Journal of Mathematical Physics

Perturbations of KMS states and noncommutative L_p-spaces

Cite as: J. Math. Phys. **60**, 081707 (2019); https://doi.org/10.1063/1.5099066
Submitted: 05 April 2019 . Accepted: 09 August 2019 . Published Online: 28 August 2019

R. Correa da Silva

Journal of Mathematical Physics ARTICLE scitation.org/journal/jmp

Perturbations of KMS states and noncommutative L_p-spaces

Cite as: J. Math. Phys. 60, 081707 (2019); doi: 10.1063/1.5099066
Submitted: 5 April 2019 • Accepted: 9 August 2019 •
Published Online: 28 August 2019

View Online Export Citation CrossMark

R. Correa da Silva[a)]

AFFILIATIONS

Department of Mathematical Physics, Institute of Physics, University of São Paulo, Sao Paulo, SP 05508-900, Brazil

[a)]Electronic mail: ricardo.correa.silva@usp.br

ABSTRACT

We extend the theory of perturbations of KMS states to a class of unbounded perturbations using noncommutative L_p-spaces. We also prove certain stability of the domain of the modular operator associated with a $\|\cdot\|_p$-continuous state. This allows us to define an analytic multiple-time KMS condition and to obtain its analyticity together with some bounds to its norm.

I. INTRODUCTION

The problem in obtaining a KMS state for a perturbed Hamiltonian is very important in several areas of physics. This problem was solved by Araki in Ref. 1 when the perturbation is bounded, but, since it is often not the case for perturbations of physical interest, a similar result including unbounded perturbations is desired.

The "expansionals" play a very important role in Araki's perturbation theory. In fact, Araki named the operator in Definition II.5 expansional because it was so called by Fujiwara. Its importance relies on its relation with Dyson's series.

One of the most interesting properties for us is that

$$e^{it(A+B)}e^{-itB}\xi = Exp_l\left(\int_0^t; e^{isB}Ae^{-isB}ds\right)\xi, \qquad \xi \in \mathcal{H}, \tag{1}$$

where A and B are bounded operators in the von Neumann algebra $\mathfrak{M} \subset B(\mathcal{H})$. This formula can be extended for unbounded operators A and B. In fact, this formula holds for vectors $\xi \in G(B)$, the set of geometric vectors with respect to B, which are defined as the vectors with the property that there exists a positive constant M_ξ such that $\|B^n\xi\| \le M_\xi^n\|\xi\|$ if the operators A and B satisfy $AG(B) \subset G(B)$ and there exists $k \in \mathbb{R}$ such that, for each $t \in \mathbb{R}$, $\tau_t(A) = \left(e^{itB}Ae^{-itB}\right)$ is a bounded operator with $\|\tau_t(A)\| \le e^{k|t|}$. It is important to notice the physical interpretation that the operator in the left-hand side takes the vector back by the dynamics defined by B and evolves it by the dynamics defined by $A + B$. It is also possible to obtain a complex version of this formula, as can be seen in Ref. 2, Secs. 1.15–1.17. It is important to stress that we are most interested in the case the imaginary part of the exponent equals $\frac{1}{2}$ since this is the case in which we obtain the vector that represents the KMS state to the perturbed dynamics as can be checked in Ref. 1, Eq. (1.10).

After Araki's original work, some improvements have been done. One of them, due to Sakai, which extends the theory for bounded perturbations bounded from below. Sakai also developed an approach to the problem using derivations in C^*-algebras. Finally, in Ref. 3, an extension of the theory was presented where unbounded perturbations may be included as perturbations.

In this paper, we will present a different approach to Araki's perturbation theory using noncommutative L_p-spaces and which includes unbounded perturbations. It is important to notice that noncommutative L_p-spaces have been successfully used in linear response and constructive quantum field theory; see Refs. 4 and 5.

The main results are Theorems III.16, III.17, and Corollary III.19.

II. BACKGROUND

The aim of this section is to fix the notation and to present the basis needed to better understand our results. For the reader interested in more details, we refer to Ref. 6.

In this entire work, we denote by $\mathfrak{A}$ a C^*-algebra and by $\mathfrak{M}$ a von Neumann algebra. In addition, $\mathcal{H}$ denotes a Hilbert space and often we will suppose $\mathfrak{A}, \mathfrak{M} \subset B(\mathcal{H})$, *i.e.*, the algebras will be thought as concrete ones.

A. Modular theory

This section is devoted to present the definitions and main properties of the modular operator and the modular conjugation. This topic is a standard subject and can be found in classical books, e.g., Refs. 7, 8, and 9 or even in Ref. 10.

Let us now define two operators in $\mathfrak{M}$, which will give rise to the operators that give name to this section. For the cyclic and separating vector Ω, define the antilinear operators,

$$S_0 : \begin{array}{ccc} \{A\Omega \in \mathcal{H} \mid A \in \mathfrak{M}\} & \to & \mathcal{H} \\ A\Omega & \mapsto & A^*\Omega \end{array} \quad , \quad F_0 : \begin{array}{ccc} \{A'\Omega \in \mathcal{H} \mid A' \in \mathfrak{M}'\} & \to & \mathcal{H} \\ A'\Omega & \mapsto & A'^*\Omega \end{array}.$$

Note that the domains of the operators are dense subspaces. It is a standard result that the operators S_0 and F_0 are closable operators. Moreover, $S_0^* = \overline{F_0}$ and $F_0^* = \overline{S_0}$. We will denote $S = \overline{S_0}$ and $F = \overline{F_0}$.

An important point to stress now is that we omitted the dependence on Ω to keep the notation clean, but we will mention it in the following.

Moreover, even though S is not a bijection, it is injective and we will write S^{-1} (which is equal to S) to denote its inverse over its range. The same holds for Δ_Ω, which will be defined soon.

Definition II.1. We denote by J_Ω and Δ_Ω the unique antilinear partial isometry and positive operator, respectively, in the polar decomposition of S, i.e., $S = J_\Omega \Delta_\Omega^{\frac{1}{2}}$. J_Ω is called the modular conjugation, and Δ_Ω is called the modular operator.

Note that the existence and uniqueness of these operators are stated in the polar decomposition theorem.

Several properties hold for the modular operator, and we refer to Refs. 10, 7, and 11 for the reader interested in this subject.

One of the most important results in modular theory is the Tomita-Takesaki theorem, which is extremely significant to both physics and mathematics. The proof and applications of this theorem can be found in Refs. 9 and 7, Theorem 2.5.14. One of the consequences of this theorem is that, for each fixed $t \in \mathbb{R}$, $A \overset{\tau_t^\Omega}{\mapsto} \Delta_\Omega^{it} A \Delta_\Omega^{-it}$ defines an isometry of the algebra. Hence, $\{\tau_t^\Omega\}_{t\in\mathbb{R}}$ is a one-parameter group of isometries.

Definition II.2 (Modular automorphism group). Let $\mathfrak{M}$ be a von Neumann algebra with cyclic and separating vector Ω, and let Δ_Ω be the associated modular operator. For each $t \in \mathbb{R}$, define the isometry $\tau^\Omega : \mathfrak{M} \to \mathfrak{M}$ by $\tau_t^\Omega(A) = \Delta_\Omega^{it} A \Delta_\Omega^{-it}$. We call the modular automorphism group the one-parameter group $\{\tau_t^\Omega\}_{t\in\mathbb{R}}$.

Notation II.3. We will denote the modular automorphism group with respect to a cyclic and separating vector Ω by $\{\tau_t^\Omega\}_{t\in\mathbb{R}}$. In addition, given a faithful normal semifinite weight ϕ on a von Neumann algebra, we will denote $\{\tau_t^\phi\}_{t\in\mathbb{R}}$ the modular automorphism group with respect to the cyclic and separating vector obtained in the Gel'fand–Naimark–Segal (GNS)-construction.

The last comment we would like to add in this section is that in relativistic quantum field theory, due to the Reeh-Schlieder theorem, it is possible to obtain a modular operator for the algebra of local observables using the vacuum state; see Ref. 12 for more details.

B. KMS states and dynamical systems

This section is devoted to establishing the basic concepts and notation about dynamical systems in the operator algebras context needed in this article.

We denote by $(\mathfrak{M}, \alpha)$ the W^*-dynamical system with α as a one-parameter group, $\mathbb{R} \ni t \mapsto \alpha_t \in Aut(\mathfrak{M})$.

A general definition of KMS states can be found in any textbook of operator algebras such as Refs. 13, 14, and 9. For a discussion in the context of equilibrium states in the thermodynamic limit, we suggest Ref. 15. We will present this definition for completeness

Definition II.4. Let $(\mathfrak{M}, \tau)$ be a W^*-dynamical system and $\beta \in \mathbb{R}$. A normal state ω over $\mathfrak{M}$ is said to be a (τ, β)-KMS state, if, for any $A, B \in \mathfrak{M}$, there exists a complex function $F_{A,B}$ which is analytic in $\mathcal{D}_\beta = \{z \in \mathbb{C} \mid 0 < \operatorname{sgn}(\beta)\operatorname{Im}(z) < |\beta|\}$ and continuous on $\overline{\mathcal{D}_\beta}$ satisfying

$$\begin{aligned} F_{A,B}(t) &= \omega(A\tau_t(B)) \ \forall t \in \mathbb{R}, \\ F_{A,B}(t + i\beta) &= \omega(\tau_t(B)A) \ \forall t \in \mathbb{R}. \end{aligned} \tag{2}$$

We finish this section mentioning a result of most importance. KMS states "survive the thermodynamical limit," which means that under topology that has physical significance in this mathematical description and under the adequate convergence and continuity hypothesis, the limit of KMS states is again a KMS state.

C. Expansionals

An extensive study of expansionals can be found in Ref. 16.

Definition II.5. Let $\mathfrak{M}$ be a von Neumann algebra and $t \mapsto A(t) \in \mathfrak{M}$ be a strongly continuous function such that $\sup_{0\leq t\leq T} \|A(t)\| = r_A(T) < \infty$ for all $T \in \mathbb{R}_+$. For each $t \in \mathbb{R}_+$, define

$$Exp_r\left(\int_0^t; A(s)ds\right) = \sum_{n=0}^{\infty} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n A(t_n)\ldots A(t_1),$$
$$Exp_l\left(\int_0^t; A(s)ds\right) = \sum_{n=0}^{\infty} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n A(t_1)\ldots A(t_n),$$

where the term for $n = 0$ is the identity.

Note that these operators are well defined since $\|A(t_i)\| \leq r_A(t)$ for every $1 \leq i \leq n$ and $\int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n = \frac{t^n}{n!}$. Thus, the series converge absolutely (and uniformly over compact sets).

It is important to mention that these operators are basically the Dyson series. In fact, if given $(t_1, \ldots, t_n) \in \mathbb{R}^n$, we set a permutation σ: $\{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $t_{\sigma(n)} \leq t_{\sigma(n-1)} \leq \ldots t_{\sigma(1)}$, and we define the operators $T, \widetilde{T} : \mathfrak{M} \to \mathfrak{M}$ by

$$T(A(t_1)\ldots A(t_n)) = A(t_{\sigma(1)})\ldots A(t_{\sigma(n)}),$$
$$\widetilde{T}(A(t_1)\ldots A(t_n)) = A(t_{\sigma(n)})\ldots A(t_{\sigma(1)});$$

then,

$$Exp_r\left(\int_0^t; A(s)ds\right) = \sum_{n=0}^{\infty} \int_0^t dt_1 \ldots \int_0^t dt_n \frac{T(A(t_n)\ldots A(t_1))}{n!},$$
$$Exp_l\left(\int_0^t; A(s)ds\right) = \sum_{n=0}^{\infty} \int_0^t dt_1 \ldots \int_0^t dt_n \frac{\widetilde{T}(A(t_n)\ldots A(t_1))}{n!}.$$

The following proposition states some interesting properties of expansionals, for example, the cocycle property. Equation (1) also can be obtained from this proposition. For the proof and more details, see Ref. 16, Propositions 2, 3, 4, and 5 or 6.

Proposition II.6. Let $\mathfrak{M}$ be a von Neumann algebra and $t \mapsto A(t) \in \mathfrak{M}$ be a strong-continuous function such that $\sup_{0\leq t\leq T} \|A(t)\| = r_A(T) < \infty$ for all $T \in \mathbb{R}_+$. Then, the following properties hold:

(i) $\frac{d}{dt}Exp_r\left(\int_0^t; A(s)ds\right) = Exp_r\left(\int_0^t; A(s)ds\right)A(t);$
$\frac{d}{dt}Exp_l\left(\int_0^t; A(s)ds\right) = A(t)Exp_l\left(\int_0^t; A(s)ds\right);$

(ii) $Exp_l\left(\int_0^t; -A(s)ds\right)Exp_r\left(\int_0^t; A(s)ds\right) = \mathbb{1};$

(iii) $Exp_r\left(\int_0^t; A(s)ds\right)Exp_l\left(\int_0^t; -A(s)ds\right) = \mathbb{1};$

(iv) $Exp_r\left(\int_0^t; A(s)ds\right)Exp_r\left(\int_0^{t'}; A(s+t)ds\right) = Exp_r\left(\int_0^{t+t'}; A(s)ds\right);$
$Exp_l\left(\int_0^{t'}; A(s+t)ds\right)Exp_l\left(\int_0^t; A(s)ds\right) = Exp_l\left(\int_0^{t+t'}; A(s)ds\right).$

In the finite-dimensional case, every state ϕ has an associated density matrix ρ_ϕ for which $\rho(A) = \mathrm{Tr}(\rho_\phi A)$ for every matrix A. In this situation, the modular automorphism group became $\tau_t^\phi(A) = \rho_\phi^{it} A \rho_\phi^{-it}$, which is a $(\tau, 1)$-KMS state. Due to the Gibbs equilibrium state expression, we call the Hamiltonian of ϕ the operator $H_\phi = -\log(\rho_\phi)$. In this situation, the relative Hamiltonian of two states ϕ and ψ is defined by $H(\phi, \psi) = H_\phi - H_\psi$. In this context, it is well known that the two vectors representing Φ and Ψ representing, respectively, the states

ϕ and ψ are related by

$$\Phi = \sum_{i=0}^{\infty}(-1)^n \int_0^{\frac{1}{2}} \int_0^{t_1} \dots \int_0^{t_{n-1}} \Delta_\Psi^{t_n} H(\phi,\psi)\Delta_\Psi^{t_{n-1}-t_n} H(\phi,\psi)\cdots\Delta_\Psi^{t_1-t_2} H(\phi,\psi)\Psi. \tag{3}$$

Since the existence of the Hamiltonian depends on the existence of the trace (more specifically, it is connected with the Radon-Nikodym derivative of the states), it is more convenient to define the relative Hamiltonian of two states ϕ and ψ in a general von Neumann algebra as the operator $H(\phi,\psi)$ such that equation (3) holds.

