ON REFLEXIVITY AND BASIS FOR $P(^mE)$

by

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ABSTRACT. We show that if E is a reflexive B-space with a.p., then the reflexivity of $P(^mE)$ is equivalent to the coincidence of $P(^mE)$ with its subspace of compact polynomials. If E has a Schauder basis, then the result can be reformulated as follows: $P(^mE)$ is reflexive if only if $P(^mE)$ admits a special Schauder basis.

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1. Introduction

In an earlier paper [1] we studied the spaces $L_N(E_1,\dots,E_m;F)$ and $L_I(E_1,\dots,E_m;F)$ of all continuous m-linear mappings $A : E_1 x \dots x E_m \to F$, respectively of Nuclear and Integral type, where $E_1,\dots,E_m$ and F denote B-spaces.

We proved the following theorem:

Theorem A: Let $E_1,\dots,E_m$ be B-spaces whose duals $E_1^*,\dots,E^*_m$ have the Radon-Nikodym property. Then for every B-space F, the spaces $L_N(E_1,\dots,E_m;F)$ and $L_I(E_1,\dots,E_m;F)$ are isometric.

In this paper we apply mainly the above theorem in order to obtain a necessary and sufficient condition for the reflexivity of $P(^mE)$, the space of m-homogeneous continuous polynomials on E.

The linear version of theorem A is in fact equivalent to the Radon-Nikodym property as established in [1]. For other variants of theorem A in the linear case we refer to Diestel and Uhl [6]. For terminology and results on the multi-linear and polynomial case of nuclear type we refer to Dineen [7]. For details and results on tensor products (m-symmetric) we refer to Ryan [14].

We introduce now the terminology and some technical lemmas. If $E_1 = E_2 = \dots = E_m = E$ and F are B-spaces, $L(\underbrace{E,\dots,E}_{m};F) = L(^mE,F)$ denotes the B-spaces of continuous m-linear mappings $A : Ex\dots xE \to F$, with the norm $A = \sup\{\|A(x_1,\dots,x_m)\| : x_j \in E\ \|x_j\| \le 1,\ 1 \le j \le m\}$. $L(^mE)$ denotes $L(^mE,K)$ where $K = R$ or C. $L^s(^mE,F)$ denotes the space of all mappings $A \in L(^mE,F)$ that are symmetric. $P(^mE,F)$ denotes the space of all continuous, m-homogeneous polynomials from E to F. We recall that an element $P \in P(^mE,F)$ is defined as $P(x) = A(x,\dots x)$ for a unique element $A \in L^s(^mE,F)$ and in this case we denote $P = \hat{A}$. On $P(^mE,F)$ we consider the norm induced by $L(^mE,F)$.

<u>Definition (a)</u>: A mapping $A \in L(^mE,F)$ is said to be nuclear if there are sequences $(\varphi_{jn})_n$ in E^*, $1 \leq j \leq m$ and (C_n) in F with

$$\sum_{n=1}^{\infty} \|\varphi_{1n}\| \ldots \|\varphi_{mn}\| \; \|C_n\| < \infty \quad \text{such that}$$

$$A(x_1,\ldots,x_m) = \sum_{n=1}^{\infty} \varphi_{1n}(x_1)\ldots\varphi_{mn}(x_m)C_n \quad \text{for all} \quad (x_1,\ldots,x_m) \in \underbrace{Ex\ldots xE}_{m}.$$

Let $L_N(^mE,F)$ denote the space of all nuclear m-linear mappings $A \in L(^mE,F)$ endowed with the nuclear norm:

$\|A\|_N = \inf \sum_{n=1}^{\infty} \|\varphi_{1n}\| \ldots \|\varphi_{mn}\| \; \|C_n\|$, where the infimum is taken over all sequences $(\varphi_{jn})_n$ and (C_n) which satisfy the definition.

Similarly, we define a polynomial $P \in P(^mE,F)$ to be nuclear if there are sequences (φ_n) in E^* and (C_n) in F with

$$\sum_{n=1}^{\infty} \|\varphi_n\|^m \; \|C_n\| < \infty \quad \text{such that} \quad P(x) = \sum_{n=1}^{\infty} \varphi_n(x)^m C_n \text{ for all } x \in E.$$

$P_N(^mE,F)$ denotes the space of all nuclear polynomials endowed with the nuclear norm:

$\|P\|_N = \inf \sum_{n=1}^{\infty} \|\varphi_n\|^m \; \|C_n\|$: ; where the infimum is taken over all sequences (φ_n) and (C_n) which satisfy the definition.

