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infinite radical cube zero

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ABSTRACT. Let A be an artin algebra and let $\text{mod}A$ denote the category of finitely generated right A -modules. We denote by $\text{rad}(\text{mod}A)$ the ideal in $\text{mod}A$ generated by all non-invertible morphisms between indecomposable modules in $\text{mod}A$. The infinite radical $\text{rad}^\infty(\text{mod}A)$ of $\text{mod}A$ is the intersection of all powers $\text{rad}^i(\text{mod}A)$, $i \geq 1$, of $\text{rad}(\text{mod}A)$. It is known that A is representation-finite if and only if $(\text{rad}^\infty(\text{mod}A))^2 = 0$. In this paper we study the representation-infinite algebras A such that $(\text{rad}^\infty(\text{mod}A))^3 = 0$. Examples of such algebras include all representation-infinite tilted algebras of Euclidean type.

Let A be an artin algebra over a commutative artin ring R . We denote by $\text{mod}A$ the category of finitely generated right A -modules and by $\text{rad}(\text{mod}A)$ the Jacobson radical of $\text{mod}A$, that is, the ideal in $\text{mod}A$ generated by all non-invertible morphisms between indecomposable modules in $\text{mod}A$. The infinite radical $\text{rad}^\infty(\text{mod}A)$ of $\text{mod}A$ is the intersection of all powers $\text{rad}^i(\text{mod}A)$, $i \geq 1$, of $\text{rad}(\text{mod}A)$.

The study of $\text{rad}^\infty(\text{mod}A)$ has had some importance in describing the category $\text{mod}A$. It is known, for instance, that an artin algebra A satisfies $\text{rad}^\infty(\text{mod}A) = 0$ if and only if it is representation-finite, that is, there exist at most finitely many non-isomorphic indecomposable A -modules (see [24, 35]). In [14] we have deepened this result by showing that $(\text{rad}^\infty(\text{mod}A))^2 = 0$ is equivalent to A being representation-finite. Also, by studying the restriction of $\text{rad}^\infty(\text{mod}A)$ to suitable subcategories of $\text{mod}A$, one can get nice informations on certain kind of components of the Auslander-Reiten quiver of A . We refer to [35] for a detailed discussion on these techniques.

We are particularly interested in describing the categories $\text{mod}A$ with $\text{rad}^\infty(\text{mod}A)$ nilpotent. This problem has been raised by O. Kerner and the last named author in [24], where it was shown that, if A is a selfinjective finite dimensional algebra over an algebraically closed field and admits a simply connected Galois covering, then $\text{rad}^\infty(\text{mod}A)$ is nilpotent if and only if A is domestic, that is, there is a positive integer m such that the indecomposable A -modules occur, in each dimension d , in a

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finite number of discrete and at most m one-parameter families. It was also recently shown in [39] that, if A is a strongly simply connected finite dimensional algebra over an algebraically closed field, then $\text{rad}^\infty(\text{mod}A)$ is nilpotent if and only if A is domestic. It is expected that, in general, $\text{rad}^\infty(\text{mod}A)$ nilpotent would imply that A is domestic.

Our approach in this paper is different. We shall study arbitrary representation-infinite artin algebras A with $(\text{rad}^\infty(\text{mod}A))^3 = 0$. As observed above such an algebra A satisfies $(\text{rad}^\infty(\text{mod}A))^2 \neq 0$. We also observe that this class of artin algebras is rather large. Let us note that all representation-infinite tilted algebras of Euclidean type [31] (also, all representation-infinite left glued algebras whose right type is a union of Euclidean diagrams [2]), and all domestic coil enlargements of tame concealed algebras [8], and hence, all domestic multicoil algebras having sincere non-directing indecomposable modules (see [4, 6, 7]), belong to this class. See section 2 for more details on these examples.

We now describe the contents of this paper. In section 2 we exhibit some classes of representation-infinite artin algebras A with $(\text{rad}^\infty(\text{mod}A))^3 = 0$. For such an algebra we show that the regular components of its Auslander-Reiten quiver Γ_A form a family of pairwise orthogonal generalized standard stable tubes. In section 3, it is shown that at most finitely many of the τ_A -orbits of Γ_A are non-periodic. Section 4 is devoted to the study of minimal representation-infinite algebras A with $(\text{rad}^\infty(\text{mod}A))^3 = 0$. Finally, in the last two sections we study algebras A satisfying $(\text{rad}^\infty(\text{mod}A))^3 = 0$ by looking at those factor algebras B of A which are tame concealed. In particular, we shall show how some components of Γ_B are embedded in Γ_A . Also, we show that there are only finitely many ideals I of A such that the quotient A/I is tame concealed.

In a forthcoming paper [15], we shall continue the study of such kind of algebras.

1. PRELIMINARIES

1.1. Let A be an artin algebra over a commutative artin ring R , that is, A is an R -algebra which is finitely generated as an R -module. By an A -module it is meant a finitely generated right A -module. We shall denote by $\text{mod}A$ the category of all (finitely generated) A -modules, and by $\text{ind}A$ the full subcategory of $\text{mod}A$ with one representative of each isomorphism class of indecomposable A -modules. Then $\text{rad}(\text{mod}A)$ denotes the Jacobson radical of $\text{mod}A$, that is, the ideal in $\text{mod}A$ generated by all non-invertible morphisms between indecomposable modules in $\text{mod}A$. The infinite radical $\text{rad}^\infty(\text{mod}A)$ of $\text{mod}A$ is defined to be the intersection of all powers $\text{rad}^i(\text{mod}A)$, $i \geq 1$, of $\text{rad}(\text{mod}A)$. We denote by D the standard duality $\text{Hom}_R(-, \bar{I}): \text{mod}A \rightarrow \text{mod}A^{\text{op}}$, where \bar{I} is the injective envelope of $R/\text{rad}R$ in $\text{mod}R$. We say that a property holds for *almost all indecomposable modules* if it holds for, up to isomorphism, all but finitely many of them. An algebra A is said

to be *representation-finite* (or *representation-infinite*) if $\text{mod}A$ contains finitely many (respectively, infinitely many) non-isomorphic indecomposable modules.

Unless otherwise stated all algebras are assumed to be basic and connected.

1.2. We shall denote by Γ_A the Auslander-Reiten quiver of A , and by $\tau_A = D\text{Tr}$ and $\tau_A^{-1} = \text{Tr}D$ the Auslander-Reiten translations in Γ_A . We shall now agree to identify the vertices of Γ_A with the corresponding A -modules in $\text{ind}A$. By a component of Γ_A we mean a connected component of Γ_A . We observe that a morphism between indecomposable modules lying in different components of Γ_A belongs to $\text{rad}^\infty(\text{mod}A)$. We shall use this fact all along this paper.

Let \mathcal{C} be a component of Γ_A . Then, \mathcal{C} is said to be *regular* if \mathcal{C} contains neither a projective module nor an injective module, and *semi-regular* if \mathcal{C} does not contain both a projective and an injective module. Following [13], a component \mathcal{C} is called a π -*component* (respectively, an ι -*component*) if (i) almost all modules in \mathcal{C} belong to τ_A -orbits of projective (respectively, injective) modules; and (ii) there are at most finitely many modules in \mathcal{C} which lie in oriented cycles (for details, we refer to [13, 2]). Postprojective (respectively, preinjective) components, that is, components \mathcal{C} without oriented cycles and such that each module in \mathcal{C} belongs to the τ_A -orbit of a projective module (respectively, an injective module) are examples of π -components (respectively, ι -components). Observe that we use the term postprojective instead of preprojective: we believe it is more suggestive. Examples of π -components which are not postprojective components can be found in [13, 2]. A component \mathcal{C} is said to be *generalized standard* if $\text{rad}^\infty(X, Y) = 0$ for all modules X and Y in \mathcal{C} [38]. Observe that a π -component (respectively, an ι -component) \mathcal{C} is generalized standard because $\text{rad}^\infty(-, X) = 0$ (respectively, $\text{rad}^\infty(X, -) = 0$) for all $X \in \mathcal{C}$. Finally, \mathcal{C} is said to be *faithful* if the intersection $\text{ann}\mathcal{C}$ of the annihilators $\text{ann}X$ of all modules X in \mathcal{C} is zero. We shall denote by \mathcal{C}_l the left stable part of \mathcal{C} obtained from \mathcal{C} by deleting the τ_A -orbits of projective modules, by \mathcal{C}_r the right stable part of \mathcal{C} obtained from \mathcal{C} by deleting the τ_A -orbits of injective modules, and by \mathcal{C}_s the stable part of \mathcal{C} obtained from \mathcal{C} by deleting the τ_A -orbits of both the projective and the injective modules. Also, a connected quiver is called *non-trivial* if it contains at least two vertices (and, hence, at least one arrow).

1.3. We recall the following results which will be useful along this paper. The first is due to Zhang [41].

Theorem. *Let Γ be a non-periodic non-trivial component of the stable part of Γ_A . Then Γ is of the form $Z\Delta$, where Δ is a valued quiver without oriented cycles.*

1.4. The following result is due to Happel-Preiser-Ringel [20], see also [25].

Theorem. *Let Γ be an infinite connected component of the stable part of Γ_A . If*

Γ contains a τ_A -periodic module, then Γ is a stable tube, that is, it is of the form $\mathbf{Z}\mathbf{A}_\infty/(\tau_A^n)$ for some $n \geq 1$.

Finally, a translation quiver is said to be *quasi-periodic* if almost all of its τ_A -orbits are periodic (see [10]). For more details on Auslander-Reiten components we refer to [9, 31].

1.5. A path $X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_t$ in Γ_A is said to be *sectional* if $\tau X_i \neq X_{i-2}$, for each $2 \leq i \leq t$. The following result proven in [23] (compare with [12]) will be useful later on.