In addition, it is known (see Ref. 1, Proposition 4.3) that the modular automorphism groups $\{\tau_t^\phi\}_{t\in\mathbb{R}}$ and $\{\tau_t^\psi\}_{t\in\mathbb{R}}$ of two states ϕ and ψ, respectively, when the relative Hamiltonian H exists, can be related as follows:

$$u_t^{\phi\psi} = Exp_r\left(\int_0^t ; -i\tau_s^\psi(H)ds\right),$$
$$\hat{u}_t^{\phi\psi} = Exp_l\left(\int_0^t ; i\tau_s^\psi(H)ds\right),$$

$$\left(u_t^{\phi\psi}\right)^* = \hat{u}_t^{\phi\psi},$$
$$u_t^{\phi\psi}\hat{u}_t^{\phi\psi} = \hat{u}_t^{\phi\psi}u_t^{\phi\psi} = \mathbb{1},$$
$$u_t^{\phi\psi}\tau_t^\psi(A) = \tau_t^\phi(A)\hat{u}_t^{\phi\psi}, \qquad A\in\mathfrak{M}.$$

D. Noncommutative L_p-Spaces

Noncommutative L_p-spaces are analogous to the Banach spaces of the p-integrable functions with respect to a measure. The study of these spaces goes back to the works of Segal[17] and Dixmier,[18] which depend on the existence of a normal faithful semifinite trace. It was just 25 years later that Haagerup in Ref. 19 proposed a generalization of the Segal-Dixmier L_p-spaces which included the type III von Neumann algebras. As a consequence of the study of spatial derivatives presented in Ref. 20, Araki and Masuda proposed a definition of noncommutative L_p-space based just on the Hilbert space of a concrete von Neumann algebra which is equivalent to Haagerup's construction.

It is interesting to note that noncommutative L_p-spaces are appearing more frequently as the best framework to describe some physical situations. See, for example, Refs. 21, 5, 22, 23, and 4.

Our interest in these spaces for a class of perturbations can be justified on two well known facts for classical L_p-spaces that still hold in the noncommutative case: they admit unbounded functions (operators) and have a useful Hölder duality property.

In this section, we will present the useful theory of noncommutative measures with respect to a normal faithful semifinite trace on a von Neumann algebra, which is the basis for the Segal-Dixmier noncommutative L_p-spaces (and, by the way, for noncommutative geometry). We use Ref. 24 very often in here.

Henceforth, we will denote by τ a trace, meaning a normal faithful semifinite trace. It is important to note that supposing the existence of such a trace restricts our options of algebras to the semifinite ones (not type III). Furthermore, from now on, $\mathfrak{M}_p$ will denote the set of all projections on the von Neumann algebra $\mathfrak{M}$, which is a complete lattice, provided the order $p\le q$ if, and only if, $pq=p$.

Given a von Neumann algebra $\mathfrak{M}\subset B(\mathcal{H})$, we say that a closed dense defined linear operator $A:\mathcal{D}(A)\to\mathcal{H}$ is affiliated to $\mathfrak{M}$ if, for every unitary operator $U\in\mathfrak{M}'$, $UAU^*=A$. We denote that an operator is affiliated to $\mathfrak{M}$ by $A\eta\mathfrak{M}$ and the set of all affiliated operators by $\mathfrak{M}_\eta$.

Definition II.7. Let $\mathfrak{M}\subset B(\mathcal{H})$ be a von Neumann algebra, τ be a normal faithful semifinite trace, and $\varepsilon,\delta>0$. Define

$$D(\varepsilon,\delta) = \left\{A\in\mathfrak{M}_\eta \;\middle|\; \begin{array}{l}\exists p\in\mathfrak{M}_p \text{such that} \\ p\mathcal{H}\subset\mathcal{D}(A), \|Ap\|\le\varepsilon \text{and } \tau(\mathbb{1}-p)\le\delta\end{array}\right\}.$$

Proposition II.8. Let $\mathfrak{M}\subset B(\mathcal{H})$ be a von Neumann algebra and τ be a normal faithful semifinite trace. A subspace $V\subset\mathcal{H}$ is τ-dense if, and only if, there exists an increasing sequence of projections $(p_n)_{n\in\mathbb{N}}\subset\mathfrak{M}_p$ such that $p_n\to\mathbb{1}$ and $\tau(\mathbb{1}-p_n)\to 0$ and $\bigcup_{n\in\mathbb{N}} p_n\mathcal{H}\subset V$.

Corollary II.9. Let $V_1, V_2\subset\mathcal{H}$ be τ-dense subspaces. Then, $V_1\cap V_2$ is τ-dense.

Definition II.10. Let $\mathfrak{M}\subset B(\mathcal{H})$ be a von Neumann algebra and τ be a normal faithful semifinite trace. A closed (densely defined) operator $A\in\mathfrak{M}_\eta$ is said τ-measurable if $\mathcal{D}(A)$ is τ-dense. We denote by $\mathfrak{M}_\tau$ the set of all τ-measurable operators.

Notice that by the previous proposition, if A is a τ-measurable operator and B extends A, we must have $A=B$. This, in turn, implies that a τ-measurable symmetric operator is self-adjoint.

Definition II.11. Let $\mathfrak{M} \subset B(\mathcal{H})$ be a von Neumann algebra and τ be a normal faithful semifinite trace. An operator $A\eta\mathfrak{M}$ is said τ-premeasurable if, $\forall\delta > 0$, there exists $p \in \mathfrak{M}_p$ such that $p\mathcal{H} \subset \mathcal{D}(A)$, $\|Ap\| < \infty$, and $\tau(\mathbb{1} - p) \leq \delta$.

An equivalent way to define a τ-premeasurable operator relies on $D(\varepsilon, \delta)$: A is τ-premeasurable if, and only if, $\forall\delta > 0$, and there exists $\varepsilon > 0$ such that $A \in D(\varepsilon, \delta)$.

Another interesting thing to notice is that a τ-premeasurable operator is densely defined since $D(A)$ must be τ-dense.

Proposition II.12. Let $\mathfrak{M} \subset B(\mathcal{H})$ be a von Neumann algebra, τ be a normal faithful semifinite trace, $A\eta\mathfrak{M}$ be a closed densely defined operator, and $\{E_{(\lambda,\infty)}\}_{\lambda\in\mathbb{R}_+}$ be the spectral resolution of $|A|$. The following are equivalent:

(i) *A is τ-measurable;*
(ii) *$|A|$ is τ-measurable;*
(iii) *$\forall\delta > 0$ $\exists\varepsilon > 0$ such that $A \in D(\varepsilon, \delta)$;*
(iv) *$\forall\delta > 0$ $\exists\varepsilon > 0$ such that $\tau\big(E_{(\varepsilon,\infty)}\big) < \delta$;*
(v) *$\lim\limits_{\lambda\to\infty} \tau\big(E_{(\lambda,\infty)}\big) = 0$;*
(vi) *$\exists\lambda_0 > 0$ such that $\tau\big(E_{(\lambda_0,\infty)}\big) < \infty$.*

Proposition II.13. $\mathfrak{M}_\tau$ provided with the usual scalar operations and involution, and the following vector operations, is a $*$-algebra:
(i) $A + B = \overline{A + B}$;
(ii) $A \times B = \overline{AB}$.

Proposition II.14. $\mathfrak{M}_\tau$ is a complete Hausdorff topological $*$-algebra with respect to the topology generated by the system of neighborhoods of zero,

$$\{\mathfrak{M}_\tau \cap D(\varepsilon, \delta)\}_{\varepsilon>0,\delta>0}.$$

Furthermore, $\mathfrak{M}$ is dense in $\mathfrak{M}_\tau$ in this topology. We will denote the balanced absorbing neighborhood of zero by $N(\varepsilon, \delta) = \mathfrak{M}_\tau \cap D(\varepsilon, \delta)$.

It is interesting to note that analyticity pervades almost every subject in von Neumann algebras. As a consequence of linearity and normality of the trace, we can use functional calculus and spectral theory to take advantage of the well known rigid behavior of analytic functions to prove the aforesaid inequalities. The details of the proofs can be found in Ref. 11, and we also refer to Ref. 25.

Lemma II.15. Let $\mathfrak{M}$ be a von Neumann algebra, τ be a normal faithful semifinite trace on $\mathfrak{M}$, $A \in \mathfrak{M}$, and $B \in \mathfrak{M}_\tau$. Then,

$$|\tau(AB)| \leq \tau(|AB|) \leq \|A\|\tau(|B|).$$

Theorem II.16 (Hölder inequality). *Let $\mathfrak{M}$ be a von Neumann algebra and τ be a normal faithful semifinite trace in $\mathfrak{M}$. Let also $A_i \in \mathfrak{M}$, $i = 1, \ldots, k$ and $\sum_{i=1}^{k} p_i > 1$ such that $\sum_{i=1}^{k} \frac{1}{p_i} = 1$, and then,*

$$\tau\left(\left|\prod_{i=1}^{k} A_i\right|\right) \leq \prod_{i=1}^{k} \tau(|A_i|^{p_i})^{\frac{1}{p_i}}.$$

The reader should keep in mind that Hölder's inequality is a very interesting result to us since it says something regarding the trace of a product and this is the case in Dyson's series. Nevertheless, it is used in the proof of the Minkowski inequality which is imperative to define a normed vector space.

Theorem II.17 (Minkowski's inequality). *Let $\mathfrak{M}$ be a von Neumann algebra, τ be a normal faithful semifinite trace in $\mathfrak{M}$, and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,*

(i) *for every $A \in \mathfrak{M}$, $\tau(|A|^p)^{\frac{1}{p}} = \sup\big\{|\tau(AB)| \,\big|\, B \in \mathfrak{M}, \tau\big(|B|^q\big) \leq 1\big\}$;*
(ii) *for every $A, B \in \mathfrak{M}$, $\|A + B\|_p \leq \|A\|_p + \|B\|_p$.*

Together, Theorems II.16 and II.17 provide us with another generalization of Hölder's inequality. This inequality is obvious in the commutative case but not in the noncommutative case.

Corollary II.18 (Hölder inequality). *Let $\mathfrak{M}$ be a von Neumann algebra and τ be a normal faithful semifinite trace in $\mathfrak{M}$, and let also $A, B \in \mathfrak{M}$ and $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then,*

$$\tau(|AB|^r)^{\frac{1}{r}} \leq \tau(|A|^p)^{\frac{1}{p}}\tau(|B|^q)^{\frac{1}{q}}.$$

Definition II.19. Let $\mathfrak{M} \subset B(\mathcal{H})$ be a von Neumann algebra and τ be a normal, faithful, and semifinite trace on $\mathfrak{M}$. We define the noncommutative L_p-space as

$$L_p(\mathfrak{M},\tau) = \left\{A \in \mathfrak{M}_\tau \;\middle|\; \tau\big(|A|^p\big) < \infty\right\},$$

provided with the norm $\|A\|_p = \tau\big(|A|^p\big)^{\frac{1}{p}}$.

We also set $L_\infty(\mathfrak{M},\tau) = \mathfrak{M}$ with $\|A\|_\infty = \|A\|$.

Now, it is easy to see that, for $p, q \geq 1$ Hölder conjugated, the Hölder and Minkowski inequalities can be extended to the whole space $L_p(\mathfrak{M},\tau)$ through an argument of density and normality of the trace. With this definition, Lemma II.15, and Corollary II.18, and Theorem II.17 can be expressed as

$$\|AB\|_1 \leq \|A\|_p \|B\|_q,$$
$$\|A+B\|_p \leq \|A\|_p + \|B\|_p,$$

and this last inequality is a triangular inequality for $\|\cdot\|_p$. It is important to notice that faithfulness guarantees $\|A\|_p = 0 \Rightarrow A = 0$; however, semifiniteness is used only at the very end of Theorem II.17 and it is completely irrelevant when talking about noncommutative L_p-spaces since the trace is never infinity on these operators.

It is possible to prove that (see Ref. 24, p 23), for a concrete von Neumann algebra,

$$\overline{\mathfrak{M} \cap L_p(\mathfrak{M},\tau)}^{\|\cdot\|_p} = L_p(\mathfrak{M},\tau). \tag{4}$$

In addition, by using a GNS construction, one can see that an equivalent definition for the noncommutative L_p-space related to an abstract von Neumann algebra is the completion of $\{A \in \mathfrak{M} \mid \tau(|A|^p) < \infty\}$ with respect to the norm $\|A\|_p = \tau\big(|A|^p\big)^{\frac{1}{p}}$.

Theorem II.20. *Let $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, the function below is an isometric isomorphism,*

$$\begin{aligned} \Xi : L_p(\mathfrak{M},\tau) &\to L_q(\mathfrak{M},\tau)^* \\ A \quad &\mapsto \quad \tau_A : \quad L_q(\mathfrak{M},\tau) \to \mathbb{C} \\ &\qquad\qquad\qquad B \mapsto \tau(AB). \end{aligned}$$

This last result is the famous identification $L_p(\mathfrak{M},\tau)^* = L_q(\mathfrak{M},\tau)$, where $p, q > 1$ are Hölder conjugated.

We do not intend to do a long presentation about the Radon-Nikodym theorem, to which the next proposition is somewhat related, but we need to write the next result since it is important to understand what is done here.