<u>Definition (b)</u>: A mapping $A \in L(^mE,F)$ is said to be integral if there exists a regular countably additive F-valued Borel measure G, of bounded variation, on the product $U^m_{E^*} = U_{E^*}x\ldots xU_{E^*}$, where the unit ball U_{E^*} of the dual space E^* is endowed with the weak * topology, such that

$$A(x_1,\ldots,x_m) = \int_{U^m_{E^*}} \varphi_1(x_1)\ldots\varphi_m(x_m)\, dG\,(\varphi_1,\ldots,\varphi_m), \text{ for all } (x_1,\ldots,x_m) \in Ex\ldots xE.$$

Let $L_I(^mE,F)$ denote the space of all integral m-linear mappings $A \in L(^mE,F)$, endowed with the integral norm:

$\|A\|_I = \inf |G|(U^m_{E*})$, where the infimum is taken over all vector measures G satisfying the definition.

A polynomial $P \in P(^mE,F)$ is said to be integral if P can be written as

$P(x) = \int_{U_{E*}} \varphi(x)^m \, dg\,(\varphi)$ for all $x \in E$, where g is an F-valued vector measure defined on U_{E*} with the same properties as before.

$P_I(^mE,F)$ denotes the space of all integral polynomials endowed with the integral norm.

$\|P\|_I = \inf |g|(U_{E*})$ where the infimum is taken over all vector measures g satisfying the definition.

Remark: The definition above was introduced by S. Dineen in [8], and only recently we introduced m-linear mappings of integral type which makes proposition 2 below necessary.

To establish the result stated in the beginning, we need the following polynomial version of theorem A

Proposition 1: Let E be a B-space such that E^* has the Radon-Nikodym property. Then $P_N(^mE,F) = P_I(^mE,F)$, with equivalent norms, for every positive integer m and every B-space F.

We need some technical results

Proposition 2: (a) Let $P \in P_I(^mE,F)$ and $A \in L^s(^mE,F)$ such that $\hat{A} = P$. Then $A \in L^s_I(^mE,F)$ and $\|A\|_I \le \|P\|_I$.

(b) Let $A \in L_I(^mE,F)$ and $P = \hat{A}$. Then $P \in P_I(^mE,F)$ and $\|P\|_I \le \frac{m^m}{m!} \|A\|_I$.

Proof: (a) Let $P \in P_I({}^mE,F)$, P has the representation

$$P(x) = \int_{U_{E*}} \langle x,\varphi\rangle^m \, dg(\varphi).$$

If $A \in L^s({}^mE,F)$ such that $\hat{A} = P$, by applying the polarization formula, we obtain:

$$A(x_1,\dots,x_m) = \int_{U_{E*}} \langle x_1,\varphi\rangle \dots \langle x_m,\varphi\rangle \, dg(\varphi).$$

Now consider the mapping:

$J : U_{E*} \to U^m_{E*}$ given by $J(\varphi) = (\varphi,\dots,\varphi)$ and define a measure $\tilde{g}$ on the Borel algebra $B(U^m_{E*})$ of U^m_{E*}, by $\tilde{g}(H) = g(J^{-1}(H))$, for every $H \in B(U^m_{E*})$.

We now calculate the variation $|\tilde{g}|$ of $\tilde{g}$.

$$|\tilde{g}|(U^m_{E*}) = \sup \{ \sum_i \|\tilde{g}(A_i)\| : (A_i) \text{ a finite partition of } U^m_{E*}\} =$$

$$= \sup \{ \sum_i \|g(J^{-1}(A_i)\| : (A_i) \dots \} \le |g| \, (U_{E*}).$$

On the other hand:

$$|g|(U_{E*}) = \sup \{ \sum \|g(B_i)\| : (B_i) \text{ finite partition of } U_{E*}\} =$$

$$= \sup \{ \sum_i \|g(J^{-1}(J(B_i)))\| : (B_i) \dots \} \le |\tilde{g}|(U^m_{E*}).$$