Proposition. *Let $X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_t$ be a sectional path between indecomposable modules, and let $f_i: X_{i-1} \rightarrow X_i$ be irreducible morphisms for $i = 1, \dots, t$. Then $f_t \cdots f_1 \in \text{rad}^t(X_0, X_t) \setminus \text{rad}^{t+1}(X_0, X_t)$. In particular, $f_t \cdots f_1$ is a non-zero morphism.*

1.6. Let H be a hereditary algebra. It is known that, in this case, we can assume that R is a field so that H is a finite dimensional algebra over R . Further, there exists a bilinear form on the Grothendieck group $K_0(H)$ of H given by

$$\langle M, N \rangle = \dim_R \text{Hom}_H(M, N) - \dim_R \text{Ext}_H^1(M, N)$$

which induces a quadratic form q_H on $K_0(H) \otimes_{\mathbf{Z}} \mathbf{Q}$. It is well-known that H is representation-finite if and only if q_H is positive definite. The algebra H is said to be of *tame* type if it is not representation-finite and q_H is positive semidefinite. For the basic representation theory of hereditary algebras we refer the reader, for instance, to [18, 19, 31]. Let now H be a representation-infinite hereditary algebra and let n denote the rank of $K_0(H)$. Let T be a multiplicity-free postprojective tilting H -module, that is: $\text{Ext}_H^1(T, T) = 0$, $\text{rad}^\infty(-, T) = 0$ (which, in this case, is equivalent to T being postprojective), and T is a direct sum of n pairwise non-isomorphic indecomposable H -modules. The algebra $B = \text{End}_H(T)$ is called a *concealed algebra*. If H is tame hereditary then B is called *tame concealed*, and otherwise *wild concealed*. It is known that the representation theory of B and H are very close to each other. Let B be a concealed algebra (which does not exclude the possibility of being hereditary). Then, Γ_B has a postprojective component \mathcal{P} containing all projectives and a preinjective component \mathcal{I} containing all injectives. If B is tame concealed, then the regular components form an infinite family of pairwise orthogonal generalized standard stable tubes \mathcal{T}_ρ , $\rho \in \Omega$, and all but finitely many of them have rank 1. Moreover, since the family \mathcal{T}_ρ , $\rho \in \Omega$, separates \mathcal{P} from \mathcal{I} , all tubes \mathcal{T}_ρ are faithful, and every map from a module in \mathcal{P} to a module in \mathcal{I} belongs to $(\text{rad}^\infty(\text{mod} B))^2$. On the other hand, if B is wild concealed then the regular components are of the form $\mathbf{Z}\mathbf{A}_\infty$ (see [30]). By [33], components of the form $\mathbf{Z}\mathbf{A}_\infty$ are not generalized standard.

1.7. We shall need the following characterisation of concealed algebras proven independently in [2](3.4) and [36](3.3).

Proposition. *The following are equivalent for a representation-infinite artin algebra A :*

- (a) A is concealed;
- (b) $pdX = 1$ and $idX = 1$ for almost all modules X in $indA$;
- (c) $rad^\infty(-, A) = 0$ and $rad^\infty(DA, -) = 0$.

2. SOME EXAMPLES

2.1. In this section we shall give some examples of representation-infinite artin algebras A satisfying $(rad^\infty(modA))^3 = 0$. Our first result describes the family of regular components of the Auslander-Reiten quiver of such algebras.

Theorem: *Let A be an artin algebra with $(rad^\infty(modA))^3 = 0$. Let $\Gamma_i, i \in I$, denote the family of all regular components of Γ_A . Then $\Gamma_i, i \in I$, are pairwise orthogonal generalized standard stable tubes, and all but finitely many of them have rank 1.*

Proof. We first show that $rad^\infty(X, Y) = 0$ for all X, Y belonging to regular components. Suppose this is not true and let $f: X \rightarrow Y$ be a non-zero morphism in $rad^\infty(X, Y)$, where X and Y belong to regular components. Consider now the projective cover $\pi: P_A(X) \rightarrow X$ of X and the injective envelope $\iota: Y \rightarrow I_A(Y)$ of Y . Observe that since X and Y belong to regular components we have that π and ι are in $rad^\infty(modA)$ (1.2). Therefore, $\iota f \pi$ is a non-zero morphism in $(rad^\infty(modA))^3$, which contradicts our hypothesis. Hence $\Gamma_i, i \in I$, are pairwise orthogonal generalized standard components.

Claim: Every regular component of Γ_A contains oriented cycles.

Suppose Γ is a regular component of Γ_A without oriented cycles. Then, by (1.3), Γ is of the form $Z\Delta$, where Δ is a valued quiver without oriented cycles. Since Γ is generalized standard, we have that $\text{Hom}_A(X, \tau_A Y) = 0$ for all X and Y in Δ and hence, by [33](2.2), Δ is finite. Moreover, according to [38](3.3) it follows that $B = A/\text{ann}\Gamma$ is a tilted algebra given by a regular tilting module and Γ is a connecting component of Γ_B . Let M be the sum of all modules in Δ . Observe that, since Γ_B contains a complete slice in a regular component, then each component of Γ_B is semiregular. It follows from tilting theory that Γ_B contains either a stable tube or a (semiregular) component whose stable part contains a component of the form ZA_∞ [26]. Suppose first that Γ_B contains a stable tube \mathcal{T} and let Z be in \mathcal{T} . Then, Z is either cogenerated or generated by M . Suppose we have the former (the other situation is dual). Consider the projective cover $\pi: P_B(Z) \rightarrow Z$ of Z , the embedding $\nu: Z \rightarrow M^r$ of Z into some power of M and the injective envelope $\iota: M^r \rightarrow I_A(M^r)$ of M^r . It follows that the composition $\nu\pi$ is a non-zero morphism in $(rad^\infty(modA))^3$, a contradiction. Suppose now that Γ_B has a semiregular component without injectives

Γ' , whose stable part contains a component of the form $\mathbf{Z}A_\infty$. It is known that there exists a sectional path $X \rightarrow \cdots \rightarrow Y$ in Γ' such that $\text{Hom}_B(X, \tau_B Y) \neq 0$ [33](2.2), and it can be chosen in such a way that there is no projective module which is a predecessor of X or Y in Γ' . Hence, $\text{rad}^\infty(X, \tau_B Y)$ contains a non-zero morphism f , because there are no oriented cycles containing X . Consider the projective cover $\pi: P_B(X) \rightarrow X$ of X and the injective envelope $\iota: \tau_B Y \rightarrow I_B(\tau_B Y)$ of $\tau_B Y$. Since the composite $\iota f \pi$ is non-zero and belongs to $(\text{rad}^\infty(\text{mod} B))^3$, which contradicts our hypothesis, the claim is proven.

By (1.4), a regular component with oriented cycles is a stable tube and so, for each $i \in I$, Γ_i is a stable tube. Let now $\text{rk} \Gamma_i$ denote the rank of the tube Γ_i . According to [38](5.10), we have the following inequality

$$\sum_{i \in I} (\text{rk} \Gamma_i - 1) \leq n - 1$$

where n is the rank of the Grothendieck group of A . Therefore, almost all of the stable tubes Γ_i , $i \in I$, have rank 1. This finishes the proof of the Theorem 2.1. \square

2.2. The next result gives a criterion for a tilted algebra B to satisfy the condition $(\text{rad}^\infty(\text{mod} B))^3 = 0$.

Proposition. *Let H be a connected representation-infinite hereditary artin algebra, T be a tilting H -module without preinjective direct summands or without postprojective direct summands, and $B = \text{End}_H(T)$. Then $(\text{rad}^\infty(\text{mod} B))^3 = 0$ if and only if H is of Euclidean type.*

Proof. It is known that if H is of Euclidean type then $(\text{rad}^\infty(\text{mod} B))^3 = 0$ (see [18, 19, 31]). Suppose now that H is of wild type. Then, by a result of Strauss [40](7.5), there is a factor algebra C of B which is wild concealed. We know, by [30], that Γ_C admits regular components of type $\mathbf{Z}A_\infty$, and hence $(\text{rad}^\infty(\text{mod} C))^3 \neq 0$ by Theorem (2.1). This implies that $(\text{rad}^\infty(\text{mod} B))^3 \neq 0$. Therefore, if $(\text{rad}^\infty(\text{mod} B))^3 = 0$, then H is of Euclidean type. \square

2.3. Let us now turn our attention to algebras which are left and right glueings of tilted algebras as defined in [2]. We shall first recall those definitions. Let B_1, \dots, B_t be representation-infinite tilted algebras having complete slices $\Sigma_1, \dots, \Sigma_t$ respectively, in the postprojective components but having no injective modules in these components. write $B = B_1 \times \cdots \times B_t$ and let C be a representation-finite algebra. An algebra A is said to be a *left glueing of B_1, \dots, B_t by C along the slices $\Sigma_1, \dots, \Sigma_t$* , or a *left glucd algebra* if:

- (i) each of B_1, \dots, B_t and C is a full convex subcategory of A and any object in A belongs to one of these subcategories;
- (ii) no projective A -module is a proper successor of the union $\Sigma_1 \cup \cdots \cup \Sigma_t$, considered as embedded in $\text{ind} A$; and

(iii) $\text{ind}B$ is cofinite in $\text{ind}A$, that is, almost all indecomposable A -modules are also B -modules.

Let A now be a left glued algebra. Then, Γ_A contains a (unique) π -component Γ consisting of all postprojective A -modules. Besides Γ , any other component of Γ_A is the image of a full embedding of a component of Γ_{B_i} , for some $1 \leq i \leq t$, and containing no projective module into Γ_A . Suppose Γ contains injectives (which is equivalent to the algebra C of the definition being non-zero). For each i , let \mathcal{I}_i be the set of all injective A -modules I such that there exists an irreducible morphism $I \rightarrow K$ with K in $\text{ind}B_i$ and such that there is no injective in Γ which is a proper successor of those summands of $I/\text{soc}I$ which belong to $\text{ind}B_i$. Let \mathcal{I} be the union of the \mathcal{I}_i 's. Finally, let Σ be the subsection of Γ consisting of the modules M such that there exists a path in Γ of length at least one from some injective in \mathcal{I} to M , and any such path is sectional. We shall call Σ the *right extremal subsection* of Γ .

Dually, we can also define *right glued algebras* and *left extremal subsection* for such algebras. We refer to [2] for details. We shall need the following result proven in [2].