Proposition II.21. Let $\mathfrak{M}$ be a von Neumann algebra and ϕ be a faithful normal semifinite weight on $\mathfrak{M}$ and $H\eta\mathfrak{M}_\tau^+$. If $(H_i)_{i\in I} \in \mathfrak{M}_\tau^+$ is an increasing net such that $\sup_{i\in I} H_i = H$, then

$$\phi_H(A) \doteq \sup_{i\in I} \phi\Big(H_i^{\frac{1}{2}} A H_i^{\frac{1}{2}}\Big), \qquad A \in \mathfrak{M}, \tag{5}$$

defines a normal semifinite weight ϕ_H on $\mathfrak{M}$, which is independent of the choice of the net $(H_i)_{i\in I}$ with $\sup_{i\in I} H_i = H$. In addition, ϕ_H is faithful if, and only if, H is nonsingular.

Moreover, if $(H_i)_{i\in I}$ is an increasing net of positive operators affiliated with $\mathfrak{M}_\tau^+$ such that $\sup_{i\in I} H_i = H$, then

$$\phi_H = \sup_{i\in I} \phi_{H_i}.$$

Note that if $\mathfrak{M} \subset B(\mathcal{H})$, the conditions on the net in the previous theorem is equivalent to $H_i \to H$ in the Strong Operator Topology (SOT) because of Vigier's theorem. Hence, the next equation holds,

$$\phi_H(A) = \lim_{H_i \to H} \phi\Big(H_i^{\frac{1}{2}} A H_i^{\frac{1}{2}}\Big) = \lim_{H_i \to H} \phi(H_i A), \qquad A \in \mathfrak{M}. \tag{6}$$

III. PERTURBATION OF p-CONTINUOUS KMS STATES

The idea of extending Araki's perturbation theory using noncommutative L_p-spaces was proposed by Jäkel and consists in a new approach to the problem. Now, we start presenting the main results of this work. All that follows is entirely new.

It is quite clear that one of the key properties used in Refs. 1 and 2 to prove the convergence of Dyson's series, or in 16 to prove the convergence of the expansional, is that $\|A_1...A_n\| \leq \|A_1\|...\|A_n\|$, which is one of the axioms of Banach algebras. Unfortunately, this property does not hold in noncommutative L_p-spaces. In fact, in Banach algebras, these are not even algebras under the induced multiplication. In particular, we have $\|Q^n\| \leq \|Q\|^n$, but no similar property holds in noncommutative L_p-spaces.

Proposition III.1. Let $\mathfrak{M}$ be a von Neumann algebra, τ be a normal faithful semifinite trace on $\mathfrak{M}$, and $A \in L_1(\mathfrak{M},\tau)$. There exists $M > 0$ such that $\tau(|A|^n) \leq M^n$ for all $n \in \mathbb{N}$, if, and only if, $A \in \mathfrak{M}$.

Proof. ($\Rightarrow$) Let us prove the contrapositive. Suppose A is unbounded, and let $|A| = \int_0^\infty \lambda dE_\lambda^{|A|}$ be the spectral decomposition of $|A|$. For every $K > M$, $E_{(K,\infty)}$ is non-null, so $\tau(E_{(K,\infty)}) > 0$. Then,

$$\tau\left(|A|^n\right) = \int_0^\infty \lambda^n \tau\left(dE_\lambda^{|A|}\right) \geq \int_K^\infty \lambda^n \tau\left(dE_\lambda^{|A|}\right) \geq K^n \tau\left(E_{[K,\infty)}\right).$$

Now, we already know that there exists $N \in \mathbb{N}$ large enough such that, for all $n \geq N$, $M^n < K^n \tau\left(E_{[K,\infty)}\right)$.
($\Leftarrow$) The case $A = 0$ is trivial. Suppose $A \neq 0$ is bounded. Then,

$$\begin{aligned}
\tau\left(|A|^n\right) &= \tau\left(|A|^{n-1}|A|\right) \\
&\leq \left\||A|^{n-1}\right\|\tau(|A|) \\
&= \|A\|^n \frac{\tau(|A|)}{\|A\|} \\
&\leq \left(\|A\| \max\left\{1, \frac{\tau(|A|)}{\|A\|}\right\}\right)^n.
\end{aligned}$$

□

The next definition captures our intentions of having a convergent Dyson's series. In this definition, one subtle difference is that the exponent cannot be passed out the trace, what is the C^*-condition for $p = \infty$. On the physical point of view, we do not want the high order terms in perturbation to affect our system too much, at least its integral.

Definition III.2. Let $\mathfrak{M}$ be a von Neumann algebra, τ be a normal faithful semifinite trace on $\mathfrak{M}$, $1 \leq p \leq \infty$, and $0 < \lambda < \infty$. An operator $A \in L_p(\mathfrak{M},\tau)$ is said to be (τ, p, λ)-exponentiable if

$$\sum_{n=1}^{\infty} \frac{\lambda^n \||A|^n\|_p}{n!} < \infty. \tag{7}$$

Furthermore, an operator $A \in L_p(\mathfrak{M},\tau)$ is said to be (τ, p, ∞)-exponentiable if

$$\sum_{n=1}^{\infty} \frac{\lambda^n \||A|^n\|_p}{n!} < \infty, \qquad \forall \lambda \in \mathbb{R}_+. \tag{8}$$

We denote

$$\mathfrak{E}_{p,\lambda}^\tau = \{A \in L_p(\mathfrak{M},\tau) \mid A \text{ is } (\tau,p,\lambda) - \text{exponentiable}\}.$$

Some properties can be seen directly from the definition. The first is that, if $\lambda \leq \lambda'$, then $\mathfrak{E}_{p,\lambda}^\tau \subset \mathfrak{E}_{p,\lambda'}^\tau$. Another very useful property that we will use to simplify our presentation is that

$$\mathfrak{E}_{p,\lambda}^\tau = \lambda\mathfrak{E}_{p,1}^\tau = \left\{\lambda A \in L_p(\mathfrak{M}) \,|A \in \mathfrak{E}_{p,1}^\tau\right\}.$$

So, the only special case is $\mathfrak{E}_{p,\infty}^\tau$, for which we have $\mathfrak{E}_{p,\infty}^\tau = \bigcap_{\lambda\in\mathbb{R}_+} \mathfrak{E}_{p,\lambda}^\tau$. Hence, it is enough to study $\mathfrak{E}_{p,1}^\tau$ and $\mathfrak{E}_{p,\infty}^\tau$.

Notation III.3. In order to simplify the notation, we will denote $\mathfrak{E}_{p,1}^\tau = \mathfrak{E}_p^\tau$ and call a $(\tau, p, 1)$-exponentiable operator just a (τ, p)-exponentiable operator.

Remark III.4. Note that Eq. (7) can be written in many forms for $1 \le p < \infty$,

$$\sum_{n=1}^{\infty} \frac{\| |A|^n \|_p}{n!} = \sum_{n=1}^{\infty} \frac{\|A\|_{np}^n}{n!} = \sum_{n=1}^{\infty} \frac{\tau(|A|^{np})^{\frac{1}{p}}}{n!} < \infty.$$

We prefer Eq. (7) because it also includes the case $p = \infty$, for which

$$\sum_{n=1}^{\infty} \frac{\| |A|^n \|_\infty}{n!} \le \sum_{n=1}^{\infty} \frac{\|A\|_\infty^n}{n!} = e^{\|A\|} - 1 < \infty.$$

Hence, we have $\mathfrak{E}_\infty^\tau = \mathfrak{M}$.

In order to simplify calculations in our examples and constructions, we prove the following lemma.

Lemma III.5. Let $\mathfrak{M}$ be a von Neumann algebra and τ be a normal faithful semifinite trace on $\mathfrak{M}$. Then, $A \in \mathfrak{E}_p^\tau$ if

$$\sum_{n=1}^{\infty} \frac{1}{n!} \tau\big(|A|^{np}\big) < \infty.$$

Proof. Define

$$N_+ = \big\{n \in \mathbb{N} \,\big|\, \tau\big(|A|^{pn}\big) > 1\big\},$$
$$N_- = \big\{n \in \mathbb{N} \,\big|\, \tau\big(|A|^{pn}\big) \le 1\big\}.$$

It is clear that

$$\begin{aligned}\sum_{n=1}^{N} \frac{\|A\|_{np}^n}{n!} &= \sum_{n=1}^{N} \frac{1}{n!} \tau\big(|A|^{pn}\big)^{\frac{1}{p}} \\ &= \sum_{\substack{n \in N_- \\ n \le N}} \frac{1}{n!} \tau\big(|A|^{pn}\big)^{\frac{1}{p}} + \sum_{\substack{n \in N_+ \\ n \le N}} \frac{1}{n!} \tau\big(|A|^{pn}\big)^{\frac{1}{p}} \\ &\le \sum_{n=1}^{\infty} \frac{1}{n!} + \sum_{n=1}^{\infty} \frac{1}{n!} \tau\big(|A|^{pn}\big).\end{aligned} \tag{9}$$

The next step is to prove that the set we just defined is big, in some sense, in $L_p(\mathfrak{M}, \tau)$.

Proposition III.6. $\mathfrak{E}_p^\tau$ and $\mathfrak{E}_\infty^\tau$ are $\|\cdot\|$-dense in $L_p(\mathfrak{M}, \tau)$.

Proof. It is enough to prove that $\mathfrak{E}_p^\tau$ is $\|\cdot\|$-dense in $L_p(\mathfrak{M}, \tau)$. Let $A \in L_p(\mathfrak{M}, \tau)$ be a positive operator, and let its spectral decomposition be $A = \int_0^\infty \lambda dE_\lambda$. Define $A_m = \int_0^m \lambda dE_\lambda$. Then, for all $n \in \mathbb{N}$,

$$\begin{aligned}\tau\big(\big(A_m^p\big)^n\big) &= \int_0^m \lambda^{pn} \tau(dE_\lambda) \\ &= \int_0^1 \lambda^{pn} \tau(dE_\lambda) + \int_1^m \lambda^{pn} \tau(dE_\lambda) \\ &\le \int_0^1 \lambda^{p} \tau(dE_\lambda) + m^{p(n-1)} \int_1^m \lambda^{p} \tau(dE_\lambda) \\ &\le m^{p(n-1)} \int_0^m \lambda^{p} \tau(dE_\lambda) \\ &= m^{p(n-1)} \tau\big(|A_m|^p\big).\end{aligned}$$

Hence, $(A_m)_{m \in \mathbb{N}}$ is a sequence of (τ, p, ∞)-exponentiable operators and

$$\tau\big(|A - A_m|^p\big) = \int_m^\infty \lambda^p \tau(dE_\lambda) \xrightarrow{n \to \infty} 0.$$

For the general case, just remember that the polarization identity implies that every operator is a linear combination of four positive operators.

Note that Proposition III.6 shows that $\mathfrak{M} \cap L_p(\mathfrak{M}, \tau) \subset \mathfrak{E}^{\tau}_{p,\infty}$ and $\|A^n\|_p \leq \max\{1, \|A\|^{n-1}\|A\|_p\}$ for $A \geq 0$. It is not difficult to see that the conclusion could also be obtained by Lemma II.15 and Eq. (4), which is in fact proved using an argument similar to what we have used above.

This comment raises doubts about the possible "triviality" of $\mathfrak{E}^{\tau}_p$ or $\mathfrak{E}^{\tau}_{p,\infty}$. I mean, although we have already proved that these sets are big enough to be dense, the set we used to prove density consists of bounded operators. The next example will answer this question.

Example III.7. Consider a function $f : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} m \text{ if } \frac{1}{(m+1)!} \leq |x| < \frac{1}{m!}, \ m \in \mathbb{N}, \\ 0 \ \text{ if } |x| \geq 1. \end{cases}$$

This is a positive unbounded integrable function with compact support in $\mathbb{R}$ and, for each $\lambda > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} (\lambda f(x))^n dx = 2\frac{\left(e^{e^{\lambda}} - 1\right)\left(e^{\lambda} - 1\right)}{e^{\lambda}}.$$

Of course, we do not need the exact result and the reader can check that it is obvious that this sum would be less than $e^{e^{\lambda}}$.

For a measurable set $K \in \mathbb{R}\setminus\{0\}$ such that 0 is an accumulation point of K, the restriction of f to K is an example of an unbounded $\left(\int_K \cdot dx, 1, \infty\right)$-exponentiable operator of $L_1\left(L_\infty(K), \int_K \cdot dx\right)$.

It is obvious that any integrable function dominated by the previous one is also $\left(\int_K \cdot dx, 1, \infty\right)$-exponentiable.

One could wonder if $\mathfrak{E}^{\tau}_p$ is a vector space or not. The following example shows the answer is negative. Another consequence of these examples is that, for $\lambda < \lambda'$, $\mathfrak{E}^{\tau}_{p,\infty} \subsetneq \mathfrak{E}^{\tau}_{p,\lambda} \subsetneq \mathfrak{E}^{\tau}_{p,\lambda'}$. This is very important and nontrivial since it means that $\left\{\mathfrak{E}^{\tau}_p\right\}_{p \in \overline{R}_+}$ or even $\left\{\mathfrak{E}^{\tau}_{p,\lambda}\right\}_{p \in \overline{\mathbb{R}}_+, \lambda \in \overline{\mathbb{R}}_+}$ are, in general, nontrivial gradations of $\mathfrak{M} = \mathfrak{E}^{\tau}_{\infty,\lambda} = \mathfrak{E}^{\tau}_{\infty,\lambda}$ for every $\lambda \in \overline{\mathbb{R}}_+$.

Example III.8. Consider a function $f : \mathbb{R}\setminus\{0\} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} m \text{ if } (2e)^{-m-1} \leq |x| < (2e)^{-m}, \ m \in \mathbb{N}, \\ 0 \ \text{ if } |x| \geq 2e. \end{cases}$$

This is a positive unbounded integrable function with compact support in $\mathbb{R}$ and

$$\sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} f(x)^n dx = \frac{2(e-1)}{e}.$$

Hence, again we have that for any measurable set $K \in \mathbb{R}\setminus\{0\}$, the restriction of f to K is an example of a $\left(\int_K \cdot dx\right)$-exponentiable operator for $L_1\left(L_\infty(K), \int_K \cdot dx\right)$, but it does not hold for $2f$. In fact,

$$\sum_{n=1}^{N} \frac{1}{n!} \int_{\mathbb{R}} f(x)^n dx = \frac{4e}{2e-1} \sum_{n=1}^{N} \frac{1}{n!} \sum_{m=1}^{\infty} (2m)^n (2e)^{-m}$$

is a divergent series.