Thus, we have $\tilde{g}$ of bounded variation and $|\tilde{g}| = |g|$. Now, since $\tilde{g}(H) = 0$ if $H \cap \Delta\,(U^m_{E*}) = \phi$, Δ denotes the diagonal, it follows that

$$\int_{U^m_{E*}} \langle x_1,\varphi_1\rangle \dots \langle x_m,\varphi_m\rangle \, d\tilde{g}(\varphi_1,\dots,\varphi_m) =$$

$$= \int_{\Delta(U^m_{E*})} \langle x_1,\varphi_1\rangle \dots \langle s_m,\varphi_m\rangle \, d\tilde{g}(\varphi_1,\dots,\varphi_m) =$$

$$= \int_{U_{E*}} \langle x_1,\varphi\rangle \ldots \langle x_m,\varphi\rangle \, d\tilde{g}(\varphi,\ldots,\varphi) =$$

$$= \int_{U_{E*}} \langle x_1,\varphi\rangle \ldots \langle x_m,\varphi\rangle \, dg(\varphi) = A(x_1,\ldots,x_m).$$

Furthermore, we have $\|A\|_I \leq \|\Gamma\|_I$.

(b) Let $A \in L_I(^mE,F)$. There exists a vector measure $G : B(U^m_{E*}) \to F$ such that

$$A(x_1,\ldots,x_m) = \int_{U^m_{E*}} \langle x_1,\varphi_1\rangle \ldots \langle x_m,\varphi_m\rangle \, dG(\varphi_1,\ldots,\varphi_m).$$

Therefore

$$P(x) = A(x,\ldots,x) = \int_{U^m_{E*}} \langle x,\varphi_1\rangle \ldots \langle x,\varphi_m\rangle \, dG(\varphi_1,\ldots,\varphi_m).$$

By the polarization formula we can write:

$$\langle x,\varphi_1\rangle \ldots \langle x,\varphi_m\rangle = \frac{1}{m!2^m} \sum_{\substack{\epsilon_i=\pm 1\\ 1\leq i\leq m}} \epsilon_1\ldots\epsilon_m \langle x, \sum_{i=1}^m \epsilon_i\varphi_i\rangle^m =$$

$$= \frac{m^m}{m!2^m} \sum_{\substack{\epsilon_i=\pm 1\\ 1\leq i\leq m}} \epsilon_1\ldots\epsilon_m \langle x, \frac{1}{m} \sum_{i=1}^m \epsilon_i\varphi_i\rangle^m$$

Therefore

$$P(x) = \frac{m^m}{m!2^m} \sum_{\substack{\epsilon_i=\pm 1\\ 1\leq i\leq m}} \epsilon_1\ldots\epsilon_m \int_{U^m_{E*}} \langle x, \frac{1}{m} \sum \epsilon_i\varphi_i\rangle^m \, dG(\varphi_1,\ldots,\varphi_m).$$

Now for each $(\varepsilon_1,\ldots,\varepsilon_m)$, we define a mapping $J_{\varepsilon_1\ldots\varepsilon_m} : U^m_{E*} \to U_{E*}$ by

$J_{\varepsilon_1\ldots\varepsilon_m}(\varphi_1,\ldots,\varphi_m) = \frac{1}{m}\sum_{i=1}^{m} \varepsilon_i\varphi_i$, and also we define a measure

$g_{\varepsilon_1\ldots\varepsilon_m} : B(U_{E*}) \to F$ by

$g_{\varepsilon_1\ldots\varepsilon_m}(B) = G(J^{-1}_{\varepsilon_1\ldots\varepsilon_m}(B))$ for every $B \in B(U_{E*})$.

We have:

$|g_{\varepsilon_1\ldots\varepsilon_m}|(U_{E*}) \le |G|(U^m_{E*})$, but

$$\int_{U^m_{E*}} \langle x, \frac{1}{m}\sum_{i=1}^{m} \varepsilon_i\varphi_i\rangle^m \, dG\,(\varphi_1,\ldots,\varphi_m) = \int_{U_{E*}} \langle x,\phi\rangle^m \, d\, g_{\varepsilon_1\ldots\varepsilon_m}(\phi)$$

Therefore

$$P(x) = \frac{m^m}{m!2^m} \sum_{\varepsilon i=\pm 1} \varepsilon_1\ldots\varepsilon_m \int_{U_{E*}} \langle x,\phi\rangle^m \, dg\, \varepsilon_1\ldots\varepsilon_m(\phi).$$

Now we define a measure

$g : B(U_{E*}) \to F$ by

$$g = \frac{m^m}{m!2^m} \sum_{\varepsilon i=\pm 1} \varepsilon_1\ldots\varepsilon_m \, g_{\varepsilon_1\ldots\varepsilon_m}.$$

Hence, it follows that $P(x)$ has the form

$P(x) = \int_{U_{E*}} \langle x,\phi\rangle^m \, dg(\phi)$ and $P \in P_I(^mE,F)$.