Lemma. (a) *Let A be a representation-infinite left glueing of B_1, \dots, B_t by the non-zero algebra C , and let Σ be its right extremal subsection. If no projective is a proper successor of Σ , the underlying graph of Σ is the disjoint union of the types of B_1, \dots, B_t .*

(b) *Let A be a representation-infinite right glueing of B_1, \dots, B_t by the non-zero algebra C , and let Σ be its left extremal subsection. If no injective is a proper predecessor of Σ , the underlying graph of Σ is the disjoint union of the types of B_1, \dots, B_t .*

2.4. The following result gives a characterisation of glued algebras A which satisfy $(\text{rad}^\infty(\text{mod}A))^3 = 0$.

Theorem. *Let A be a representation-infinite finite-dimensional algebra over an algebraically closed field.*

(a) *If A is a left glueing of B_1, \dots, B_t by the non-zero algebra C , and if there is no projective module which is a proper successor of the right extremal subsection Σ then $(\text{rad}^\infty(\text{mod}A))^3 = 0$ if and only if Σ is a disjoint union of Euclidean graphs.*

(b) *If A is a right glueing of B_1, \dots, B_t by the non-zero algebra C , and if there is no injective module which is a proper predecessor of the left extremal subsection Σ then $(\text{rad}^\infty(\text{mod}A))^3 = 0$ if and only if Σ is a disjoint union of Euclidean graphs.*

Proof. We shall prove only (a), since (b) will follow from a dual argument.

(a) *Sufficiency.* Let Γ be the (unique) π -component of Γ_A and denote by B the product $B_1 \times \dots \times B_t$. By (2.3), the underlined graph associated with Σ is the union of the types of B_1, \dots, B_t and then for each i , B_i is a tilted algebra of Euclidean

type having a complete slice in the postprojective component. Therefore, by (2.2), it follows that $(\text{rad}^\infty(\text{mod}B_i))^3 = 0$ for each i . Observe now that $\text{ind}B$ is cofinite in $\text{ind}A$ and that all indecomposable A -modules which are not B -modules belong to Γ . Suppose now that $(\text{rad}^\infty(\text{mod}A))^3 \neq 0$. Then, there are morphisms in $\text{rad}^\infty(\text{mod}A)$ between indecomposable A -modules

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

such that the composite hgf is a non-zero morphism in $(\text{rad}^\infty(\text{mod}A))^3$. Since $f \in \text{rad}^\infty(\text{mod}A)$, we infer that $\text{rad}^\infty(-, Y) \neq 0$, and hence Y does not belong to Γ . Therefore, there is an i such that $Y \in \text{ind}B_i$. Then, Y does not belong to the postprojective component of Γ_{B_i} , and Z and W also belong to $\text{ind}B_i$. But since Y is not postprojective, we get that one of the maps g or h is not in $\text{rad}^\infty(\text{mod}B_i)$, a contradiction.

Necessity. Let us assume now that $(\text{rad}^\infty(\text{mod}A))^3 = 0$. In particular, we have that $(\text{rad}^\infty(\text{mod}B_i))^3 = 0$ for all i . Observe that, for each i , B_i is a tilted algebra in the conditions of (2.2) and then the types of B_1, \dots, B_t are all Euclidean. It follows that Σ is a disjoint union of Euclidean quivers, and this finishes the proof. \square

2.5. In the representation theory of tame simply connected algebras over an algebraically closed field an important role is played by coil enlargements of tame concealed algebras (see [35]), introduced and studied in [4, 5, 8].

Let A be a finite dimensional algebra over an algebraically closed field k , and Γ be a component of Γ_A . For an indecomposable module X in Γ , called the *pivot*, three types of *admissible operations* are defined, yielding in each case a modified algebra A' of A , and a modified component Γ' of Γ :

(ad1) If the support of $\text{Hom}_A(X, -)|_\Gamma$ is of the form

$$X = X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \dots$$

we set A' to be the one-point extension $(A \times D)[X \oplus Y]$, where D is the full $t \times t$ lower triangular matrix algebra and Y is the unique projective-injective indecomposable D -module. In this case, Γ' is obtained by inserting in Γ a rectangle consisting of the modules $Z_{ij} = (k, X_i \oplus Y_j, \begin{pmatrix} 1 \\ 1 \end{pmatrix})$, for $i \geq 0, 1 \leq j \leq t$, and $X'_i = (k, X_i, 1)$ for $i \geq 1$, where $Y_j, 1 \leq j \leq t$, denote the indecomposable injective D -modules. If $t = 0$, we set $A' = A[X]$, and the rectangle reduces to the ray formed by modules of the form X'_i .

(ad2) If the support of $\text{Hom}_A(X, -)|_\Gamma$ is of the form

$$Y_t \longleftarrow \dots \longleftarrow Y_1 \longleftarrow X = X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \dots$$

with $t \geq 1$ (so that X is injective), we set $A' = A[X]$. In this case, Γ' is obtained by inserting in Γ a rectangle consisting of the modules $Z_{ij} = (k, X_i \oplus Y_j, \begin{pmatrix} 1 \\ 1 \end{pmatrix})$, for $i \geq 1, 1 \leq j \leq t$, and $X'_i = (k, X_i, 1)$ for $i \geq 0$.

(ad3) If the support of $\text{Hom}_A(X, -)|_\Gamma$ is of the form

$$\begin{array}{ccccccc} Y_1 & \longrightarrow & Y_2 & \longrightarrow & \cdots & \longrightarrow & Y_t \\ \uparrow & & \uparrow & & & & \uparrow \\ X = X_0 & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_{t-1} \longrightarrow X_t \longrightarrow \cdots \end{array}$$

with $t \geq 2$ (so that X_{t-1} is injective), we set $A' = A[X]$. In this case, Γ' is obtained by inserting in Γ a rectangle consisting of the modules $Z_{ij} = (k, X_i \oplus Y_j, \begin{pmatrix} 1 \\ 1 \end{pmatrix})$, for $i \geq 1, 1 \leq j \leq t$, and $X'_i = (k, X_i, 1)$ for $i \geq 0$.

The dual coextension operations (ad1*), (ad2*) and (ad3*) are also called *admissible*.

Let C be a tame concealed algebra over k and $T = (T_\lambda)_{\lambda \in \mathbf{P}_1(k)}$ be the family of all stable tubes in Γ_C . Following [8], an algebra B is said to be a *coil enlargement* of C if there is a finite sequence of algebras $C = B_0, B_1, \dots, B_m = B$ such that for each $0 \leq j < m$, B_{j+1} is obtained from B_j by an admissible operation with the pivot in a stable tube of T or in a component of Γ_{B_j} , obtained from a stable tube of T by means of the sequence of the admissible operations done so far. Then, for each $\lambda \in \mathbf{P}_1(k)$, all modules of T_λ are contained in one component C_λ of Γ_B , called a coil. It follows that coil enlargements of C using only the operations of type (ad1) (respectively of type (ad1*)) are just tubular extensions (respectively, tubular coextensions) in the sense of [17, 31]. We then have the following fact proved in [8](3.5).

Proposition. *Let B be a coil enlargement of a tame concealed algebra. Then, in the above notation the following hold:*

- (a) *There exists a unique maximal tubular extension B^+ of C which is a convex subalgebra of B such that B is obtained from B^+ by a sequence of admissible operations of types (ad1*), (ad2*) or (ad3*).*
- (b) *There exists a unique maximal tubular coextension B^- of C which is a convex subalgebra of B such that B is obtained from B^- by a sequence of admissible operations of types (ad1), (ad2) or (ad3).*

2.6. The following result gives a characterisation of coil enlargements B of concealed algebras which satisfies $(\text{rad}^\infty(\text{mod } B))^3 = 0$.

Theorem. *Let B be a coil enlargement of a tame concealed algebra C . Then the following conditions are equivalent:*

- (a) $(\text{rad}^\infty(\text{mod}B))^3 = 0$;
- (b) $\text{rad}^\infty(\text{mod}B)$ is nilpotent;
- (c) B is domestic;
- (d) B^- and B^+ are tilted algebras of Euclidean type.

Proof. We know from [24](1.7) that if B is wild or tubular in the sense of [24](5.1) then $\text{rad}^\infty(\text{mod}B)$ is not nilpotent. Then, the theorem is a direct consequence of [8](4.1) or (4.2). \square

3. COMPONENTS ARE QUASI-PERIODIC

3.1. Let A be an artin algebra such that $(\text{rad}^\infty(\text{mod}A))^3 = 0$. The main purpose of this section is to prove that almost all τ_A -orbits in Γ_A are periodic or, in other words that Γ_A is a quasi-periodic translation quiver. In order to do so we shall prove first some general results.

Proposition: *Let A be an arbitrary artin algebra and Γ be a connected stable translation subquiver of Γ_A of the form $\mathbf{Z}\Delta$, where Δ is an infinite locally finite quive without oriented cycles. Let M be an indecomposable module in Γ . Then*

- (a) *there exists an infinite sequence of irreducible morphisms*

$$\cdots \longrightarrow M_{r+1} \xrightarrow{f_r} M_r \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{f_1} M_1 \xrightarrow{g} M_0 = M$$

where for each i , $M_i \in \Gamma$, f_i is an epimorphism and $gf_1 \cdots f_i$ is a composition of morphisms in a sectional path through M_1, \dots, M_{i-1} .

- (b) *there exists an infinite sequence of irreducible morphisms*

$$M = M'_0 \xrightarrow{g'} M'_1 \xrightarrow{f'_1} M_2 \longrightarrow \cdots \longrightarrow M'_r \xrightarrow{f'_r} M_{r+1} \longrightarrow \cdots$$

where for each i , $M'_i \in \Gamma$, f'_i is a monomorphism and $f'_i \cdots f'_1 g'$ is a composition of morphisms in a sectional path through M'_1, \dots, M'_{i-1} .

Proof. We shall only prove (a), since the proof of (b) is dual.

- (a) Let Γ and M as above. Consider first the set \mathcal{P} of all paths

$$N_k \longrightarrow \cdots \longrightarrow N_1 \longrightarrow N_0 = M$$

in Γ ending at M and such that $\tau_A^{-1}N_k$ is not a predecessor of M in Γ .

