turns $\mathbf{x}^{-1}$ ($(g \circ \mathbf{x})^{-1}$) a diffeomorphism, then the maps

$$\overline{X^1_{d\omega}} : C^\omega_0 \to C^\omega_0 \qquad \text{and} \qquad \overline{X^1_{d\eta}} : C^\eta_0 \to C^\eta_0$$

are conjugated by the homeomorphism $h = \mathbf{x}^{-1} \circ g^{-1} \circ h_0 \circ \mathbf{x}$. Denote by $[\mathbf{K}^\eta_\varepsilon]$ the subset of $\mathcal{U}$ which contains the positive orbits of $g_* k(\eta^\times)$ with initial conditions in S^{s-1}, the negative orbits with initial conditions in S^{u-1}, the origin $(0,0)$ and for each initial condition $(x, y) \in S^{s-1} \times (\overline{D^u_\varepsilon})_\bullet$ the positive arc of orbit which ends on $E \times S^{u-1}$. This subset is a compact disk neighborhood of $(0,0)$ with the property that $\frac{[\mathbf{K}^\eta_\varepsilon]_\bullet}{g_* k(\eta^\times)} = \mathbf{K}^*_\varepsilon$. The next step is to extend h to a homeomorphism

$h : \mathbf{x}^{-1}[\mathbf{K}_\varepsilon^\omega] \to \mathbf{x}^{-1} \circ g^{-1}[\mathbf{K}_\varepsilon^\eta]$. At this moment we substitute $X_{d\eta}^1$ by $Y_{d\eta}^1$ because, according to Lemma 4.4, they are equivalent, but $Y_{d\eta}^1$ has the advantage of being a contraction. The rest of the proof imitates the first case ∎

I think that the non-resonant condition can be erased from the hypothesis of the Theorem 4.5. A geometric consequence of this theorem is that in the neighborhood of a structurally stable circle leaf there are at least $(m-1)$ and at most $2(m-1)$ leaves containing L in its closure and whose intersection with the neighborhood are topologically cylinders.

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