Furthermore, we have

$$|g| \leq \frac{m^m}{m!2^m} 2^m|G| = \frac{m^m}{m!} |G|, \text{ therefore}$$

$$\|P\|_I \leq \frac{m^m}{m!} \|A\|_I$$

Remark: Similar results hold for nuclear mappings and nuclear polynomials.

The following are clear:

Corollary 3: If $A \in L_I(^mE,F)$, then $A^s \in L_I^s(^mE,F)$ and $\|A^s\|_I \leq \frac{m^m}{m!} \|A\|_I$ (A^s denotes the symmetrization of A).

Corollary 4: Let $A \in L_I(^mE,F)$ and A^s its symmetrization. Given $\varepsilon > 0$ there is a measure g, F-valued and of bounded variation and σ-additive defined on $B(U_{E^*})$ such that

$$A^s(x_1,\ldots,x_m) = \int_{U_{E*}} \langle x_1,\phi\rangle \ldots \langle x_m,\phi\rangle \, dg(\phi) \text{ and } |g| \leq \frac{m^m}{m!} \|A\|_I + \varepsilon .$$

Proof of proposition 1: We always have the inclusion $P_N(^mE,F) \subset P_I(^mE,F)$.

For the opposite inclusion, let $P \in P_I(^mE,F)$. Then there is $A \in L^s(^mE,F)$ such that $\hat{A} = P$. By proposition 2 (a) we have that $A \in L_I^s(^mE,F)$. By theorem A it follows that $A \in L_N^s(^mE,F)$, but it is easy to check that $\hat{A} \in P_N(^mE,F)$, hence $P_N(^mE,F) = P_I(^mE,F)$. Recalling the remark after proposition 2, we see that their norms are equivalent.

2. Main Result

Before establishing our main result, theorems 7 and 8, we need some more terminology and some definitions.

A scalar valued polynomial $P \in P(^mE)$ is said to be of finite type if P has the form

$P(x) = \sum_{j=1}^{r} \varphi_j^m(x)$ for every $x \in E$, where $\varphi_j \in E^*$, $1 \leq j \leq r$. $P_f(^mE)$ denotes the space of all polynomials of finite type. $P_c(^mE)$ denotes the closure of $P_f(^mE)$

in $P(^mE)$ with respect to the norm induced by $L(^mE)$ (for definitions and results on $P_c(^mE,F)$, vector valued polynomials, we refer to Aron and Prolla [4]) and is called the space of compact polynomials.

Now we follow the terminology of Ryan [14].

If E is a B-space from, $h_\pi^m E$ denotes the subspace of $\underbrace{E\otimes_\pi \dots \otimes_\pi E}_{m}$, the projective tensor product, span by the elements $x^{(m)} = \underbrace{x\otimes \dots \otimes x}_{m}$, $x \in E$.

If $x_1,\dots,x_m$ E, $x_1 \circ \dots \circ x_m$ denotes the element of $E\otimes \dots \otimes E$, given by

$x_1 \circ \dots \circ x_m = \frac{1}{m!} \sum_{\sigma \in S_m} x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(m)}$, S_m denotes the permutations of $\{1,2,\dots,m\}$, (symmetric tensor product).

Suppose now (e_j) is a Schauder basis for E, if $((j_1,\dots,j_m)_j)_{j=1}^{\infty}$ denotes the square order of N^m (the m-fold cartesian product of the natural numbers), then $(e_{j1}\otimes \dots \otimes e_{jm})_j$ is a Schauder basis for $\underbrace{E\hat{\otimes}_\pi \dots \hat{\otimes}_\pi E}_{m}$, (the completed tensor product). For details see Ryan [14] and also Gelbaum-Lamadrid [10].

Let Q_m be the set of $(j_1,\dots,j_m)$ in N^m such that $j_1 \geq j_2 \geq \dots \geq j_m$. If we consider the square order induced on Q_m, then Ryan has proved the following:

"For every positive integer m, $(e_{j1}\circ \dots \circ e_{jm})_{J\in Q_m}$ is a Schauder basis for $\hat{h}_\pi^m E$ (the complete B-space)". Also from Ryan, we point out the following result: The spaces $P(^mE)$ and $(\hat{h}_\pi^m E)^*$ are isometric, for every B-space E, (E not necessarily with Schauder basis).