Observe that if $N_k \longrightarrow \cdots \longrightarrow N_1 \longrightarrow N_0 = M$ is a path from \mathcal{P} , then Γ has n paths from $\tau_A^{-1}N_i$ to N_j , where $0 \leq i, j \leq k$.

Indeed, if

$$\tau_A^{-1}N_i = V_0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_p = N_j$$

is a path in Γ , then Γ admits a path

$$\tau_A^{-1}N_k \longrightarrow \tau_A^{-1}N_{k-1} \longrightarrow \cdots \longrightarrow \tau_A^{-1}N_i = V_0 \longrightarrow \cdots \longrightarrow V_p = N_j \longrightarrow \cdots \longrightarrow N_0,$$

which is a contradiction.

Further, since Γ is of the form $Z\Delta$, with Δ infinite, \mathcal{P} admits paths of arbitrarily large length. Moreover, all paths from \mathcal{P} are sectional. Indeed, if $N_l \rightarrow \dots \rightarrow N_0 = M$ is a path in \mathcal{P} which is not sectional, then there is an i , $2 \leq i \leq l$, such that $\tau_A^{-1}M_i = M_{i-2}$ and so a predecessor of M contradicting our hypothesis on \mathcal{P} . Since Γ is locally finite, by König's Lemma, there exists an infinite sectional path

$$\dots \rightarrow L_r \rightarrow L_{r-1} \rightarrow \dots \rightarrow L_1 \rightarrow L_0 = M$$

such that, for each $i \geq 1$, the path

$$L_i \rightarrow \dots \rightarrow L_1 \rightarrow L_0 = M$$

belongs to \mathcal{P} . Consider now

$$\mathcal{M} = \{L_i : i \in \mathbb{N}\} \cup \{X \in \Gamma : \exists \text{ a path } L_r \rightarrow \dots \rightarrow X \rightarrow \dots \rightarrow L_s \text{ in } \Gamma, \text{ for some } r, s\}$$

Let now

$$L_r = U_0 \rightarrow U_1 \rightarrow \dots \rightarrow U_t = L_s \tag{*}$$

be a path in Γ . Clearly, $U_i \in \mathcal{M}$ for all i . Observe also that $(*)$ is sectional. In fact, if it were not so, then $\tau_A^{-1}U_l = U_{l+2}$ for some $0 \leq l \leq t-2$, and hence

$$\tau_A^{-1}L_r = \tau_A^{-1}U_0 \rightarrow \tau_A^{-1}U_1 \rightarrow \dots \rightarrow \tau_A^{-1}U_l \rightarrow U_{l+3} \rightarrow \dots \rightarrow U_t = L_s$$

is a path from $\tau_A^{-1}L_r$ to L_s , which contradicts the above claim. Therefore, it follows that each path between modules in \mathcal{M} is sectional.

We now define inductively a sequence of modules M_i and subsets \mathcal{M}_i of \mathcal{M} . Let first $M_0 = \mathcal{M}$. Let M_1 be a module in \mathcal{M}_0 of minimal length and

$$\mathcal{M}_1 = \{X \in \mathcal{M} : \text{there is a path from } X \text{ to } M_1\}.$$

Suppose now M_i and \mathcal{M}_i are defined for all $i < j$. We take M_j to be a module in $\mathcal{M}_{j-1} \setminus \{M_{j-1}\}$ of minimal length and

$$\mathcal{M}_j = \{X \in \mathcal{M} : \text{there is a path from } X \text{ to } M_j\}.$$

Observe that, if X and Y belong to \mathcal{M}_j for some j , then any path from X to Y is sectional. Fix a $j \geq 1$ and consider a morphism $h: M_j \rightarrow M_{j-1}$ which is a composite of irreducible morphisms given by a path in Γ . We will show that h is an epimorphism. Suppose this is not true. Then, there are an epimorphism

$$\alpha = {}^t(\alpha_1, \dots, \alpha_r): M_j \rightarrow Y_1 \oplus \dots \oplus Y_r$$

and a (proper) monomorphism

$$\beta = (\beta_1, \dots, \beta_r): Y_1 \oplus \dots \oplus Y_r \rightarrow M_{j-1}$$

such that $Y_i \in \text{ind}A$ and

$$h = \sum_{i=1}^r \beta_i \alpha_i.$$

Since β is a proper monomorphism, then for all $i = 1, \dots, t$, $l(Y_i) < l(M_{j-1})$, and hence $Y_i \notin \mathcal{M}_{j-1}$. We shall show now that for each i either α_i or β_i belongs to $\text{rad}^\infty(\text{mod}A)$. If this is not the case, then for some t , there exists a path

$$M_j \longrightarrow \cdots \longrightarrow Y_t \longrightarrow \cdots \longrightarrow M_{j-1}$$

with $Y_t \notin \mathcal{M}_{j-1}$, which is a contradiction. Therefore, for each i , the morphism $\beta_i \alpha_i$ belongs to $\text{rad}^\infty(M_j, M_{j-1})$ and so $h = \sum \beta_i \alpha_i \in \text{rad}^\infty(M_j, M_{j-1})$. The contradiction comes from the fact that h is a composite of irreducible morphisms from a sectional path and hence, by (1.5), it does not belong to $\text{rad}^\infty(\text{mod}A)$. Therefore, the morphisms $f_i: M_{i+1} \longrightarrow M_i$ can now be chosen, for each i , to be any composite of irreducible morphisms in a sectional path from M_{i+1} to M_i . This finishes the proof. \square

3.2. We are now ready to prove the main result in this section.

Theorem. *Let A be an artin algebra such that $(\text{rad}^\infty(\text{mod}A))^3 = 0$. Then almost all τ_A -orbits in Γ_A are periodic.*

Proof. Since all τ_A -orbits in a regular component are periodic (by (2.1)) and since almost all components are regular, it suffices to prove that any non-regular component is quasi-periodic, that is, almost all of its τ_A -orbits are periodic. \blacksquare

Let Γ be a non-regular component of Γ_A and suppose it is not quasi-periodic. Then, by duality, we can also assume that there exists a connected component Γ' of the right stable part of Γ with infinitely many non-periodic τ_A -orbits. It follows from [27] that Γ' is a subquiver of $Z\Delta'$, closed under successors, where Δ' is an infinite locally finite quiver without oriented cycles. Then, Γ' has a (stable) subquiver of the form $Z\Delta$, where Δ is an infinite locally finite quiver without oriented cycles. Let M be a module in $Z\Delta$ such that $\tau_A M$ is not a predecessor of an injective module in Γ . By (3.1) there exists an infinite sequence of irreducible morphisms

$$\cdots \longrightarrow M_{r+1} \xrightarrow{f_r} M_r \longrightarrow \cdots \longrightarrow M_2 \xrightarrow{f_1} M_1 \xrightarrow{g} M_0 = M \quad (*)$$

where for each i , $M_i \in \Gamma$, f_i is an epimorphism and $gf_1 \cdots f_i$ is a composite of morphisms in a sectional path through M_1, \dots, M_{i-1} . Since the path $(*)$ is infinite and consists of pairwise non-isomorphic modules (by [11]), it follows from [33](Lemma2) that for some i and j there is a non-zero morphism $f: M_i \longrightarrow \tau_A M_j$, which is in fact in $\text{rad}^\infty(\text{mod}A)$ because Γ' has no oriented cycles and M is not a predecessor of an injective module in Γ . Observe that any morphism from a projective to a module M_l is in $\text{rad}^\infty(\text{mod}A)$ because it factors through all M_k , $k \geq l$. In particular, consider the projective cover $\pi: P_A(M_i) \longrightarrow M_i$ of M_i . Consider also the injective envelope $\iota: \tau_A M_j \longrightarrow I_A(\tau_A M_j)$ of $\tau_A M_j$, which is also in $\text{rad}^\infty(\text{mod}A)$ because $\tau_A M_j$ is not a predecessor of an injective module in Γ . Now, the composite $\iota f \pi$ is

a non-zero morphism and belongs to $(\text{rad}^\infty(\text{mod}A))^3 = 0$, a contradiction with the hypothesis. \square

Corollary. *Let A be an artin algebra such that $(\text{rad}^\infty(\text{mod}A))^3 = 0$. Then, for each $d \geq 1$, almost all indecomposable A -modules of length d are τ_A -invariant.*

Proof. It follows from Theorem 3.2 that all components of Γ_A are quasi-periodic. Fix now a positive integer d . It is known that a quasi-periodic component has at most finitely many indecomposable modules of length d [10](5.5) and [38](2.6). Since there are at most finitely many components of Γ_A which are not stable tubes of rank greater than 1 we infer that almost all indecomposable modules of length d belong to stable tubes of rank 1 (that is, homogeneous tubes), hence the result. \square

4. MINIMAL REPRESENTATION-INFINITE ALGEBRAS

4.1. Our main aim in this section is to show that an artin algebra A is tame concealed if and only if $(\text{rad}^\infty(\text{mod}A))^3 = 0$ and Γ_A contains a faithful regular component. In order to do so we shall first establish some slightly more general results. The first one is an easy consequence of (1.7).

Lemma. *A representation-infinite artin algebra A is tame concealed if and only if $\text{rad}^\infty(-, A) = 0$, $\text{rad}^\infty(DA, -) = 0$ and $(\text{rad}^\infty(\text{mod}A))^3 = 0$.*

Proof. Since the necessity is clear, we shall prove the sufficiency. Suppose that A is such that (i) $\text{rad}^\infty(-, A) = 0$, (ii) $\text{rad}^\infty(DA, -) = 0$ and (iii) $(\text{rad}^\infty(\text{mod}A))^3 = 0$. Conditions (i) and (ii) imply that A is concealed by (1.7). If A is a wild concealed algebra then it contains a component of the form $\mathbf{Z}A_\infty$ (1.6), which contradicts (2.1). \square

4.2. We shall now prove a very useful criterion for deciding when an artin algebra A with $(\text{rad}^\infty(\text{mod}A))^3 = 0$ is tame concealed. We need the following lemma.