Although we have presented examples just for $p = 1$, it is enough to take the pth root of f to obtain examples for any $p > 1$.

Example III.9. In order to construct an example in a noncommutative von Neumann algebra, it is sufficient that there exists a monotonic decreasing sequence of projections $(P_n)_{n\in\mathbb{N}} \in \mathfrak{M}_p$ such that $P_n \xrightarrow{\|\cdot\|_1} 0$, which is true if there exists any τ-measurable unbounded operator.

In fact, fix $1 \leq p$. If there exists such a sequence, we can suppose without loss of generality, by taking a subsequence if necessary, that $\tau(P_n) \leq \frac{1}{(e^n-1)2^n}$. Define the positive unbounded τ-measurable operator

$$A = \sum_{n=1}^{\infty} n^{\frac{1}{p}} (P_n - P_{n+1}).$$

It follows from the definition that

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{1}{m!} \tau(|A|^{pm}) &= \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{n=1}^{\infty} n^m \tau(P_n - P_{n+1}) \\ &\leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n^m}{m!} \frac{1}{(e^n-1)2^n} \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} = 1. \end{aligned}$$

Thus, $A \in \mathfrak{E}_p^\tau$.

It is not difficult to see, with the help of the spectral decomposition, that if an operator is in $L_p(\mathfrak{M},\tau) \cap L_q(\mathfrak{M},\tau)$ with $1 \le p < q < \infty$, then it is in $L_r(\mathfrak{M},\tau)$ for every $p \le r \le q$. More than that, it follows by analyticity and the three-line theorem, a special case of the Riesz-Thorin theorem, that

$$\begin{aligned} &\|A\|_r \le \|A\|_p^{\frac{p}{(q-p)}\left(\frac{q}{r}-1\right)} \|A\|_q^{\frac{q}{(q-p)}\left(1-\frac{p}{r}\right)}, && \text{if } q < \infty; \\ &\|A\|_r \le \|A\|_p^{\frac{p}{r}} \|A\|_\infty^{1-\frac{p}{r}}, && \text{if } q = \infty. \end{aligned} \tag{10}$$

An analogous property holds for (τ, p)-exponentiable operators.

Proposition III.10. Let $1 \le p < q \le \infty$, and then $\mathfrak{E}_p^\tau \cap \mathfrak{E}_q^\tau \subset \mathfrak{E}_r^\tau$ for every $p \le r \le q$.

Proof. Let $A \in \mathfrak{E}_p^\tau \cap \mathfrak{E}_q^\tau$, in particular, $|A|^n \in L_p(\mathfrak{M},\tau) \cap L_q(\mathfrak{M},\tau)$ for all $n \in \mathbb{N}$.
Using Eq. (10), we get that, for all $p \le r \le q$,

$$\||A|^n\|_r \le \max\left\{\||A|^n\|_p, \||A|^n\|_q\right\} \le \||A|^n\|_p + \||A|^n\|_q,$$

$$\begin{aligned} \sum_{n=1}^{N} \frac{\||A|^n\|_r}{n!} &= \sum_{n=1}^{N} \frac{\||A|^n\|_p + \||A|^n\|_q}{n!} \\ &= \sum_{n=1}^{\infty} \frac{\||A|^n\|_p}{n!} + \sum_{n=1}^{\infty} \frac{\||A|^n\|_q}{n!} \\ &< \infty. \end{aligned}$$

Although $\mathfrak{E}_p^\tau$, in general, are not vector spaces, they still have a very convenient geometric structure for perturbations. □

Proposition III.11. (i) *$\mathfrak{E}_p^\tau$ is a balanced and convex set;*

(ii) *for every $A \in \mathfrak{E}_p^\tau$ and $B \in \mathfrak{M}$ with $\|B\| \le 1$, $BA \in \mathfrak{E}_p^\tau$;*

(iii) *if $1 \le p, q, r \le \infty$ are such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $A, B \in \mathfrak{M}_\tau$,*

$$\sum_{n=1}^{\infty} \frac{\tau(|A|^{np})}{n!} \quad , \quad \sum_{n=1}^{\infty} \frac{\tau(|B|^{nq})}{n!} < \infty \Rightarrow \sum_{n=1}^{\infty} \frac{\tau(|AB|^{nr})}{n!} < \infty;$$

(iv) *$\mathfrak{E}_{p,\infty}^\tau$ is a subspace of $L_p(\mathfrak{M})$.*

Proof. (i) It is obvious that, for $A \in \mathfrak{E}_p^\tau$ and $|\lambda| \le 1$, we have

$$\sum_{n=1}^{\infty} \frac{\|\lambda A\|_{np}^n}{n!} = \sum_{n=1}^{\infty} \frac{|\lambda|^n \|A\|_{np}^n}{n!} \le \sum_{n=1}^{\infty} \frac{\|A\|_{np}^n}{n!}.$$

Let $A, B \in \mathfrak{E}_p^\tau$, and let $0 < \lambda < 1$. Then, thanks to convexity of $x \mapsto x^n$ for each $n \in \mathbb{N}$,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\|\lambda A + (1-\lambda)B\|_{np}^n}{n!} &\le \sum_{n=1}^{\infty} \frac{1}{n!}\Big(\lambda\|A\|_{np} + (1-\lambda)\|B\|_{np}\Big)^n \\ &\le \sum_{n=1}^{\infty} \frac{1}{n!}\Big(\lambda\|A\|_{np}^n + (1-\lambda)\|B\|_{np}^n\Big) \\ &\le \sum_{n=1}^{\infty} \frac{1}{n!}\|A\|_{np}^n + \sum_{n=1}^{\infty} \frac{1}{n!}\|B\|_{np}^n. \end{aligned}$$

(ii) It follows trivially from (i) in Theorem II.17.

(iii) It follows from Corollary II.18 that

$$\begin{aligned}\sum_{n=1}^{N}\frac{\tau(|AB|^{nr})}{n!} &\leq \sum_{n=1}^{N}\frac{1}{n!}\tau\left(|A|^{np}\right)^{\frac{r}{p}}\tau\left(|B|^{nq}\right)^{\frac{r}{q}}\\ &= \sum_{n=1}^{N}\left(\frac{\tau(|A|^{np})}{n!}\right)^{\frac{r}{p}}\left(\frac{\tau(|B|^{nq})}{n!}\right)^{\frac{r}{q}}\\ &\leq \left(\sum_{n=1}^{N}\frac{\tau(|A|^{np})}{n!}\right)^{\frac{r}{p}}\left(\sum_{n=1}^{N}\frac{\tau(|B|^{nq})}{n!}\right)^{\frac{r}{q}}.\end{aligned}$$

(iv) Note that $A \in \mathfrak{E}^{\tau}_{p,\infty}$ if, and only if, $\lambda A \in \mathfrak{E}^{\tau}_{p}$ for every $\lambda \in \mathbb{R}$. If $\alpha = \beta = 0$, the result is obvious; otherwise, it follows from item (*i*) that, if $\alpha, \beta \in \mathbb{C}$ and $A, B \in \mathfrak{E}^{\tau}_{p,\infty}$,

$$\begin{aligned}\alpha A + \beta B &= (|\alpha| + |\beta|)\left(\frac{|\alpha|}{|\alpha|+|\beta|}\left(\frac{\alpha}{|\alpha|}A\right) + \frac{|\beta|}{|\alpha|+|\beta|}\left(\frac{\beta}{|\beta|}B\right)\right)\\ &\in (|\alpha|+|\beta|)\mathfrak{E}^{\tau}_{p} \subset \mathfrak{E}^{\tau}_{p,\infty}.\end{aligned}$$

□

The following lemma justifies the choice of the name "exponentiable" for such operators.

Lemma III.12. For each $A \in \mathfrak{E}^{\tau}_{p,\lambda}$ and $B\eta\mathfrak{M}$ self-adjoint, define $A(t) = B^{it}AB^{-it}$. Then, for $0 \leq t < \lambda$,

$$\mathbb{1} - Exp_r\left(\int_0^t; A(s)ds\right) \quad and \quad \mathbb{1} - Exp_l\left(\int_0^t; A(s)ds\right) \in L_p(\mathfrak{M},\tau).$$

Proof. Since $A \in \mathfrak{M}_\tau \Rightarrow A(t) \in \mathfrak{M}_\tau$, Proposition II.14 implies that each term in the definition of these operators is in $\mathfrak{M}_\tau$ except for the identity.

In addition, using Theorem II.16 for $p_i = n$, $i = 1, \ldots, n$, we have, for every $N > M$, that

$$\begin{aligned}&\left\|\sum_{n=N}^{M}\int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n A(t_n)\ldots A(t_1)\right\|_p\\ &\leq \sum_{n=N}^{M}\left\|\int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n A(t_n)\ldots A(t_1)\right\|_p\\ &\leq \sum_{n=N}^{M}\tau\left(\left|\int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n A(t_n)\ldots A(t_1)\right|^p\right)^{\frac{1}{p}}\\ &\leq \sum_{n=N}^{M}\int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \tau\left(|A(t_n)\ldots A(t_1)|^p\right)^{\frac{1}{p}}\\ &= \sum_{n=N}^{M}\int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \tau\left(|A|^{pn}\right)^{\frac{1}{p}}\\ &= \sum_{n=N}^{M}\frac{t^n}{n!}\tau\left(|A|^{pn}\right)^{\frac{1}{p}},\end{aligned} \tag{11}$$

which shows simultaneously that each term is in $L_p(\mathfrak{M},\tau)$ and the partial sum is a $\|\cdot\|_p$-Cauchy sequence. The thesis follows by completeness.

□

To clarify the next definition, remember that $\mathfrak{M} \cap L_p(\mathfrak{M},\tau)$ [or even $\mathfrak{M} \cap L_1(\mathfrak{M},\tau)$] is $\|\cdot\|_p$-dense in $L_p(\mathfrak{M},\tau)$.

Definition III.13. Let $\mathfrak{M}$ be a von Neumann algebra and τ be a faithful normal semifinite trace on $\mathfrak{M}$. We say that a state ϕ on $\mathfrak{M}$ is $\|\cdot\|_p$-continuous if it is continuous on $\left(\mathfrak{M} \cap L_p(\mathfrak{M},\tau), \|\cdot\|_p\right)$.

Of course, such a weight can be continuously extended to $\left(L_p(\mathfrak{M},\tau), \|\cdot\|_p\right)$ in a unique way.

Hitherto, we have defined a set, namely, $\mathfrak{E}_p^\tau$, that we assert is the right set to take our perturbation, but the reader should be warned after so many comments about the duality relations between L_p-spaces that we will demand some extra "dual" property on the original state. This motivates our next definition.

Proposition III.14. Let $\mathfrak{M}$ be a von Neumann algebra and τ be a faithful normal semifinite trace on $\mathfrak{M}$ and $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. A state ϕ on $\mathfrak{M}$ is $\|\cdot\|_p$-continuous if, and only if, there exists $H \in L_q(\mathfrak{M}, \tau)$, $H\eta\mathfrak{M}_{\tau^\phi}$, such that

$$\phi(A) = \tau_H(A) \quad \forall A \in \mathfrak{M},$$

in the sense of Proposition II.21.

Proof. As mentioned in Definition III.14, we can continuously extend ϕ to $L_p(\mathfrak{M}, \tau)$. By the dual relation between $L_p(\mathfrak{M}, \tau)$ and $L_q(\mathfrak{M}, \tau)$ stated in Theorem II.20, there exists $H \in L_q(\mathfrak{M}, \tau)$ such that $\phi(A) = \tau(HA)$ for all $A \in L_p(\mathfrak{M}, \tau)$. Such H must be affiliated with $\mathfrak{M}_{\tau^\phi}$.

In particular, $\phi(A) = \tau(HA)$ for all $A \in \mathfrak{M} \cap L_p(\mathfrak{M}, \tau)$, but $\mathfrak{M} \cap L_p(\mathfrak{M}, \tau)$ is WOT-dense in $\mathfrak{M}$ since the trace is semifinite.

The cases $p = 1, \infty$ are analogous, and the other part of the equivalence is trivial. □

The next two theorems can be seen as the key to guarantee Dyson's series that is convergent. It is time to stress how important Araki's multiple-time KMS condition is for the theory. Here, it is used with the same purposes of the original Araki's article, Ref. 1. Mentioning an interesting connection, this property is also used in Araki's noncommutative L_p-spaces, what makes us believe there is a natural way to extend this result.

Notation III.15. (i) Let $\Omega \subset \mathbb{R}^n$ be a convex domain (i.e., an open convex set). We define the tube over Ω by

$$T(\Omega) = \left\{z \in \mathbb{C}^n \,\middle|\, \mathrm{Im}(z) \in \Omega\right\}.$$

The following convex domain will play a relevant role to our purposes:

$$S_\alpha^n \doteq \left\{(t_1, \ldots, t_n) \in \mathbb{R}^n \,\middle|\, t_i < 0,\ 1 \leq i \leq n, \text{and } -\alpha < \sum_{i=1}^n t_i < 0\right\}.$$

(ii) $\mathfrak{M}_\mathcal{A}$ will denote the set of analytic elements for the (one-parameter) modular automorphism group.