The elements of the sequence $(Z_J)_{J\in Q_m}$, given by the biorthogonal system

$[(e_{j_1}\otimes\ldots\otimes e_{j_m})_J, Z_J]$ are called the m-homogeneous monomials.

Now we recall the following results:

Proposition 5: (Dineen [8]). Let E be a reflexive B-space. Then the spaces $(P_I({}^mE^*), \| \|_I)$ and $P_c({}^mE)^*$ are isometric, for every positive integer m.

Proposition 6: (Gupta [11]). Let E be a B-space such that E^* has the approximation property. Then the spaces $(P_N({}^mE), \| \|_N)^*$ and $P({}^mE^*)$ are isometric for every positive integer m.

Now we state and prove our main results:

Theorem 7: Let E be a reflexive B-space with the approximation property. Then the space $P({}^mE)$ is reflexive if only if $P({}^mE) = P_c({}^mE)$.

In case E has Schauder basis, theorem 7 can be reformulated as follows:

Theorem 8: Suppose E is a reflexive B-space with Schauder basis. Then $P({}^mE)$ is reflexive if only if the monomials define a Schauder basis for $P({}^mE)$.

Proof of Theorem 7: First of all, we note that E has the Radon-Nikodym property, since E is reflexive. From proposition 1, we have that

$(P_N({}^mE^*), \| \|_N) = (P_I({}^mE^*), \| \|_I)$ topologically. Now combining this result with proposition 5 and 6 we obtain:

$P({}^mE) = (P_N({}^mE^*), \| \|_N)^* = (P_I({}^mE^*), \| \|_I) = P_c({}^mE)^{**}$. To conclude it is enough to observe that $P_c({}^mE)$ is a closed subspace of $P({}^mE)$.

Proof of Theorem 8: Let E be a reflexive B-space with Schauder basis (e_j).

Suppose $P({}^mE)$ reflexive, from Ryan's results we have that

$(e_{j_1}\otimes\ldots\otimes e_{j_m})_{J\in Q_m}$ is a Schauder basis for $\hat{\otimes}_\pi^m E$ and $(\hat{\otimes}_\pi^m E)^* = P({}^mE)$, therefore the basis $(e_{j_1}\otimes\ldots\otimes e_{j_m})_{J\in Q_m}$ is shrinking, which means that the monomials $(Z_J)_{J\in Q_m}$ is a Schauder basis for $P({}^mE)$. Conversely suppose the monomials

(Z_J) form a Schauder basis for $P(^mE)$. Hence if $P \in P(^mE)$ then $P = \sum_{J\in Q_m} \lambda_J Z_J$.

Now, if $\varepsilon > 0$, there is an integer J_0 such that

$$\|P - \sum_{J\leq J_0} \lambda_J Z_J\| \leq \varepsilon.$$

Since each Z_J is a polynomial of finite type it follows from the definition of $P_c(^mE)$, that $P \in P_c(^mE)$, thus $P(^mE) = P_c(^mE)$ and now we apply theorem 7.

<u>Final Remarks</u>: One may ask if there is an example of infinite dimensional B-space for which the conditions of theorem 7 and 8 hold. In a previous paper [2], we show that $P(^mE)$ is reflexive B-space when E is the original Tsirelson space T^* (an infinitive dimensional reflexive B-space with unconditional Schauder basis and containing no l_p, $1 < p < \infty$). Also in [3], we show that $P(^mT^*)$ is a Tsirelson-like space in the sense that it is reflexive, contains no l_p, but lacks the unconditionality of its basis, that is, the monomials $(Z_J)_J$ cannot form an unconditional basis for $P(^mT^*)$.

Finally, we would like to thank Professor Richard Aron for a proof of the vector-valued version of theorem 7, where using some results of ε-product, the reflexivity of $P(^mE,F)$ can be reduced to the reflexivity of $L(^mE,F)$, for E and F reflexive B-spaces and F with the approximation property (see also Aron-Schottenloher [5]). However, we notice that the approach in the scalar case is distinct from the vector-valued case, where the compact mappings (in the usual sense) play an important role.

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