Lemma: *Let \mathcal{T} be a faithful stable tube in Γ_A . Then almost all indecomposable modules in \mathcal{T} are faithful.*

Proof. Let M_1, \dots, M_r be indecomposable modules in \mathcal{T} such that $M = M_1 \oplus \dots \oplus M_r$ is a faithful A -module. For each i , $1 \leq i \leq r$, denote by Ω_i the sectional path in \mathcal{T} from infinity to M_i . Let Σ be an arbitrary ray in \mathcal{T} . For each i , $1 \leq i \leq r$, take a module N_i in $\Sigma \cap \Omega_i$, and put $N = N_1 \oplus \dots \oplus N_r$. Clearly, for each i , $1 \leq i \leq r$, there is a path of irreducible epimorphisms from N_i to M_i , and so we have an epimorphism $N \rightarrow M$. Then $\text{ann}N \subset \text{ann}M$, and so $\text{ann}N = 0$ because $\text{ann}M = 0$. Let X be the module of maximal quasi-length among the modules N_1, \dots, N_r . We then have a monomorphism $N \rightarrow X^r$, and so $\text{ann}X = \text{ann}X^r \subset \text{ann}N = 0$. This shows that X is a faithful A -module. Let now

$$X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \dots$$

be the subpath of Σ from X to infinity. Since there are monomorphisms $X \rightarrow X_j$, $j \geq 1$, we get that the modules X_j are faithful, because X is faithful. Observe now that for almost all modules Y in \mathcal{T} there is a path of irreducible epimorphisms

$$Y = Y_0 \rightarrow Y_1 \rightarrow \cdots \rightarrow Y_j = X_j$$

for some j . Therefore, almost all modules in \mathcal{T} are faithful, which completes the proof. \square

4.3. Proposition: *Let A be an artin algebra with $(\text{rad}^\infty(\text{mod}A))^3 = 0$. The following conditions are equivalent:*

- (a) A is tame concealed;
- (b) There exist a faithful indecomposable A -module M such that there are an infinite sequence of proper monomorphisms

$$M = X_0 \xrightarrow{f_1} X_1 \rightarrow \cdots \rightarrow X_{i-1} \xrightarrow{f_i} X_i \rightarrow \cdots$$

and an infinite sequence of proper epimorphisms

$$\cdots \rightarrow Y_i \xrightarrow{g_i} Y_{i-1} \rightarrow \cdots \rightarrow Y_1 \xrightarrow{g_1} Y_0 = M$$

with all X_i and Y_i in $\text{ind}A$.

Proof. (a) \implies (b) If A is a tame concealed algebra then, by (1.6), Γ_A admits a faithful stable tube \mathcal{T} . Then, by Lemma 4.2 there is a faithful module M in \mathcal{T} . Choose now an infinite sequence of irreducible maps

$$M = X_0 \xrightarrow{f_1} X_1 \rightarrow \cdots \rightarrow X_{i-1} \xrightarrow{f_i} X_i \rightarrow \cdots$$

corresponding to the sectional path in \mathcal{T} from M to infinity and an infinite sequence of irreducible maps

$$\cdots \rightarrow Y_i \xrightarrow{g_i} Y_{i-1} \rightarrow \cdots \rightarrow Y_1 \xrightarrow{g_1} Y_0 = M$$

corresponding to the sectional path in \mathcal{T} from infinity to M . Then, the maps f_i (respectively, g_i) are proper monomorphisms (respectively, proper epimorphisms), and (b) follows.

(b) \implies (a) According to Lemma (4.1), it is enough to prove that $\text{rad}^\infty(-, A) = 0$ and $\text{rad}^\infty(DA, -) = 0$. Let M be as in the statement. We first observe that any non-zero morphism $f: P \rightarrow M$, where P is a projective, is in $\text{rad}^\infty(\text{mod}A)$. Indeed, for each $i \geq 1$ there is an $\phi_i: P \rightarrow Y_i$ such that $f = g_1 \cdots g_i \phi_i$ and so $f \in \text{rad}^i(\text{mod}A)$, which proves our claim. Similarly, any non-zero morphism $g: M \rightarrow I$, where I is injective, belongs to $\text{rad}^\infty(\text{mod}A)$. Suppose now that $\text{rad}^\infty(-, A) \neq 0$ and take $0 \neq h \in \text{rad}^\infty(Z, A)$ for some $Z \in \text{mod}A$. Since M is a faithful A -module, there is a monomorphism $v: A \rightarrow M^r$, for some $r \geq 1$. Therefore, there exists a projection $p: M^r \rightarrow M$ such that $pvh \neq 0$. Consider now the injective envelope $\iota: M \rightarrow I_A(M)$ of M . The composite $\iota(pv)h$ is non-zero and, by the above remarks we infer

that it belongs to $(\text{rad}^\infty(\text{mod } A))^3$, a contradiction. Hence, $\text{rad}^\infty(-, A) = 0$ and by dual arguments we also have that $\text{rad}^\infty(DA, -) = 0$. This finishes the proof. \square

4.4. Let A be an artin algebra. The following lemma, though straightforward, will be very useful.

Lemma. *Let A be an artin algebra. Then*

- (a) $\text{rad}^\infty(-, A) = 0$ if and only if all indecomposable projective modules lie in a π -component.
- (b) $\text{rad}^\infty(DA, -) = 0$ if and only if all indecomposable injective modules lie in a ι -component.

Proof. We shall prove only (a), since the proof of (b) is dual.

(a) Observe that if all indecomposable projective modules lie in a π -component then there are at most finitely many non-isomorphic indecomposable modules X such that $\text{Hom}_A(X, A) \neq 0$ (1.2). Hence, it follows from Harada-Sai Lemma [22] that $\text{rad}^\infty(-, A) = 0$. Conversely, if $\text{rad}^\infty(-, A) = 0$, then $\text{rad}^\infty(A, A) = 0$ and clearly $\text{rad}^n(A, A) = 0$, for some natural number n . We claim that $\text{rad}^n(X, A) = 0$ for all indecomposable modules X . Indeed, suppose there is an $X \in \text{ind } A$ and $0 \neq f \in \text{rad}^n(X, A)$ and consider the projective cover $\pi: P_A(X) \rightarrow X$ of X . The composite $f\pi$ is non-zero and clearly belongs to $\text{rad}^n(P_A(X), A)$, a contradiction and this proves the claim. It follows now that there are at most finitely many indecomposable modules mapping to an indecomposable projective and, hence, by [13](1.2) (see also [2]) all indecomposable projective modules lie in π -components. The result follows now from the fact that A is connected. \square

4.5. An artin algebra A is said to be *minimal representation-infinite* if A is representation-infinite but any factor algebra A/I , where I is a non-zero ideal of A , is representation-finite. Observe first that if A is a minimal representation-infinite algebra then each component Γ of Γ_A is faithful, that is, $\text{add } \Gamma$ contains a faithful module.

Lemma: *Let A be a minimal representation-infinite algebra. Then*

- (a) $\text{rad}^\infty(-, A) = 0$ if and only if all indecomposable projective modules lie in generalized standard components.
- (b) $\text{rad}^\infty(DA, -) = 0$ if and only if all indecomposable injective modules lie in generalized standard components.

Proof. We shall prove only (a) since the proof of (b) is dual.

(a) *Necessity.* It follows from Lemma (4.4) that all indecomposable projective modules lie in a π -component, which is clearly generalized standard.

Sufficiency. Suppose that $\text{rad}^\infty(-, A) \neq 0$. It is not difficult to see that there are indecomposable projective modules P and Q such that $\text{rad}^\infty(P, Q) \neq 0$. Let now Γ

and Γ' be the components of Γ_A containing P and Q , respectively. Since, by hypothesis, Γ and Γ' are generalized standard, we conclude that they are distinct. Observe that Γ is a faithful component because A is minimal representation-infinite. Hence, there exists a module $Z \in \text{add}\Gamma$ and a monomorphism $\iota: Q \rightarrow Z$. Let now $0 \neq g \in \text{rad}^\infty(P, Q)$. The composite ιg is non-zero and belongs to $\text{rad}^\infty(P, Z)$, which is a contradiction to the fact that Γ is generalized standard and the result is proven. \square

4.6. We can now establish our main result of this section.

Theorem: *Let A be an artin algebra. Then the following are equivalent:*

- (i) *A is tame concealed;*
- (ii) *$(\text{rad}^\infty(\text{mod}A))^3 = 0$ and there exists a faithful regular component;*

Moreover, if A is not wild concealed, then (i) and (ii) are equivalent to:

- (iii) *A is minimal representation-infinite and $\text{rad}^\infty(-, A) = 0$;*
- (iv) *A is minimal representation-infinite and every indecomposable projective module lies in a generalized standard component;*
- (v) *A is minimal representation-infinite and $\text{rad}^\infty(DA, -) = 0$;*
- (vi) *A is minimal representation-infinite and every indecomposable injective module lies in a generalized standard component;*

Proof. The equivalences between (iii) and (iv) and between (v) and (vi) are given by Lemma (4.5). On the other hand, (i) implies (ii), (iii) and (v) trivially. We shall first prove that (ii) implies (i). Then, it will be enough to show that, if A is not wild concealed, (iii) implies (i) because the implication (v) \implies (i) will follow from a dual argument.

(ii) \implies (i) We first show that $\text{rad}^\infty(-, A) = 0$ and $\text{rad}^\infty(DA, -) = 0$. Indeed, by hypothesis there exists a faithful module Z , whose indecomposable summands lie in a regular component \mathcal{C} . If $\text{rad}^\infty(-, A) \neq 0$ then there exist a non-zero morphism $f \in \text{rad}^\infty(P, Q)$ for some indecomposable projective modules P and Q . Since Z is faithful, there exists a monomorphism $v: Q \rightarrow Z^r$ which belongs to $\text{rad}^\infty(\text{mod}A)$ because Q does not belong to \mathcal{C} . Consider now the injective envelope $\iota: Z^r \rightarrow I_A(Z^r)$ of Z^r , which is also in $\text{rad}^\infty(\text{mod}A)$. The contradiction comes from the fact that $v\iota\pi$ is a non-zero morphism in $(\text{rad}^\infty(\text{mod}A))^3$ and so $\text{rad}^\infty(-, A) = 0$. Dually, we can show that $\text{rad}^\infty(DA, -) = 0$. It follows from (4.1) that A is a tame concealed algebra.