Theorem III.16. *Let $\mathfrak{M}$ be a von Neumann algebra, ϕ be a faithful state in $\mathfrak{M}$, and $n \in \mathbb{N}$. Let also $(\mathcal{H}_\phi, \Phi, \pi_\phi)$ be the GNS representation throughout ϕ, τ be a normal faithful semifinite trace on $B(\mathcal{H}_\phi)$, and $Q_i, J_\phi Q_i J_\phi \in L_{2mq}(B(\mathcal{H}_\phi), \tau)$ such that $\|J_\phi Q_i J_\phi\|_{2mq} = \|Q_i\|_{2mq}$ for all $1 \leq i, m \leq n$, and suppose $\phi = \tau_H$ is $\|\cdot\|_p$-continuous. Then, if $\Phi \in \mathcal{D}(Q_1)$ and $\Delta_\Phi^{iz_{j-1}} Q_{j-1} \ldots \Delta_\Phi^{iz_1} Q_1 \Phi \in \mathcal{D}(Q_j)$ for every $-\frac{1}{2} \leq \mathrm{Im}(z_j) \leq 0$ and for every $2 \leq j \leq n$,*

$$Q_n \Delta_\Phi^{iz_{n-1}} Q_{n-1} \ldots \Delta_\Phi^{iz_1} Q_1 \Phi \in \mathcal{D}\left(\Delta_\Phi^{iz}\right) \text{ for } -\frac{1}{2} \leq \mathrm{Im}(z) \leq 0 \text{ and}$$

$$A^n(z_1, \ldots, z_n)\Phi \doteq \Delta_\Phi^{iz_n} Q_n \Delta_\Phi^{iz_{n-1}} Q_{n-1} \ldots \Delta_\Phi^{iz_1} Q_1 \Phi$$

is analytic on $T\left(S_{\frac{1}{2}}^n\right)$ and bounded on its closure by

$$\left\|A^n(z_1, \ldots, z_n)\Phi\right\| \leq \|H\|_p^{\frac{1}{2}} \prod_{j=1}^n \|Q_i\|_{2nq}.$$

Proof. Let us proceed by induction on n.

For $n = 1$, let $Q_1 = U|Q_1|$ be the polar decomposition of Q_1 and $|Q_1| = \int_0^\infty \lambda dE_\lambda^{|Q_1|}$ be the spectral decomposition of $|Q_1|$. Since $\Phi \in \mathcal{D}(Q_1)$, $Q_1\Phi = U \lim_{k\to\infty} Q_{1,k}\Phi$, where $Q_{1,k} = \int_0^k \lambda dE_\lambda^{|Q_1|}$. Define the following functionals on $\mathfrak{M}_\mathcal{A}\Phi$:

$$f_k^z(A\Phi) \doteq \left\langle UQ_{1,k}\Phi, \Delta_\Phi^{-i\bar{z}} A\Phi\right\rangle_\phi,$$

$$f^z(A\Phi) \doteq \lim_{k\to\infty} f_k^z(A\Phi) = \left\langle Q_1\Phi, \Delta_\Phi^{-i\bar{z}} A\Phi\right\rangle_\phi.$$

Of course, for fixed $A\Phi$, $\bar{f}_k(z) = \overline{f_k^z(A\Phi)}$ is entire analytic and

$$\begin{aligned}
\left|\bar{f}(t)\right| &= \lim_{k\to\infty}\left|\bar{f}_k(t)\right| \\
&= \lim_{k\to\infty}\left|\left\langle \Delta_\Phi^{-it} A\Phi, UQ_{1,k}\Phi\right\rangle_\phi\right| \\
&\le \left\|\Delta_\Phi^{-it} A\Phi\right\| \lim_{k\to\infty} \|UQ_{1,k}\Phi\| \\
&\le \|A\Phi\| \lim_{k\to\infty} \phi\left(Q_{1,k}U^* UQ_{1,k}\right)^{\frac{1}{2}} \\
&\le \|A\Phi\| \lim_{k\to\infty} \tau\left(H^{\frac{1}{2}} Q_{1,k}U^* UQ_{1,k}H^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&\le \|A\Phi\| \|H\|_p^{\frac{1}{2}} \||Q_1|^2\|_q^{\frac{1}{2}} \\
&\le \|A\Phi\| \|H\|_p^{\frac{1}{2}} \|Q_1\|_{2q}, \quad \forall t \in \mathbb{R}.
\end{aligned} \tag{12}$$

Moreover,

$$\begin{aligned}
\left|\bar{f}\left(t + \frac{1}{2}it\right)\right| &= \lim_{k\to\infty}\left|\bar{f}_k\left(t + \frac{1}{2}it\right)\right| \\
&= \lim_{k\to\infty}\left\langle \Delta_\Phi^{\frac{1}{2}}\Delta_\Phi^{-it} A\Phi, UQ_{1,k}\Phi\right\rangle_\phi \\
&\le \left\|\Delta_\Phi^{-it} A\Phi\right\| \lim_{k\to\infty} \|J_\Phi Q_{1,k}U^*\Phi\| \\
&= \|A\Phi\| \lim_{k\to\infty} \|Q_{1,k}U^*\Phi\| \\
&\le \|A\Phi\| \lim_{k\to\infty} \phi\left(UQ_{1,k}Q_{1,k}U^*\right)^{\frac{1}{2}} \\
&\le \|A\Phi\| \lim_{k\to\infty} \tau\left(H^{\frac{1}{2}} UQ_{1,k}^2U^* H^{\frac{1}{2}}\right)^{\frac{1}{2}} \\
&\le \|A\Phi\| \|H\|_p^{\frac{1}{2}} \||Q_1^*|^2\|_q^{\frac{1}{2}} \\
&\le \|A\Phi\| \|H\|_p^{\frac{1}{2}} \||Q_1|^2\|_q^{\frac{1}{2}} \\
&\le \|A\Phi\| \|H\|_p^{\frac{1}{2}} \|Q_1\|_{2q}, \quad \forall t \in \mathbb{R},
\end{aligned} \tag{13}$$

which proves that the functional concerned is bounded for $-\frac{1}{2} \le \mathrm{Im}(z) \le 0$ due to the maximum modulus principle. This bound also proves that if $\bar{f}_k \to \bar{f}$ uniformly for $-\frac{1}{2} \le \mathrm{Im}(z) \le 0$, then $\bar{f}$ is analytic for $-\frac{1}{2} < \mathrm{Im}(z) < 0$ and bounded for $-\frac{1}{2} \le \mathrm{Im}(z) \le 0$.

Using first the Hahn-Banach theorem to obtain an extension (also denoted by f_z) to the whole Hilbert space in such a way that $\|f_z\| \le \|H\|_p^{\frac{1}{2}} \|Q_1\|_{2q}$, we know by the Riesz representation theorem that there exists a $\Omega(z) \in \mathcal{H}_\phi$ such that $f_z(\cdot) = \langle \Omega(z), \cdot\rangle_\phi$. Since $\mathfrak{M}_{\mathcal{A}}\Phi$ is dense, $\Omega(z)$ is unique.

So far, we have that $Q_1\Phi \in \mathcal{D}\left(\left(\Delta_\Phi^{-i\bar{z}}\right)^*\right) = \mathcal{D}\left(\Delta_\Phi^{iz}\right)$ and $\bar{f}(z) = \left\langle A\Phi, \Delta_\Phi^{iz} Q_1\Phi\right\rangle_\phi$ is analytic on $\left\{z \in \mathbb{C} \,\middle|\, -\frac{1}{2} < \mathrm{Im}(z) < 0\right\}$ and continuous on its closure, for every $A \in \mathfrak{M}_{\mathcal{A}}$.

Since $\overline{\mathfrak{M}_{\mathcal{A}}\Phi}^{\|\cdot\|} = \overline{\mathfrak{M}\Phi}^{\|\cdot\|} = \mathcal{H}_\phi$, the vector-valued function $A(z)\Phi \doteq \Delta_\Phi^{iz} Q_1\Phi$ is weak analytic, hence strong analytic on $\left\{z \in \mathbb{C} \,\middle|\, -\frac{1}{2} < \mathrm{Im}(z) < 0\right\}$ and

$$\|A(z)\Phi\| \le \|H\|_p^{\frac{1}{2}} \|Q_1\|_{2q} \quad \forall z \in \left\{z \in \mathbb{C} \,\middle|\, -\frac{1}{2} \le \mathrm{Im}(z) \le 0\right\}.$$

Suppose now the hypothesis hold for $n \in \mathbb{N}$. We will use the same ideas: let $Q_{n+1} = U|Q_{n+1}|$ be the polar decomposition of Q_{n+1} and $|Q_{n+1}| = \int_0^\infty \lambda dE_\lambda^{|Q_{n+1}|}$ be the spectral decomposition of $|Q_{n+1}|$. Since $\Phi \in \mathcal{D}(Q_{n+1})$, $Q_{n+1}\Phi = U \lim_{k\to\infty} Q_{n+1,k}\Phi$,

$$f^{(z_1,\ldots,z_{n+1})}(A\Phi) = \left\langle Q_{n+1}\Delta_\Phi^{iz_n} Q_n \ldots \Delta_\Phi^{iz_1} Q_1\Phi, \Delta_\Phi^{-i\bar{z}_{n+1}} A\Phi\right\rangle_\phi.$$

Since $\bar{f}_k(z_1,\ldots,z_{n+1}) \doteq \overline{f_k^{(z_1,\ldots,z_{n+1})}(A\Phi)}$ is an analytic function, it attains its maximum at an extremal point of $S_{\frac{1}{2}}^{n+1}$ (see Ref. 1 Corollary 2.2). Denoting $z_j = x_j + ity_j$, $x_j, y_j \in \mathbb{R}$, for all $1 \le j \le n+1$, and repeating the calculations in Eqs. (12) and (13), first for the extremal points with $\mathrm{Im}(z_j) = 0$ for all $1 \le j \le n+1$, we get

(i) if $\mathrm{Im}(z_i) = 0,\ 1 \le i \le n$,

$$\begin{aligned}\left|\bar{f}_k(z_1,\ldots,z_{n+1})\right| &= \left|\left\langle Q_{n+1}\Delta_\Phi^{ix_n}Q_n\ldots\Delta_\Phi^{ix_1}Q_1\Phi, \Delta_\Phi^{-ix_{n+1}}A\Phi\right\rangle_\phi\right|\\ &\le \left\|Q_{n+1}\tau_{x_n}^\phi(Q_n)\ldots\tau_{x_n+\cdots+x_1}^\phi(Q_1)\Phi\right\|\left\|\Delta_\Phi^{-ix_{n+1}}A\Phi\right\|\\ &\le \tau\left(\left|Q_{n+1}\tau_{x_n}^\phi(Q_n)\ldots\tau_{x_n+\cdots+x_1}^\phi(Q_1)H^{\frac{1}{2}}\right|^2\right)^{\frac{1}{2}}\|A\Phi\|\\ &\le \left\|H^{\frac{1}{2}}\right\|_{2p}\prod_{i=1}^{n+1}\|Q_i\|_{2nq}.\end{aligned}$$

(ii) if $\mathrm{Im}(z_i) = 0,\ 1 \le i \le n,\ i \ne k$, and $\mathrm{Im}(z_k) = -\frac{1}{2}$, where $x_i = \mathrm{Re}(z_i)$,

$$\begin{aligned}&\left|\bar{f}_k(z_1,\ldots,z_{n+1})\right|\\ &= \left|\left\langle Q_{n+1}\Delta_\Phi^{ix_n}Q_n\ldots\Delta_\Phi^{ix_{k-1}}Q_{k-1}\Delta_\Phi^{\frac{1}{2}}Q_k\Delta_\Phi^{ix_1}Q_1\Phi, \Delta_\Phi^{-ix_{n+1}}A\Phi\right\rangle_\phi\right|\\ &\le \|A\Phi\|\|H\|_p^{\frac{1}{2}}\prod_{i=1}^{k-1}\left\|\tau_{x_n+\cdots+x_i}^\phi(Q_i)\right\|_{2nq}\prod_{i=k}^{n+1}\left\|J_\phi\tau_{x_n+\cdots+x_i}^\phi(Q_i)J_\phi\right\|_{2nq}\\ &= \|A\Phi\|\|H\|_p^{\frac{1}{2}}\prod_{i=1}^{n+1}\|Q_i\|_{2nq}.\end{aligned}$$

The previous result depends a lot on the possibility of "extending" the trace, which is originally defined only in the algebra, to the algebra generated by $\mathfrak{M}\cup\mathfrak{M}'$. One may try to define

$$\tau(J_\Phi|A|J_\Phi B) = \tau(|A|)\tau(|B|),$$

but it immediately fails, in general, since the application of this formula with either $A = \mathbb{1}$ or $B = \mathbb{1}$, due to $\tau(\mathbb{1}) = \infty$.

In order to relax the condition on the possibility of having a trace in all the *GNS*-represented algebra, we have to demand more regularity on the perturbation. The next theorem shows almost the same result as the previous one, with a little more restricted perturbation.

Theorem III.17. *Let $\mathfrak{M} \subset B(H)$ be a von Neumann algebra, τ be a normal faithful semifinite trace on $\mathfrak{M}$, $\phi(\cdot) = \langle\Phi,\cdot\Phi\rangle$ be a state on $\mathfrak{M}$, and $n \in \mathbb{N}$. Let also $n \in \mathbb{N}$, $p, q \ge 1$ with $\frac{1}{p}+\frac{1}{q} = 1$, $Q_i \in L_{4mq}(\mathfrak{M},\tau)$ for all $1 \le i, m \le n$, and suppose $\phi = \tau_H$ is $\|\cdot\|_p$-continuous.*

Then, if $\Phi \in \mathcal{D}(Q_1)$ and $\Delta_\Phi^{iz_{j-1}}Q_{j-1}\ldots\Delta_\Phi^{iz_1}Q_1\Phi \in \mathcal{D}(Q_j)$ for every $-\frac{1}{2} \le \mathrm{Im}(z_j) \le 0$ and for every $2 \le j \le n$, $Q_n\Delta_\Phi^{iz_n}Q_{n-1}\ldots\Delta_\Phi^{iz_1}Q_1\Phi \in \mathcal{D}\left(\Delta_\Phi^{iz}\right)$ for $-\frac{1}{2} \le \mathrm{Im}(z) \le 0$ and

$$A^n(z_1,\ldots,z_n)\Phi \doteq Q\Delta_\Phi^{iz_n}Q_n\ldots\Delta_\Phi^{iz_1}Q_1\Phi$$

is analytic on $T\left(S_{\frac{1}{2}}^n\right)$ and bounded on its closure by

$$\left\|A^n(z_1,\ldots,z_n)\Phi\right\| \le \|H\|_p^{\frac{1}{2}}\max_{0\le l\le n-1}\left\{\underbrace{\left(\prod_{j=1}^{l}\|Q_j\|_{4lq}\right)}_{=1 if\ l=0}\left(\prod_{j=l+1}^{n}\|Q_j\|_{4(n-l)q}\right)\right\}.$$

Proof. Let us proceed by induction on n.