(iii) \implies (i) By Lemma (4.4), all the indecomposable projective modules lie in a π -component Γ . On the other hand, since A is minimal representation-infinite there are no injective modules belonging to Γ . Indeed, if $I = D(eA)$ is an injective in Γ , we will have that $\text{Hom}_A(X, I) = 0$ for almost all indecomposable modules and then A/AeA is representation-infinite. By [13](6.7), Γ is then a postprojective component containing all the projectives but no injectives. Hence, it contains a complete slice and then A is a tilted algebra (see, for instance, [2](2.8)). According to [40](7.7), Γ_A has also a preinjective component \mathcal{I} . We claim that \mathcal{I} contains all the injective

modules. In fact, if there exists an indecomposable injective I which is not in \mathcal{I} , then we will have that $\text{Hom}_A(X, I) = 0$, for all $X \in \mathcal{I}$, contradicting the minimality of A . Therefore, \mathcal{I} is a preinjective component containing all injectives but no projectives and hence it also contains a complete slice. Therefore, A is a concealed algebra because it contains complete slices in two different components (see [32, 33]). Since A is minimal representation-infinite but not wild concealed we conclude that A is tame concealed and the result is proven. \square

4.7. Corollary: *Let A be a connected finite dimensional algebra over an algebraically closed field. Then A is tame concealed if and only if A is minimal representation-infinite and $(\text{rad}^\infty(\text{mod}A))^3 = 0$.*

Proof. The necessity is well-known. Assume now that A is minimal representation-infinite and $(\text{rad}^\infty(\text{mod}A))^3 = 0$. The second assumption implies (see [24](1.7)) that A is tame, and then, by [16](CorollaryF), Γ_A admits a stable tube \mathcal{T} . Clearly, \mathcal{T} is faithful because A is minimal representation-infinite. Therefore, by (4.6), A is tame concealed. \square

Examples: (1) Let A be the radical square zero algebra over a field k given by the following quiver

$$\bullet \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\quad} \\ \xrightarrow{\quad} \end{array} \bullet$$

Then $(\text{rad}^\infty(\text{mod}A))^3 = 0$ and A/AeA is representation-finite for every non-zero idempotent $e \in A$. However, A is not minimal representation-infinite because if I is generated by the arrow α , then A/I is still representation-infinite.

(2) Let k be a field and let $A = k[x, y]/(x, y)^2$. It is known that A is minimal representation-infinite, $(\text{rad}^\infty(\text{mod}A))^4 \neq 0$ and $(\text{rad}^\infty(\text{mod}A))^5 = 0$.

5. EMBEDDINGS OF TAME CONCEALED ALGEBRAS

5.1. Let A be an artin algebra with $(\text{rad}^\infty(\text{mod}A))^3 = 0$. We know from Theorem 4.6 that, for any regular component \mathcal{C} in Γ , the factor algebra $A/\text{ann}\mathcal{C}$ is tame concealed. In this section we are mainly interested in studying A by looking at those factor algebras of A which are tame concealed. Also, in the next section we shall prove that for such an algebra A there are only finitely many ideals I such that A/I is tame concealed.

Let B be a tame concealed factor algebra of A . We shall show first how some components of Γ_B are embedded in Γ_A . We start by proving some lemmata.

5.2. Lemma: *Let A be an artin algebra with $(\text{rad}^\infty(\text{mod}A))^3 = 0$. Assume that B is a tame concealed factor algebra of A and $f: M \rightarrow N$ is a map in $\text{ind}B$ which does not belong to $\text{rad}^\infty(\text{mod}B)$. Then f does not belong to $\text{rad}^\infty(\text{mod}A)$.*

Proof. Observe first that M and N belong to the same component, say \mathcal{C} , of Γ_B . Let $\pi: P_B(M) \rightarrow M$ be the projective cover of M and $\iota: M \rightarrow I_B(M)$ be the injective envelope in $\text{mod}B$. Then $\iota f \pi \neq 0$, because $f \neq 0$. Moreover, $\iota \in (\text{rad}^\infty(\text{mod}B))^2$ if \mathcal{C} is postprojective, $\pi \in (\text{rad}^\infty(\text{mod}B))^2$ if \mathcal{C} is preinjective, and $\iota, \pi \in \text{rad}^\infty(\text{mod}B)$ if \mathcal{C} is a stable tube of Γ_B . Since $(\text{rad}^\infty(\text{mod}A))^3 = 0$, we conclude then that f does not belong to $\text{rad}^\infty(\text{mod}A)$. \square

Corollary: *Let A be an artin algebra with $(\text{rad}^\infty(\text{mod}A))^3 = 0$ and B be a tame concealed factor algebra of A . Then, for every component \mathcal{C} in Γ_B , all modules from \mathcal{C} are contained in one component of Γ_A . In particular, if an indecomposable B -module M lies on an oriented cycle in Γ_B , then it also lies on an oriented cycle in Γ_A .*

5.3. Lemma: *Let A be an artin algebra with $(\text{rad}^\infty(\text{mod}A))^3 = 0$ and B be a tame concealed factor algebra of A . Assume that \mathcal{T} is a stable tube of Γ_A which contains a B -module. Then \mathcal{T} consists entirely of B -modules.*

Proof. We first observe that \mathcal{T} contains infinitely many B -modules. In fact, since \mathcal{T} contains a B -module Y then, by the Corollary (5.2), it also contains the entire component \mathcal{C} of Γ_B which contains Y , which is infinite. Let now X be an arbitrary module in \mathcal{T} . Since \mathcal{T} is a stable tube, there are a path $X = X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_n = Y$ of irreducible monomorphisms and a path $Z = Y_t \rightarrow Y_{t-1} \rightarrow \dots \rightarrow Y_0 = Y$ of irreducible epimorphisms with Z an indecomposable B -module. Then Y and hence also X are B -modules. This proves our claim. \square

5.4. Lemma: *Let A be an artin algebra with $(\text{rad}^\infty(\text{mod}A))^3 = 0$ and let Γ be a left stable translation subquiver of Γ_A containing no oriented cycles and closed under predecessors in Γ_A . Assume that B is a tame concealed factor algebra of A , and M is an indecomposable B -module lying in Γ . Then M is a preinjective B -module.*

Proof. Observe that M does not belong to stable tubes of Γ_B . Indeed, if M lies in a stable tube, then it would lie on an oriented cycle in Γ_B , and consequently, by (5.2), M lies also on an oriented cycle in Γ_A . But this is impossible because M is in Γ . Suppose now that M is a postprojective B -module. Let $\pi: P_A(M) \rightarrow M$ be the projective cover of M in $\text{mod}A$ and $\iota: M \rightarrow I_B(M)$ be the injective envelope of M in $\text{mod}B$. Since M is a postprojective B -module, ι belongs to $(\text{rad}^\infty(\text{mod}B))^2$, and hence, also to $(\text{rad}^\infty(\text{mod}A))^2$. Moreover, π belongs to $\text{rad}^\infty(\text{mod}A)$, because Γ has no projective modules and it is closed under predecessors in Γ_A . Therefore, $\iota\pi$ is non-zero and belongs to $(\text{rad}^\infty(\text{mod}A))^3$, a contradiction. Hence, M is preinjective. \square

5.5. Dually, we have the following.

Lemma: *Let A be an artin algebra with $(\text{rad}^\infty(\text{mod}A))^3 = 0$ and let Γ be a right stable translation subquiver of Γ_A containing no oriented cycles and closed under successors in Γ_A . Assume that B is a tame concealed factor algebra of A , and M is an indecomposable B -module lying in Γ . Then M is a postprojective B -module.*

5.6. Let A be an artin algebra with $(\text{rad}^\infty(\text{mod}A))^3 = 0$ and B be a tame concealed factor algebra of A . We can now describe the components of Γ_A containing the postprojective and preinjective B -modules.

Proposition: *Let A be an artin algebra with $(\text{rad}^\infty(\text{mod}A))^3 = 0$ and B be a tame concealed factor algebra of A . Then*

- (i) *there is a left stable connected full translation subquiver \mathcal{C} of Γ_A containing no oriented cycles and closed under predecessors in Γ_A such that all but finitely many indecomposable preinjective B -modules lie in \mathcal{C} .*
- (ii) *there is a right stable connected full translation subquiver \mathcal{D} of Γ_A containing no oriented cycles and closed under successors in Γ_A such that all but finitely many indecomposable postprojective B -modules lie in \mathcal{D} .*