For $n = 1$, let $Q_1 = U|Q_1|$ be the polar decomposition of Q_1 and $|Q_1| = \int_0^\infty \lambda dE_\lambda^{|Q_1|}$ be the spectral decomposition of $|Q_1|$. Since $\Phi \in \mathcal{D}(Q_1)$, $Q_1\Phi = U\lim_{k\to\infty}Q_{1,k}\Phi$, where $Q_{1,k} = \int_0^k \lambda dE_\lambda^{|Q_1|}$. Define the following functionals on $\mathfrak{M}_A\Phi$:

$$\begin{aligned}f_k^z(A\Phi) &\doteq \left\langle UQ_{1,k}\Phi, \Delta_\Phi^{-i\bar{z}}A\Phi\right\rangle_\phi,\\ f^z(A\Phi) &\doteq \lim_{k\to\infty}f_k^z(A\Phi) = \left\langle Q_1\Phi, \Delta_\Phi^{-i\bar{z}}A\Phi\right\rangle_\phi.\end{aligned}$$

Of course, for fixed $A\Phi$, $\bar{f}_k(z) = \overline{f_k^z(A\Phi)}$ is entire analytic and, in the two lines of extremal points, we have

$$\begin{aligned} \left|\bar{f}(t)\right| &= \lim_{k\to\infty}\left|\bar{f}_k(t)\right| \\ &\leq \|A\Phi\|\|H\|_p^{\frac{1}{2}}\|Q_1\|_{2q} \quad \forall t\in\mathbb{R}, \\ \left|\bar{f}\left(t+\frac{1}{2}it\right)\right| &= \lim_{k\to\infty}\left|\bar{f}_k\left(t+\frac{1}{2}it\right)\right| \\ &= \lim_{k\to\infty}\left\langle \Delta_\Phi^{\frac{1}{2}}\Delta_\Phi^{-it}A\Phi, UQ_{1,k}\Phi\right\rangle_\phi \\ &\leq \|A\Phi\|\|H\|_p^{\frac{1}{2}}\|Q_1\|_{2q} \quad \forall t\in\mathbb{R}, \end{aligned} \tag{14}$$

which proves that the functional concerned is bounded for $-\frac{1}{2}\leq \mathrm{Im}(z)\leq 0$ due to the maximum modulus principle. This bound also proves that if $\bar{f}_k\to\bar{f}$ uniformly for $-\frac{1}{2}\leq \mathrm{Im}(z)\leq 0$, then $\bar{f}$ is analytic for $-\frac{1}{2}<\mathrm{Im}(z)<0$ and bounded for $-\frac{1}{2}\leq\mathrm{Im}(z)\leq 0$.

Using first the Hahn-Banach theorem to obtain an extension, also denoted by f_z, to the whole Hilbert space in such a way that $\|f_z\|\leq \|H\|_p^{\frac{1}{2}}\|Q_1\|_{2q}$, we know by Riesz's representation theorem that there exists a $\Omega(z)\in\mathcal{H}_\phi$ such that $f_z(\cdot) = \langle\Omega(z),\cdot\rangle_\phi$. Since $\mathfrak{M}_{\mathcal{A}}\Phi$ is dense, $\Omega(z)$ is unique.

So far, we have that $Q_1\Phi\in\mathcal{D}\left(\left(\Delta_\Phi^{-i\bar{z}}\right)^*\right) = \mathcal{D}\left(\Delta_\Phi^{iz}\right)$ and $\bar{f}(z) = \left\langle A\Phi,\Delta_\Phi^{iz}Q_1\Phi\right\rangle_\phi$ is analytic on $\left\{z\in\mathbb{C}\,\middle|\,-\frac{1}{2}<\mathrm{Im}(z)<0\right\}$ and continuous on its closure, for every $A\in\mathfrak{M}_{\mathcal{A}}$.

Since $\overline{\mathfrak{M}_{\mathcal{A}}\Phi}^{\|\cdot\|} = \overline{\mathfrak{M}\Phi}^{\|\cdot\|} = \mathcal{H}_\phi$, the vector-valued function $A(z)\Phi \doteq \Delta_\Phi^{iz}Q_1\Phi$ is weakly analytic, hence strongly analytic on $\left\{z\in\mathbb{C}\,\middle|\,-\frac{1}{2}<\mathrm{Im}(z)<0\right\}$ and

$$\|A(z)\Phi\|\leq\|H\|_p^{\frac{1}{2}}\|Q_1\|_{2q}\quad \forall z\in\left\{z\in\mathbb{C}\,\middle|\,-\frac{1}{2}\leq\mathrm{Im}(z)\leq 0\right\}.$$

Suppose that now the hypothesis holds for $n\in\mathbb{N}$. We will use the same ideas: we can define the sequence $Q_i^{k_i} = U_i\int_0^{k_i}\lambda dE_\lambda^{|Q_i|}$, where $Q_i = U_i|Q_i|$ is the polar decomposition of Q_i for every $i\leq i\leq n+1$. Define

$$f_{k_1,\ldots,k_{n+1}}^{(z_1,\ldots,z_{n+1})}(A\Phi) = \left\langle Q_{n+1}\Delta_\Phi^{iz_n}Q_n\ldots\Delta_\Phi^{iz_1}Q_1\Phi,\Delta_\Phi^{-i\bar{z}_{n+1}}A\Phi\right\rangle_\phi.$$

For now, we will omithe superscript index on the operators to not overload the notation.

Since $\bar{f}(z_1,\ldots,z_n)\doteq\overline{f_{k_1,\ldots,k_{n+1}}^{(z_1,\ldots,z_{n+1})}(A\Phi)}$ is an analytic function, it attains its maximum at an extremal point of S (see Ref. 1 Corollary 2.2). Denoting $z_j = x_j+ity_j$ and using now the Tomita-Takesaki theorem in the similar calculation we made in Eq. (14), we get

(i) for the extremal points with $\mathrm{Im}(z_j)=0$ for all $1\leq j\leq n+1$, we get

$$\begin{aligned} &\left|\bar{f}(x_1,\ldots,x_{n+1})\right|\leq \\ &\|A\Phi\|\left\langle Q_{n+1}\tau_{x_n}^\phi(Q_n)\ldots\tau_{x_n+\cdots+x_1}^\phi(Q_1)\Phi, Q_{n+1}\tau_{x_n}^\phi(Q_n)\ldots\tau_{x_n+\cdots+x_1}^\phi(Q_1)\Phi\right\rangle^{\frac{1}{2}} \\ &\leq\|A\Phi\|\|H\|_p^{\frac{1}{2}}\prod_{j=1}^{n+1}\|Q_j\|_{4(n+1)q},\quad\forall(x_1,\ldots,x_{n+1})\in\mathbb{R}^{n+1}. \end{aligned} \tag{15}$$

(ii) now for $\mathrm{Im}(z_j)=0$ for all $i\neq l$ and $\mathrm{Im}(z_l) = -\frac{1}{2}$, where $l\neq n+1$,

$$\begin{aligned} &\left|\bar{f}\left(x_1,\ldots,x_l-\frac{1}{2}i,\ldots,x_{n+1}\right)\right|\leq\|A\Phi\|\times \\ &\left\|\tau_{x_{n+1}}^\phi(Q_{n+1})\ldots\tau_{x_{n+1}+\cdots+x_{l+1}}^\phi(Q_{l+1})J_\Phi\tau_{x_{n+1}+\cdots+x_1}^\phi(Q_1^*)\ldots\tau_{x_{n+1}+\cdots+x_l}^\phi(Q_l^*)\Phi\right\| \\ &\leq\|A\Phi\|\|H\|_p^{\frac{1}{2}}\left(\prod_{j=1}^{l}\|Q_j\|_{4lq}\right)\left(\prod_{j=l+1}^{n+1}\|Q_j\|_{4(n+1-l)q}\right)\forall(x_1,\ldots,x_{n+1})\in\mathbb{R}^{n+1}. \end{aligned} \tag{16}$$

(iii) finally, for $\mathrm{Im}(z_j) = 0$ for all $i \neq n+1$ and $\mathrm{Im}(z_{n+1}) = -\frac{1}{2}$,

$$\begin{aligned}
&\left|\bar{f}\left(x_1,\dots,x_l,\dots,x_n,x_{n+1}-\frac{1}{2}i\right)\right| \\
&\quad= \left|\left\langle Q_{n+1}\Delta_\Phi^{ix_n}Q_n\dots\Delta_\Phi^{ix_1}Q_1\Phi,\Delta_\Phi^{-ix_{n+1}}\Delta_\Phi^{\frac{1}{2}}A\Phi\right\rangle_\phi\right| \\
&\quad= \left|\left\langle \tau^\phi_{x_{n+1}}(Q_{n+1})\dots\tau^\phi_{x_{n+1}+\dots+x_1}(Q_1)\Phi, J_\Phi A^*\Phi\right\rangle_\phi\right| \\
&\quad\leq \|A\Phi\|\|H\|_p^{\frac{1}{2}}\prod_{j=1}^{n+1}\|Q_j\|_{4(n+1)q}, \forall(x_1,\dots,x_{n+1})\in\mathbb{R}^{n+1},
\end{aligned} \tag{17}$$

where the last line follows by the same calculation done in Eq. (15).

The last step is to remember that we omitted the superscripts and note that

$$\begin{aligned}
\lim_{k_{n+1}\to\infty}\dots\lim_{k_1\to\infty}f^{(z_1,\dots,z_{n+1})}_{k_1,\dots,k_{n+1}}(A\Phi) &= \left\langle Q^{k_{n+1}}_{n+1}\Delta_\Phi^{iz_n}Q_n\dots\Delta_\Phi^{iz_1}Q_1^{k_1}\Phi,\Delta_\Phi^{-i\bar{z}_{n+1}}A\Phi\right\rangle_\phi \\
&= \left\langle Q_{n+1}\Delta_\Phi^{iz_n}Q_n\dots\Delta_\Phi^{iz_1}Q_1\Phi,\Delta_\Phi^{-i\bar{z}_{n+1}}A\Phi\right\rangle_\phi,
\end{aligned}$$

so $\bar{f}(z_1,\dots,z_{n+1}) = \overline{\left\langle Q_{n+1}\Delta_\Phi^{iz_n}Q_n\dots\Delta_\Phi^{iz_1}Q_1\Phi,\Delta_\Phi^{-i\bar{z}_{n+1}}A\Phi\right\rangle_\phi}$ is the limit of a sequence of analytic functions uniformly bounded, thus analytic on $S^{n+1}_{\frac{1}{2}}$ and bounded on its closure, as desired.

As we saw in Eq. (17), the term $\Delta_\Phi^{z_{n+1}}$ does not interfere with the conclusion for $-\frac{1}{2} < \mathrm{Im}(z_{n+1}) < 0$ and, by the very same argument used above to obtain a continuous linear extension of $\bar{f}(z_1,\dots,z_{n+1})$, it follows that $Q_n\Delta_\Phi^{iz_n}Q_{n-1}\dots\Delta_\Phi^{iz_1}Q_1\Phi\in\mathcal{D}(\Delta_\Phi^{iz})$ for $-\frac{1}{2} < \mathrm{Im}(z) < 0$.

Remark III.18. In contrast to what we did in Eq. (15), for $\mathrm{Im}(z_i) = 0$ for all $1\leq i\leq n+1$, it holds that

$$\begin{aligned}
\left|\bar{f}_k(z_1,\dots,z_{n+1})\right| &= \lim_{k\to\infty}\left|\bar{f}_k(z_1,\dots,z_{n+1})(t)\right| \\
&= \lim_{k\to\infty}\left|\left\langle Q_{n+1,k}\Delta_\Phi^{ix_n}Q_n\dots\Delta_\Phi^{ix_1}Q_1\Phi,\Delta_\Phi^{-ix_{n+1}}A\Phi\right\rangle_\phi\right| \\
&\leq \left\|\Delta_\Phi^{-it}A\Phi\right\|\lim_{k\to\infty}\left\|UQ_{n+1,k}\Delta_\Phi^{ix_n}Q_n\dots\Delta_\Phi^{ix_1}Q_1\Phi\right\| \\
&\leq \left\|\Delta_\Phi^{-it}A\Phi\right\|\lim_{k\to\infty}\left\|UQ_{n+1,k}\tau^\phi_{x_n}(Q_n)\dots\tau^\phi_{x_1+\dots+x_n}(Q_1)\Phi\right\| \\
&\leq \|A\Phi\|\lim_{k\to\infty}\phi\left(UQ_{n+1,k}\tau^\phi_{x_n}(Q_n)\dots\tau^\phi_{x_1+\dots+x_n}(Q_1)\right)^{\frac{1}{2}} \\
&\leq \|A\Phi\|\lim_{k\to\infty}\tau\Big(H^{\frac{1}{2}}\tau^\phi_{x_1+\dots+x_n}(Q_1^*)\dots\tau^\phi_{x_n}(Q_n^*)Q_{n+1,k}U^* \\
&\qquad UQ_{n+1,k}\tau^\phi_{x_n}(Q_n)\dots\tau^\phi_{x_1+\dots+x_n}(Q_1)H^{\frac{1}{2}}\Big)^{\frac{1}{2}} \\
&\leq \|A\Phi\|\|H\|_p^{\frac{1}{2}}\lim_{k\to\infty}\left(\tau\left(|Q_{n+1,k}|^{2nq}\right)^{\frac{1}{nq}}\prod_{j=1}^{n}\tau\left(|Q_i|^{2nq}\right)^{\frac{1}{nq}}\right)^{\frac{1}{2}} \\
&\leq \|A\Phi\|\|H\|_p^{\frac{1}{2}}\prod_{j=1}^{n+1}\|Q_i\|_{2nq}\quad\forall t\in\mathbb{R}.
\end{aligned} \tag{18}$$

Corollary III.19. Let $\mathfrak{M}\subset B(\mathcal{H})$ be a von Neumann algebra, τ be a normal faithful semifinite trace on $\mathfrak{M}$, and $\phi(\cdot) = \langle\Phi,\cdot\Phi\rangle$ be a state on $\mathfrak{M}$. Let also $\frac{1}{p}+\frac{1}{q} = 1$, $\lambda\in\overline{\mathbb{R}}_+$, and $Q\in\mathfrak{E}^\tau_{4q,\lambda}$, and suppose $\phi = \tau_H$ is $\|\cdot\|_p$-continuous. Then, if $\Delta_\Phi^{iz_{j-1}}Q\dots\Delta_\Phi^{iz_1}Q\Phi\in\mathcal{D}(Q)$ for every $-\frac{1}{2}\leq\mathrm{Im}(z_j)\leq 0$ and for every $j\in\mathbb{N}$,

$$\Phi(Q) \doteq \sum_{n=0}^{\infty}\int_{S^n_{\frac{1}{2}}}dt_1\dots dt_n\Delta_\Psi^{t_n}Q\Delta_\Psi^{t_{n-1}}Q\dots\Delta_\Psi^{t_1}Q\Phi$$

is absolutely and uniformly convergent.

Proof. By Theorem III.17, $\Delta_\Psi^{iz_n}Q\Delta_\Psi^{iz_{n-1}}Q\dots\Delta_\Psi^{iz_1}QA\Phi$ is well defined and

$$
\begin{aligned}
&\left\|\Delta_\Psi^{iz_n}Q\Delta_\Psi^{iz_{n-1}}Q\dots\Delta_\Psi^{iz_1}QA\Phi\right\| \\
&\le \|A\Phi\|\|H\|_p^{\frac{1}{2}}\max_{0\le l\le n}\left\{\underbrace{\left(\prod_{j=1}^{l}\|Q_j\|_{4lq}\right)}_{=1\text{ if } l=0}\left(\prod_{j=l+1}^{n}\|Q_j\|_{4(n-l)q}\right)\right\} \\
&= \|A\Phi\|\|H\|_p^{\frac{1}{2}}\max_{0\le l\le n}\left\{\underbrace{\|Q\|_{4lq}^{l}}_{=1\text{ if } l=0}\|Q\|_{4(n-l)q}^{n-l}\right\} \\
&= \|A\Phi\|\|H\|_p^{\frac{1}{2}}\max_{0\le l\le\lfloor\frac{n}{2}\rfloor}\left\{\underbrace{\|Q\|_{4lq}^{l-1}}_{=1\text{ if } l=0}\|Q\|_{4(n-l)q}^{n-l}\right\} \\
&= \|A\Phi\|\|H\|_p^{\frac{1}{2}}\max_{0\le l\le\lfloor\frac{n}{2}\rfloor}\left\{\underbrace{\tau\left(|Q|^{4lq}\right)^{\frac{l}{4lq}}}_{=1\text{ if } l=0}\tau\left(|Q|^{4(n-l)q}\right)^{\frac{n-l}{4(n-l)q}}\right\}.
\end{aligned}
$$

Note that, for $Q_m = \int_0^m \lambda dE_\lambda^{|Q|}$, we have that

$$f_m(z) \doteq \tau\left(Q_m^{4zq}\right)^{\frac{1}{4q}}\tau\left(Q_m^{4(n-z)q}\right)^{\frac{1}{4q}}$$

is an analytic function in the region $\{z\in\mathbb{C} \mid 1\le \mathrm{Re}(z)\le n-1\}$ since this region is a strip that does not intercept the negative real line. Hence, its modulus in the region mentioned is assumed when $\mathrm{Re}(z) = 1$ or $\mathrm{Re}(z) = n-1$ by the maximum modulus principle. In these cases, we have

$$|f_m(1+it)| = |f_m(n-1+it)| = \|Q_m\|_{4q}\|Q_m\|_{4(n-1)q}^{n-1}.$$

As usual, taking the limit $m\to\infty$, we obtain, for all $z\in\{w\in\mathbb{C}\mid 1\le\mathrm{Re}(w)\le n-1\}$,

$$\tau\left(Q^{4zq}\right)^{\frac{1}{4q}}\tau\left(Q^{4(n-z)q}\right)^{\frac{1}{4q}} \le \|Q\|_{4q}\|Q\|_{4(n-1)q}^{n-1}. \tag{19}$$

Finally, the series

$$\sum_{n=0}^{\infty}\int_0^t dt_1\dots\int_0^{t_n} dt_n\Delta_\Psi^{t_n}Q\Delta_\Psi^{t_{n-1}}Q\dots\Delta_\Psi^{t_1}QA\Phi$$

is $\|\cdot\|$-convergent. In fact, considering first the case $Q\in\mathfrak{E}_{4q,\lambda}^{\tau}$ with $0<\lambda<\infty$, there exists $N\in\mathbb{N}$ such that, for all $k, l > N$, $\lambda\sum_{n=k}^{l}\frac{\lambda^{n-1}\|Q\|_{4(n-1)q}^{n-1}}{(n-1)!} < \frac{\epsilon}{2}$ and $\frac{\|Q\|_{4q}}{N} < 1$; thus,

$$
\begin{aligned}
&\left\|\sum_{n=k}^{l}\int_0^t dt_1\dots\int_0^{t_{n-1}} dt_n\Delta_\Psi^{t_n}Q\Delta_\Psi^{t_{n-1}}Q\dots\Delta_\Psi^{t_1}QA\Phi\right\| \\
&\le \sum_{n=k}^{l}\int_0^t dt_1\dots\int_0^{t_{n-1}} dt_n\left\|\Delta_\Psi^{t_n}Q\Delta_\Psi^{t_{n-1}}Q\dots\Delta_\Psi^{t_1}QA\Phi\right\| \\
&\le \sum_{n=k}^{l}\frac{t^n\max\left\{\|Q\|_{4q}\|Q\|_{4(n-1)q}^{n-1}, \|Q\|_{4nq}^{n}\right\}}{n!} \\
&\le \sum_{n=k}^{l}\frac{t^n\left(\|Q\|_{4q}\|Q\|_{4(n-1)q}^{n-1}+\|Q\|_{4nq}^{n}\right)}{n!} \\
&\le \sum_{n=k}^{l}\frac{\|Q\|_{4q}}{n}\frac{t^n\|Q\|_{4(n-1)q}^{n-1}}{(n-1)!}+\frac{t^n\|Q\|_{4nq}^{n}}{n!} \\
&< \epsilon.
\end{aligned}
$$

For the case $\lambda = \infty$, just remember that $\mathfrak{E}^{\tau}_{p,\infty} = \bigcap_{\lambda \in \overline{\mathbb{R}}_+} \mathfrak{E}^{\tau}_{p,\lambda}$. □

Proposition III.20. Let $(Q_n)_{n\in\mathbb{N}} \subset \mathfrak{E}^{\tau}_{4q,\lambda}$ be a sequence such that $Q_n \overset{\|\cdot\|_{4mq}}{\longrightarrow} Q \in \mathfrak{E}^{\tau}_{4q,\lambda}$, $\|Q_n\|_{4mq} \le \|Q\|_{4mq}$, and $\|Q - Q_n\|_{4mq} \le M$ for all $m \in \mathbb{N}$. In addition, suppose that, for each fixed $n \in \mathbb{N}$, $\Phi \in \mathcal{D}(Q_1)$ and $\Delta_{\Phi}^{iz_{j-1}} Q_{j-1} \ldots \Delta_{\Phi}^{iz_1} Q_1 \Phi \in \mathcal{D}(Q_j)$ for every $-\frac{1}{2} \le \mathrm{Im}(z_j) \le 0$ and for every $2 \le j \le n$, where Q_j can be either Q_n or Q. Then,

$$Exp_{l,r}\left(\int_0^t ; Q_n(s)ds\right)\Phi \overset{n\to\infty}{\longrightarrow} Exp_{l,r}\left(\int_0^t ; Q(s)ds\right)\Phi\,, \quad t < \lambda.$$

Proof. First, note that Q and Q_n, $n \in \mathbb{N}$, are densely defined closed operators. Furthermore, they have a common core due to the increasing hypothesis and the properties of τ-dense subsets.

Define

$$A_j^m(z_1,\ldots,z_n) = \Delta_{\Phi}^{iz_m} Q \ldots \Delta_{\Phi}^{iz_{j-1}} Q \Delta_{\Phi}^{iz_j}(Q - Q_n)\Delta_{\Phi}^{iz_{j+1}} Q_n \ldots \Delta_{\Phi}^{iz_1} Q_n \Phi.$$

Using a telescopic sum argument, we have, for $m > 1$,

$$\begin{aligned}
&\left\|\Delta_{\Phi}^{iz_m} Q \ldots \Delta_{\Phi}^{iz_1} Q\Phi - \Delta_{\Phi}^{iz_n} Q_n \ldots \Delta_{\Phi}^{iz_1} Q_n \Phi\right\| \\
&\quad = \left\|\sum_{j=1}^{m} A_j^m(z_1,\ldots,z_n)\right\| \\
&\quad \le \sum_{j=1}^{m} \left\|A_j^m(z_1,\ldots,z_n)\right\| \\
&\quad \le \sum_{j=1}^{m} \|H\|_p^{\frac{1}{2}} \max_{0\le l\le m-1}\left\{\|Q\|_{4lq}^{l} \|Q - Q_n\|_{4(m-l)q} \|Q_n\|_{4(m-l)q}^{m-l-1}\right\} \\
&\quad \le m\,\|H\|_p^{\frac{1}{2}} \max_{0\le l\le m-1}\left\{\|Q\|_{4lq}^{l} \|Q\|_{4(m-l)q}^{m-l-1} \|Q - Q_n\|_{4(m-l)q}\right\} \\
&\quad = m\,\|H\|_p^{\frac{1}{2}} \max_{0\le l\le m-1}\left\{\|Q\|_{4lq}^{l} \|Q\|_{4(m-l)q}^{m-l} \frac{\|Q - Q_n\|_{4(m-l)q}}{\|Q\|_{4(m-l)q}}\right\}.
\end{aligned}$$

Applying Eq. (19) to the inequality above, we get

$$\begin{aligned}
&\left\|\Delta_{\Phi}^{iz_m} Q \ldots \Delta_{\Phi}^{iz_1} Q\Phi - \Delta_{\Phi}^{iz_n} Q_n \ldots \Delta_{\Phi}^{iz_1} Q_n \Phi\right\| \\
&\quad \le m\,\|H\|_p^{\frac{1}{2}} \|Q\|_{4q} \|Q\|_{4(m-1)q}^{m-1} \max_{0\le l\le m-1}\left\{\frac{\|Q - Q_n\|_{4(m-l)q}}{\|Q\|_{4(m-l)q}}\right\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left\|Exp_{l,r}\left(\int_0^t ; Q(s)ds\right)\Phi - Exp_{l,r}\left(\int_0^t ; Q_n(s)ds\right)\Phi\right\| \\
&\le \|H\|_p^{\frac{1}{2}} \|Q - Q_n\|_{4q} + \\
&\qquad + \sum_{m=2}^{\infty} \frac{t^m}{m!} m\,\|H\|_p^{\frac{1}{2}} \|Q\|_{4q} \|Q\|_{4(m-1)q}^{m-1} \max_{0\le l\le m-1}\left\{\frac{\|Q - Q_n\|_{4(m-l)q}}{\|Q\|_{4(m-l)q}}\right\} \\
&= \|H\|_p^{\frac{1}{2}} \|Q - Q_n\|_{4q} + \\
&\qquad + \|H\|_p^{\frac{1}{2}} \|Q\|_{4q}\, t \sum_{m=2}^{\infty} \frac{t^{m-1}}{(m-1)!} \|Q\|_{4(m-1)q}^{m-1} \max_{0\le l\le m-1}\left\{\frac{\|Q - Q_n\|_{4(m-l)q}}{\|Q\|_{4(m-l)q}}\right\}.
\end{aligned} \tag{20}$$

Finally, let $\epsilon > 0$ be given. Since $Q \in \mathfrak{E}^{\tau}_{4q,\lambda}$, there exists $m_0 \in \mathbb{N}$ such that, for all $m \le m_0$,

$$\sum_{m=M}^{\infty} \frac{t^{m-1}}{(m-1)!} \|Q\|_{4(m-1)q}^{m-1} < \frac{\epsilon}{3M}.$$

By hypothesis, there also exists $n_0 \in \mathbb{N}$ such that

$$\frac{\|Q - Q_n\|_{4mq}}{\max\left\{\|Q\|_{4nq}, 1\right\}} < \epsilon\left[3\lambda\|H\|_p^{\frac{1}{2}} \sum_{m=2}^{\infty} \frac{t^{m-1}}{(m-1)!}\|Q\|_{4(m-1)q}^{m-1}\right]^{-1}, \quad \forall n \geq n_0.$$

It follows from Eq. (20) that

$$\left\|Exp_{l,r}\left(\int_0^t ; Q(s)ds\right)\Phi - Exp_{l,r}\left(\int_0^t ; Q_n(s)ds\right)\Phi\right\| < \epsilon, \quad \forall n \geq n_0.$$

□

One of the consequences of Proposition III.20 is that the sequence of Araki's perturbations obtained by the upper cut in the spectral decomposition of the modulus of a $\mathfrak{E}^{\tau}_{4q,\lambda}$-perturbation converges to the perturbation described in Corollary III.19.

Moreover, Proposition III.20 gives us an interpretation for the parameter λ in $\mathfrak{E}^{\tau}_{p,\lambda}$.

It is important to mention the notorious similarity between our approach and Sakai's geometric vectors [which were mentioned right after Eq. (1)]. In the direction of Eq. (1), we can obtain the following result:

Corollary III.21. Let $\mathfrak{M} \subset B(\mathcal{H})$ be a von Neumann algebra, τ be a normal faithful semifinite trace on $\mathfrak{M}$, and $\phi(\cdot) = \langle\Phi, \cdot\Phi\rangle$ be a state on $\mathfrak{M}$. Let also $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda \in \overline{\mathbb{R}}_+$, and $Q = Q^* \in \mathfrak{E}^{\tau}_{4q,\lambda}$, and suppose ϕ is $\|\cdot\|_p$-continuous. In addition, suppose that $\Delta_\Phi^{iz_{j-1}} Q \ldots \Delta_\Phi^{iz_1} Q\Phi \in \mathcal{D}(Q)$ for every $-\frac{1}{2} \leq \mathrm{Im}(z_j) \leq 0$ and for every $j \in \mathbb{N}$. Then, for every $z \in \mathbb{C}$ with $0 < \mathrm{Re}(z) < \frac{1}{2}$, $\Phi \in \mathcal{D}\left(e^{z(H+Q)}\right)$ and

$$e^{z(H_\Phi + Q)}\Phi = \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \int_{S^n_{\frac{1}{2}}} dt_1 \ldots dt_n \Delta_\Psi^{t_n} Q \Delta_\Psi^{t_{n-1}} Q \ldots \Delta_\Psi^{t_1} Q\Phi,$$

where $H_\Phi = \log \Delta_\Phi$.