Proof. We shall prove only (i), since the proof of (ii) is dual. We know from Lemma (5.2) that all indecomposable preinjective B -modules belong to one component, say Γ , of Γ_A . From Theorem (3.2), Γ admits at most finitely many non-periodic τ_A -orbits. Further, by Lemma (5.5), we know also that, if \mathcal{D} is a right stable translation subquiver of Γ_A containing no oriented cycles and closed under successors in Γ_A , then \mathcal{D} does not contain preinjective B -modules. Observe that there is at most one left stable full translation connected subquiver \mathcal{C} of Γ without oriented cycles and closed under predecessors in Γ containing infinitely many indecomposable preinjective B -modules. Indeed, suppose that there are two such subquivers, say \mathcal{C} and \mathcal{C}' , in Γ . Then, there are indecomposable preinjective B -modules X in \mathcal{C} and X' in \mathcal{C}' such that there is a path from X to X' in Γ_B . But then, by Lemma (5.2), there is a path from X to X' in Γ_A , a contradiction because \mathcal{C}' is closed under predecessors in Γ_A . Suppose now that, for every left stable connected full translation subquiver \mathcal{C} of Γ_A without oriented cycles and closed under predecessors, \mathcal{C} does not contain infinitely many indecomposable preinjective B -modules. Then, by the above remarks, we infer that either the left stable part Γ_l or the right stable part Γ_r of Γ admits an infinite connected component, say Ω , with oriented cycles and containing infinitely many indecomposable preinjective B -modules. Without loss of generality we may assume that Ω is a component of Γ_l . By (1.4) and [27] we know that Ω is either a stable tube or there is an infinite sectional path

$$\cdots \longrightarrow \tau_A^{2t} X_1 \longrightarrow \tau_A^t X_s \longrightarrow \cdots \longrightarrow \tau_A^t X_2 \longrightarrow \tau_A^t X_1 \longrightarrow X_s \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1$$

in Ω with $t > s$ such that $\{X_1, \dots, X_s\}$ is a complete set of representatives of τ_A -orbits in Ω . In both cases, since Ω has infinitely many preinjective B -modules, there are two sectional paths

$$M = U_0 \longrightarrow U_1 \longrightarrow \dots \longrightarrow U_p = N \text{ and}$$

$$N = V_0 \longrightarrow V_1 \longrightarrow \dots \longrightarrow V_q = M$$

in Ω such that M is an indecomposable preinjective B -module and, for any $1 \leq j \leq q$, there is an infinite sectional path

$$\dots \longrightarrow Z_2^j \longrightarrow Z_1^j \longrightarrow Z_0^j = V_j$$

in Ω with $Z_1^j \neq V_{j-1}$. Choose now irreducible morphisms $f_i: U_{i-1} \longrightarrow U_i$, $1 \leq i \leq p$, and $g_j: V_{j-1} \longrightarrow V_j$, $1 \leq j \leq q$. We know from (1.7) that $f_p \cdots f_1 \in \text{rad}^p(M, N) \setminus \text{rad}^{p+1}(M, N)$. Moreover, by [25](1.6), the maps g_j , $1 \leq j \leq q$, are of infinite left degree. Therefore $g_q \cdots g_1 f_p \cdots f_1 \in \text{rad}^{p+q}(M, M) \setminus \text{rad}^{p+q+1}(M, M)$. In particular, we get that $\text{rad}(M, M) \neq 0$. On the other hand, $\text{End}_A(M) = \text{End}_B(M)$ is a division ring, because M is an indecomposable preinjective B -module. Hence, we get a contradiction. This finishes our proof of (i). \square

5.7. Let A be an artin algebra with $(\text{rad}^\infty(\text{mod}A))^3 = 0$ and B be a tame concealed factor algebra of A . We know from Corollary (5.2) that every component of Γ_B is embedded into a component of Γ_A . Moreover, by (2.1), every regular component of Γ_A is a stable tube. The next result shows that, in fact, these two components coincide in almost all instances.

Proposition: *Let A be an artin algebra with $(\text{rad}^\infty(\text{mod}A))^3 = 0$ and B be a tame concealed factor algebra of A . Then, all but finitely many stable tubes of Γ_B are full components of Γ_A .*

Proof. By our previous remarks and (5.3), it is enough to show that every non-regular component of Γ_A contains at most finitely many stable tubes of Γ_B . Suppose that there is a component \mathcal{C} in Γ_A containing infinitely many stable tubes, say $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \dots$, of Γ_B . Observe that \mathcal{C} has no sectional path

$$X = Z_0 \longrightarrow Z_1 \begin{array}{c} \uparrow \\ \longrightarrow \end{array} \dots \longrightarrow Z_r = Y$$

with $X \in \mathcal{T}_i$, $Y \in \mathcal{T}_j$, and $i \neq j$. Indeed, if this is the case, and $f_i: Z_{i-1} \longrightarrow Z_i$, $1 \leq i \leq r$, are irreducible morphisms, then, by (1.7), $f_r \cdots f_1$ is non-zero, and hence $\text{Hom}_B(X, Y) = \text{Hom}_A(X, Y) \neq 0$. However, this contradicts the fact that different tubes in Γ_B are orthogonal. Hence, if Γ is a stable tube in the stable part \mathcal{C}_s of \mathcal{C} , then Γ admits modules from at most finitely many tubes \mathcal{T}_i . We know, by (3.2), that all but infinitely many τ_A -orbits in \mathcal{C} are periodic. Further, by (1.4), if Γ' is an infinite component of \mathcal{C}_s containing a periodic module, then Γ' is a stable tube. Therefore, we conclude that there is a non-periodic τ_A -orbit \mathcal{O} in \mathcal{C} containing

modules from infinitely many tubes \mathcal{T}_i . We may assume that there is a module X in \mathcal{O} such that $\{\tau_A^t X, t \geq 0\}$ contains modules from infinitely many tubes \mathcal{T}_i , and that the connected component, say \mathcal{D} , of \mathcal{C}_l containing X is non-trivial. Observe that \mathcal{D} contains oriented cycles. Indeed, if this is not the case, then, there is a connected full translation quiver \mathcal{E} of \mathcal{D} which is closed under predecessors in Γ_A and contains an indecomposable module from a stable tube of Γ_B , a contradiction with Corollary (5.2). Therefore, \mathcal{D} has oriented cycles. Since \mathcal{D} is infinite and not a stable tube, by [27], there is an infinite sectional path

$$\cdots \rightarrow \tau_A^{2t} X_1 \rightarrow \tau_A^t X_s \rightarrow \cdots \rightarrow \tau_A^t X_2 \rightarrow \tau_A^t X_1 \rightarrow X_s \rightarrow \cdots \rightarrow X_2 \rightarrow X_1$$

in \mathcal{D} with $t > s$ and $\{X_1, \dots, X_s\}$ is a complete set of representatives of τ_A -orbits in \mathcal{C} . But then there is a sectional path

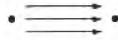
$$\tau_A^{t_1} X \rightarrow \cdots \rightarrow \tau_A^{t_2} X$$

in \mathcal{D} with $\tau_A^{t_1} X$ and $\tau_A^{t_2} X$ lying in different tubes of Γ_B , a contradiction with the fact proven above. This finishes our proof. \square

6. NUMBER OF IDEALS

6.1. Let A be an artin algebra. In order to study the complexity of A it is interesting to look at the number of ideals I such that A/I is tame concealed. The following example shows that they are not necessarily finite.

Example. Let A be the wild hereditary algebra given by the following quiver



It is not difficult to see that there are infinitely many ideals (a) generated by different linear combinations of arrows such that the quotients $A/(a)$ are the Kronecker algebras, thus concealed.

The main purpose of this section is to show that for an artin algebra A satisfying $(\text{rad}^\infty(\text{mod} A))^3 = 0$ the number of ideals I such that A/I is tame concealed is finite.

6.2. We shall first prove the following general lemma.

Lemma: *Let A be an artin algebra and I and J be two ideals in A such that $I \subset J$. Assume that A/I and A/J are tame concealed algebras. Then $I = J$.*

Proof. Observe that A/J is the quotient of A/I by J/I . Since A/I is minimal representation-infinite and A/J is representation-infinite, we get that $J/I = 0$, and so $I = J$. \square

6.3. Our next immediate concern are the algebras with two simple modules. We first observe that every concealed algebra B with two simples is hereditary. Indeed, observe that one of the simple B -modules is projective and the other is injective. In order to prove our next result we shall also need the following proposition (see [29])

Proposition. *Let A be a tame concealed algebra with two simple modules S_1 and S_2 . Then*

$$A = \begin{pmatrix} F & {}_F M_G \\ 0 & G \end{pmatrix}$$

where $F = \text{End}_A(S_2)$ and $G = \text{End}_A(S_1)$ are division algebras, and ${}_F M_G$ is an F - G -bimodule such that

$$\dim_F({}_F M_G) \times \dim_G({}_F M_G) = 4.$$

6.4. Lemma: *Let A be an artin algebra with two simple modules and $(\text{rad}^\infty(\text{mod}A))^3 = 0$. Assume that I and J are ideals of A contained in $\text{rad}A$ and such that A/I and A/J are tame concealed algebras. Then $I = J$.*

Proof. Suppose that $I \neq J$ and put $B = A/I$, $C = A/J$. Replacing, if necessary, A by $A/I \cap J$ we may assume, by Lemma (6.2), that $I \cap J = 0$. Let S_1 and S_2 be two non-isomorphic simple A -modules. Since I and J are contained in $\text{rad}A$, we get that S_1 and S_2 are both B -modules and C -modules. We may assume that S_1 is a simple projective B -module and S_2 is a simple injective B -module. We claim then that S_1 is also projective in $\text{mod}C$ and S_2 is also injective in $\text{mod}C$. Indeed, suppose that S_2 is projective in $\text{mod}C$. Consider the projective cover $f: P_B(S_2) \rightarrow S_2$ of S_2 in $\text{mod}B$ and the injective envelope $g: S_2 \rightarrow I_C(S_2)$ of S_2 in $\text{mod}C$. Then $f \in (\text{rad}^\infty(\text{mod}B))^2$ and $g \in (\text{rad}^\infty(\text{mod}C))^2$. Hence gf is a non-zero morphism in $\text{mod}A$ which belongs to $(\text{rad}^\infty(\text{mod}A))^4$, a contradiction. This proves our claim.