Proof. Note that for $z = itt$, $t \in \mathbb{R}$, one identifies the right-hand side with $Exp_l\left(\int_0^t ; \tau_t^\Phi(Q)ds\right)$. Thus, for purely imaginary z, we have the equality.

Now, since we know by Theorem III.19 that the left-hand side is analytic in the region $0 < \mathrm{Re}(z) < \frac{1}{2}$, 16, Proposition 4.12 guarantees the thesis.

□

We can use all previous results to conclude with a general theorem. Note that, for the KMS condition, the interesting case is the case $\lambda = \frac{1}{2}$ as one can see in the definition of the expansional, in Ref. 16.

Theorem III.22. *Let $\mathfrak{M} \subset B(\mathcal{H})$ be a von Neumann algebra, τ be a normal faithful semifinite trace on $\mathfrak{M}$, $(\mathfrak{M}, \alpha)$ be a W^*-dynamical system, H_Φ be the Hamiltonian of α, and $\phi(\cdot) = \langle\Phi, \cdot\Phi\rangle$ be a (τ, β)-KMS state on $\mathfrak{M}$. Let also $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda \in \overline{\mathbb{R}}_+$, and $Q = Q^* \in \mathfrak{E}^{\tau}_{4q,\lambda}$, and suppose ϕ is $\|\cdot\|_p$-continuous. In addition, suppose that $\Delta_\Phi^{iz_{j-1}} Q \ldots \Delta_\Phi^{iz_1} Q\Phi \in \mathcal{D}(Q)$ for every $-\frac{1}{2} \leq \mathrm{Im}(z_j) \leq 0$ and for every $j \in \mathbb{N}$. Then, $\Phi \in \mathcal{D}\left(e^{-\frac{\beta}{2}(H_\Phi + Q)}\right)$ and $\phi^Q(A) = \left\langle\Psi^Q, A\Psi^Q\right\rangle$ is as (τ^Q, β)-KMS state for the perturbed dynamics defined by $\alpha_t^Q(A) = e^{it(H_\omega + Q)}Ae^{-it(H_\omega + Q)}$, where $\Psi^Q = \frac{e^{-\frac{\beta}{2}(H_\omega + Q)}\Omega_\omega}{\|e^{-\frac{\beta}{2}(H_\omega + Q)}\Omega_\omega\|}$.*

We would add at this point that, in contrast to Araki's treatment in Ref. 1, we do not believe our results prove any kind of stability of KMS states. This belief is based on the necessity to add a "dual" continuity property on the state, i.e., we need an additional $\|\cdot\|_p$-continuity hypothesis with index p Hölder-conjugated with the one used to control the perturbation. Hence, all the stability we proved seems to be a consequence of that continuity. An important exception is the stability of the domain of the modular operator, which allows us to extend the multiple-time KMS condition to unbounded operators, proved in Theorems III.16 and III.17.

Regarding the other paper in the literature of perturbation of KMS states,[3] also extend the theory to unbounded perturbations. Their approach does not use Araki's techniques; in particular, it does not use expansionals. One advantage of using expansionals, as mentioned in Ref. 3, is that expansionals give explicitly a way to obtain the perturbed state and its vector representative. The intrinsic differences between 3 and this work make it very difficult to compare the methodologies, but some similarities can be spotted, for example, the condition $\Delta_\Phi^{iz_{j-1}} Q \ldots \Delta_\Phi^{iz_1} Q\Phi \in \mathcal{D}(Q)$ and $Q \in \mathfrak{E}^{\tau}_{4q,\lambda}$ in Corollary III.19 seems to play the same kind of role as Assumptions 3.1 and 3.2, and 5.1 in Ref. 3, respectively. On the other hand, the noncommutative *LP*-space approach required a hypothesis that has no counterpart in Ref. 3, namely, the $\|p\|$-continuity of the state. In fact, we have already mentioned that our results reduce to Araki's theorem in the case $q = \infty$ and makes a gradation of the result for $1 \leq q < \infty$. This is not the case in Ref. 3, which does not add any hypothesis on the state. Been a little speculative, it seems also possible to generalize the results presented here using the methodology of Ref. 3.

Although all the applications of noncommutative L_p-spaces to physics known by the author are restricted to semifinite von Neumann algebras, see, for example, Refs. 21, 5, 22, and 4, we finish saying that the author is aware of the limitations imposed by the existence of a faithful normal semifinite trace on the algebra.

Several results have been proved about the type of the algebras in relativistic AQFT in the past decades, showing that, under some physical reasonably assumptions, the algebra of observables of a diamond has to be of type III, see Ref. 26, Secs. V.6 and Ref. 27, Proposition 3.2.

Fortunately, for general von Neumann algebras, either in the Haagerup or in the Araki-Masuda construction, there is a natural trace related with the noncommutative L_p-space. This suggests that our ideas can be generalized. In fact, roughly speaking, Haagerup's generalization of noncommutative L_p-spaces for general von Neumann algebras uses several identifications between a von Neumann algebra (and other objects related to it) and the crossed product $\mathfrak{M} \rtimes_{\tau^\varphi} \mathbb{R}$, where $\tau^\varphi = \{\tau_t^\varphi\}_{t\in\mathbb{R}}$ is the modular automorphism group obtained throughout the faithful normal and semifinite weight φ. Among these identifications, we highlight (i) the one due to Ref. 24, II, Lemma 1, which says that the mapping $\phi \mapsto \tilde{\phi}$, where $\tilde{\phi} = \hat{\phi} \circ T$ and $\hat{\phi}$ is a natural extension of ϕ to $\widehat{\mathfrak{M}}_+$ (the extended positive part of $\mathfrak{M}$) as described in Ref. 28, Proposition 1.10, is a bijection from the set of all normal semifinite weights on $\mathfrak{M}$ onto the set of all normal semifinite weights ψ on $\mathfrak{M} \rtimes_{\tau^\varphi} \mathbb{R}$ satisfying $\psi \circ \theta_t = \psi \quad \forall t \in \mathbb{R}$, and (ii) the one due to Ref. 19, Theorem 1.2, which says that

$$\left\{H_\phi \in (\mathfrak{M} \rtimes_{\tau^\varphi} \mathbb{R})_\eta \,\middle|\, \phi \text{ is normal and semifinite}\right\} = \left\{H_\phi \in (\mathfrak{M} \rtimes_{\tau^\varphi} \mathbb{R})_\eta \,\middle|\, \theta_t H_\phi = e^{-t} H_\phi\right\},$$

where $H_\phi \eta(\mathfrak{M} \rtimes_{\tau^\varphi} \mathbb{R})$ is the Radon-Nikodym derivative for the normal semifinite weight ϕ on $\mathfrak{M}$ with respect to the trace τ, i.e., H_ϕ is the operator affiliated with $\mathfrak{M} \rtimes_{\tau^\varphi} \mathbb{R}$ such that $\tilde{\phi} = \tau_{H_\phi}$. That means that we can consider the state ϕ as a state in $\mathfrak{M} \rtimes_{\tau^\varphi} \mathbb{R}$.

It is important to note that $\mathfrak{M} \rtimes_{\tau^\varphi} \mathbb{R}$ has a natural trace τ,[29] Lemma 5.2, which is used in the definition of the noncommutative L_p-space, namely,

$$L_p(\mathfrak{M}) \doteq \left\{H \in (\mathfrak{M} \rtimes_{\tau^\varphi} \mathbb{R})_\tau \,\middle|\, \theta_t H = e^{-\frac{t}{p}} H,\ \forall t \in \mathbb{R}\right\},$$
$$L_\infty(\mathfrak{M}) \doteq \left\{H \in (\mathfrak{M} \rtimes_{\tau^\varphi} \mathbb{R})_\tau \,\middle|\, \theta_t H = H,\ \forall t \in \mathbb{R}\right\}.$$

The trace τ is not used to define the norm on these spaces, instead a positive linear function, $\mathrm{tr}(\cdot)$, is defined on $L_1(\mathfrak{M})$. This linear function satisfies a tracelike property $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, where $A \in L_p(\mathfrak{M})$ and $B \in L_q(\mathfrak{M})$ and $\frac{1}{p} + \frac{1}{q} = 1$.

In addition, it is also easy to see that there is a natural inclusion of the operators affiliated with $\mathfrak{M}$ in the operators affiliated with $\mathfrak{M} \rtimes_{\tau^\varphi} \mathbb{R}$ because of the spectral decomposition.

Again, we stress that, if in one hand the development presented here is not appropriated to deal with perturbations of KMS states in relativist algebraic quantum field theory, it is a natural framework to deal with statistical mechanics, linear response, and information theory.

ACKNOWLEDGMENTS

We are grateful to Christian Jäkel and João Barata for all fruitful discussions during the development of this work.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior-Brasil (CAPES)-Finance Code 001.

REFERENCES

[1] H. Araki, "Relative Hamiltonian for faithful normal states of a von Neumann algebra," Publ. Res. I. Math. Sci. **9**, 165–209 (1973).
[2] S. Sakai, *Operator Algebras in Dynamical Systems* (Cambridge University Press, 1991), Vol. 41.
[3] J. Dereziński, C.-A. Pillet, and V. Jakšić, "Perturbation theory of W^*-dynamics, Liouvilleans and KMS states," Rev. Math. Phys. **15**, 447–489 (2003).
[4] G. De Nittis and M. Lein, "Linear response theory: A modern analytic-algebraic approach," e-print arXiv:1612.01710 (2016).
[5] J.-B. Bru and W. de Siqueira Pedra, "Universal bounds for large determinants from non-commutative Hölder inequalities in fermionic constructive quantum field theory," Math. Models Methods Appl. Sci. **27**, 1963–1992 (2017).
[6] R. Correa da Silva, "Noncommutative L_p-spaces and perturbation of KMS states," Ph.D. thesis, University of São Paulo, 2018; in *Digital Library of Theses and Dissertations* (University of Sao Paulo, 2018).
[7] O. Bratteli and D. W. R. Robinson, *Operator Algebras and Quantum Statistical Mechanics 1*, 2nd ed. (Springer, Berlin, 1987).
[8] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras, Part I* (Academic Press, New York, 1983).
[9] M. Takesaki, *Theory of Operator Algebras I*, Encyclopedia of Mathematical Sciences Vol. 124 (Springer-Verlag, Berlin, Heidelberg, 2002).
[10] H. Araki, "Some properties of modular conjugation operator of von Neumann algebras and a non-commutative Radon-Nikodym theorem with a chain rule," Pac. J. Math. **50**, 309–354 (1974).
[11] R. Correa da Silva, "Lecture notes on noncommutative L_p-spaces," e-print arXiv:1803.02390 (2018).
[12] H. Araki, *Mathematical Theory of Quantum Fields* (Oxford University Press on Demand, 1999), Vol. 101.
[13] O. Bratteli and D. W. R. Robinson, *Operator Algebras and Quantum Statistical Mechanics 2*, 2nd ed. (Springer, Berlin, 1997).
[14] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras, Part II* (Academic Press, New York, 1986).
[15] R. Haag, N. M. Hugenholtz, and M. Winnink, "On the equilibrium states in quantum statistical mechanics," Commun. Math. Phys. **5**, 215–236 (1967).
[16] H. Araki, "Expansional in Banach algebras," Ann. Sci. Ec. Norm. Super. **6**, 67–84 (1973).
[17] I. E. Segal, "A non-commutative extension of abstract integration," Ann. Math. **57**, 401–457 (1953).

[18] J. Dixmier, "Formes linéaires sur un anneau d'opérateurs," Bull. Soc. Math. Fr. **81**, 9–39 (1953).
[19] U. Haagerup, "L_p-spaces associated with an arbitrary von Neumann algebra," in *Algebres d'Opérateurs et leurs applications en physique mathématique Proc. Colloq., Marseille, 1977* (CNRS, 1979), Vol. 274, pp. 175–184.
[20] M. Hilsum, "Les espaces L_p d'une algèbre de von neumann définies par la dérivée spatiale," J. Funct. Anal. **40**, 151–169 (1981).
[21] J.-M. Bouclet, F. Germinet, A. Klein, and J. H. Schenker, "Linear response theory for magnetic Schrödinger operators in disordered media," J. Funct. Anal. **226**, 301–372 (2005).
[22] N. Dombrowski and F. Germinet, "Linear response theory for random Schrödinger operators and noncommutative integration," preprint arXiv:1103.5498 (2011).
[23] V. Jaksic, Y. Ogata, Y. Pautrat, and C.-A. Pillet, "Entropic fluctuations in quantum statistical mechanics. An introduction," preprint arXiv:1106.3786 (2011).
[24] M. Terp, *L_p spaces associated with von Neumann algebras*, Notes, Math. Institute (Copenhagen University, 1981), No. 3.
[25] M. B. Ruskai, "Inequalities for traces on von Neumann algebras," Commun. Math. Phys. **26**, 280–289 (1972).
[26] R. Haag, *Local Quantum Physics: Fields, Particles, Algebras* (Springer Science & Business Media, 2012).
[27] D. Buchholz, C. D'Antoni, and K. Fredenhagen, "The universal structure of local algebras," Commun. Math. Phys. **111**, 123–135 (1987).
[28] U. Haagerup, "Operator valued weights in von Neumann algebras, I," J. Funct. Anal. **32**, 175–206 (1979).
[29] U. Haagerup, "Operator valued weights in von Neumann algebras, II," J. Funct. Anal. **33**, 339–361 (1979).