By (6.3) we know that B and C are of the form

$$B = \begin{pmatrix} F & {}_F M'_G \\ 0 & G \end{pmatrix} \quad C = \begin{pmatrix} F & {}_F M''_G \\ 0 & G \end{pmatrix}$$

where $F = \text{End}_A(S_2)$ and $G = \text{End}_A(S_1)$ are division algebras, and ${}_F M'_G, {}_F M''_G$ are F - G -bimodules such that

$$\dim_F({}_F M'_G) \times \dim_G({}_F M'_G) = 4 \quad \text{and} \quad \dim_F({}_F M''_G) \times \dim_G({}_F M''_G) = 4.$$

Hence, since $I \cap J = 0$, we deduce that A is of the form

$$A = \begin{pmatrix} F & {}_F M_G \\ 0 & G \end{pmatrix}$$

where ${}_F M_G$ is an F - G -bimodule such that

$${}_F M'_G \cong ({}_F M_G)/I \quad \text{and} \quad {}_F M''_G \cong ({}_F M_G)/J.$$

Moreover, we have then the inequality $\dim_F({}_F M_G) \times \dim_G({}_F M_G) \geq 5$, and so A is a wild hereditary algebra. In particular, Γ_A admits a component of type \mathbf{ZA}_∞ . But then, by Theorem (2.1), we have a contradiction with our assumption $(\text{rad}^\infty(\text{mod}A))^3 = 0$. \square

6.5. We shall now prove the main result of this section.

Theorem: *Let A be an artin algebra with $(\text{rad}^\infty(\text{mod}A))^3 = 0$. Then there exist at most finitely many ideals I in A such that A/I is tame concealed.*

Proof. Suppose there exists an infinite sequence of pairwise different ideals I_1, I_2, I_3, \dots in A such that the algebras $B_j = A/I_j$, $j \geq 1$, are tame concealed. Without loss of generality we may assume that $I_j \subset \text{rad}A$ for any $j \geq 1$. We infer from Lemma (6.2) that I_j is not a subset of I_l and I_l is not a subset of I_j , for any $j \neq l$. Moreover, by Lemma (6.4), the rank n of $\text{K}_0(A)$ is at least 3. This implies that, for any $i \geq 1$, Γ_{B_i} admits a stable tube of rank at least two [31]. Further, we know from (5.2) that if \mathcal{T} is a stable tube of Γ_{B_i} , then either \mathcal{T} is a full component of Γ_A or all modules of \mathcal{T} belong to one non-regular component of Γ_A .

Let $(\Gamma_\lambda)_{\lambda \in \Omega}$ be the family of all regular components of Γ_A . Then, by (2.1), Γ_λ , $\lambda \in \Omega$, are pairwise orthogonal generalized standard stable tubes. Denote by $\text{rk}\Gamma_\lambda$ the rank of the tube Γ_λ . Then, by [38](5.10), we have the following inequality

$$\sum_{\lambda \in \Omega} (\text{rk}\Gamma_\lambda - 1) \leq n - 1$$

Hence, since Γ_A admits only finitely many non-regular components, there exists a non-regular component \mathcal{C} in Γ_A such that for infinitely many $i \geq 1$ there exists a stable tube of rank ≥ 2 in Γ_{B_i} , whose modules all are contained in \mathcal{C} . We know from (5.4) and (5.5) that if Γ' is a left (respectively, right) stable translation subquiver of Γ_A containing no oriented cycles and closed under predecessors (respectively, successors) in Γ_A , then Γ' has no module belonging to a stable tube of Γ_{B_i} , $i \geq 1$. Further, all but finitely many τ_A -orbits in \mathcal{C} are periodic (3.2). Then, there is a connected component \mathcal{D} of either \mathcal{C}_l or \mathcal{C}_r containing oriented cycles, and there is an infinite sequence $1 \leq i_1 < i_2 < i_3 < \dots$ such that, for each $s \geq 1$, the Auslander-Reiten quiver of B_{i_s} admits a stable tube \mathcal{T}_s of rank greater or equal to 2 whose modules all belong to \mathcal{D} . By duality, we may assume that \mathcal{D} is a component of \mathcal{C}_l . Then either \mathcal{D} is a stable tube, if it contains a periodic module (1.4), or there exists, by [26](2.3), an infinite sectional path

$$\dots \longrightarrow \tau_A^{2t} X_1 \longrightarrow \tau_A^t X_s \longrightarrow \dots \longrightarrow \tau_A^t X_2 \longrightarrow \tau_A^t X_1 \longrightarrow X_s \longrightarrow \dots \longrightarrow X_2 \longrightarrow X_1$$

in \mathcal{D} with $l > s$, where at least one of the modules X_j is not stable, and where $\{X_1, \dots, X_s\}$ is a complete set of representatives of τ_A -orbits in \mathcal{D} . In both cases, there is an infinite sectional path

$$\dots \longrightarrow Z_3 \longrightarrow Z_2 \longrightarrow Z_1 \quad (*)$$

in \mathcal{D} such that every Z_i has exactly two direct predecessors (respectively, successors) in \mathcal{C} , and there are two integers $p \neq q$ such that $(*)$ contains infinitely many modules from \mathcal{T}_p and infinitely many modules from \mathcal{T}_q . We know that all but finitely many modules in \mathcal{T}_p (respectively, \mathcal{T}_q) are faithful B_{i_p} -modules (respectively, B_{i_q} -modules). Hence, there is a subpath

$$M = Z_0 \longrightarrow Z_1 \longrightarrow Z_2 \longrightarrow \dots \longrightarrow Z_m \longrightarrow Z_{m+1} \longrightarrow \dots \longrightarrow Z_l = N$$

of $(*)$ such that M and N are faithful B_{i_p} -modules from \mathcal{T}_p and Z_m is a faithful B_{i_q} -module from \mathcal{T}_q . Choose irreducible morphisms $f_j: Z_{j-1} \longrightarrow Z_j$, $1 \leq j \leq l$, in $\text{mod} A$, and put $g = f_l \cdots f_1$. Then, by (1.5), we get that $g \in \text{rad}^l(M, N) \setminus \text{rad}^{l+1}(M, N)$. Hence, if some Z_j is a B_{i_p} -module, then it belongs to \mathcal{T}_p . Therefore, we may assume that the modules Z_1, \dots, Z_{l-1} do not belong to \mathcal{T}_p . We claim that g is an irreducible morphism in $\text{mod} B_{i_p}$. Suppose this is not the case. Then, since \mathcal{T}_p is generalized standard in the Auslander-Reiten quiver of B_{i_p} , g is a sum of composites of irreducible morphisms between modules from \mathcal{T}_p . But then $g \in \text{rad}^{l+1}(M, N)$ because the above path in \mathcal{D} from M to N is a shortest one, a contradiction. This proves that g is irreducible in $\text{mod} B$, and hence is either a monomorphism or an epimorphism.

Observe that g factors through the faithful B_{i_q} -module Z_m , and consequently I_{i_q} is contained in I_{i_p} , because $\text{ann} M = I_{i_p} = \text{ann} N$ and $\text{ann} Z_m = I_{i_q}$. Therefore, we get a contradiction as required, and this finishes our proof. \square

6.6. We have the following consequences of Theorem (6.5).

Corollary: *Let A be an artin algebra with $(\text{rad}^\infty(\text{mod} A))^3 = 0$. Then there is a finite number of tame concealed factor algebras B_1, \dots, B_r of A such that, if \mathcal{T} is a stable tube of Γ_A , then \mathcal{T} is a component of some Γ_{B_i} , $1 \leq i \leq r$.*

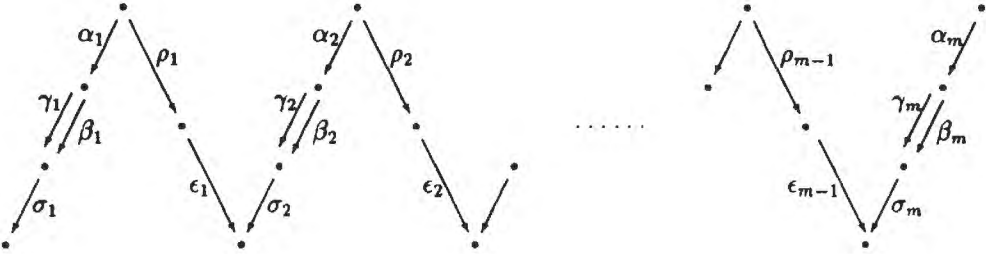
Proof. We know that if \mathcal{T} is a stable tube of Γ_A , then $B = A/\text{ann} \mathcal{T}$ is a tame concealed algebra, and clearly \mathcal{T} is a component of Γ_B . \square

6.7. Corollary: *Let A be a finite dimensional algebra over an algebraically closed field such that $(\text{rad}^\infty(\text{mod} A))^3 = 0$. Then A is domestic.*

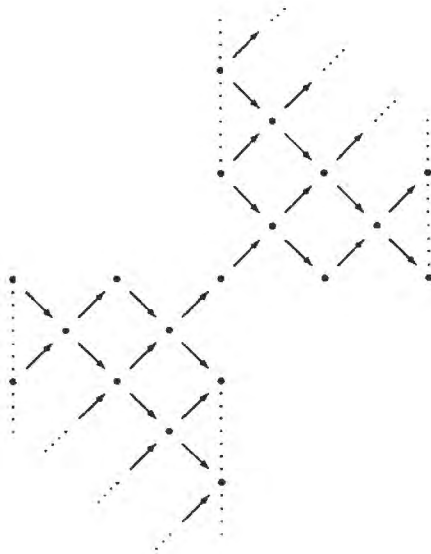
Proof. Since $(\text{rad}^\infty(\text{mod} A))^3 = 0$, by [24](1.7), we infer that A is tame. Then, by [16](Corollary F), for any dimension d , all but a finite number of isomorphism classes of indecomposable A -modules of dimension d lie in stable tubes of rank 1. From the above Corollary we know that the stable tubes of Γ_A belong to a finite number of tubular families given by tame concealed algebras. Therefore, A is domestic. \square

6.8. The following example shows that an algebra A with $(\text{rad}^\infty(\text{mod}A))^3 = 0$ may contain an arbitrary large number of tame concealed factor algebras.

Example. Let A be the bound quiver algebra kQ/I over an algebraically closed field k given by the quiver Q of the form



and the ideal I in kQ generated by the elements $\sigma_i\gamma_i$, $\beta_i\alpha_i$, $1 \leq i \leq m$, and $\epsilon_i\rho_i$, $1 \leq i \leq m - 1$. Then, it is easy to see that $(\text{rad}^\infty(\text{mod}A))^3 = 0$ and Γ_A consists of postprojective components, preinjective components, tubes, and components of the form



where we identify along the vertical dotted lines.

Clearly, the path algebras $H_i = k\Delta_i$, $1 \leq i \leq m$, given by the subquivers

$$\Delta_i : \bullet \begin{array}{c} \xleftarrow{\gamma_i} \\ \xleftarrow{\beta_i} \end{array} \bullet$$

are factor algebras of A . Moreover, each of the above glued tubes contains one stable tube of rank 1 of Γ_{H_i} , and one stable tube of rank 1 of $\Gamma_{H_{i+1}}$, for some $1 \leq i \leq m-1$.

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