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IMPROVED SIGNIFICANCE PROBABILITIES OF THE WILCOXON TEST

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Improved Significance Probabilities of the Wilcoxon Test

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Abstract

An Edgeworth approximation for accurate significance probabilities for the Wilcoxon two-sample test is substantially simplified. A method is developed that allows quick and easy calculations of very accurate probabilities. Exact formulas are given for most of the remaining cases. Tables are presented comparing the accuracy of the new simplification relative to the most likely alternatives.

Improved Significance Probabilities of the Wilcoxon Test

Accurate p-values provide a precise method for testing the significance of the test statistic. Such probabilities are also useful for applications of Bonferroni procedures. Biased probabilities will add up to even greater bias in the simultaneous Type I error probability. Accuracy is needed in moderate sized p-values as well as the more extreme p-values for the purpose of combining p's from different studies.

Without loss of generality we label the sample sizes m and n so that

$$m \leq n \tag{1}$$

It will suffice to consider the one-sided test, the null distribution being symmetric. We shall use the Mann-Whitney version of the test statistic, U, being the number of times that items from one sample outrank items from the other. On observing $U = u$, we seek $P = P_H(U \leq u)$.

The straightforward approach would be to table P as a function of m, n, and u. Equivalently, one might table

$$I = \binom{m+n}{m} P; \tag{2}$$

this would have the advantage of providing exact results, at the price of requiring use of a table of binomial coefficients. Unfortunately, an adequate table in either of these styles must be very large. The statistic U has $mn + 1$ values for given m and n, so that the number of entries grows like n^4. To cover samples of sizes 25 or less, even though exact values are not given, the Montreal table (Buckle, Kraft, & van Eeden, 1969) runs to nearly 300 pages.

A normal approximation can be given for P. Since

$$E(U) = mn/2 \tag{3}$$

and

$$Var(U) = mn(m + n + 1)/12, \tag{4}$$

this calls for

$$P \approx \Phi(z), \tag{5}$$

where

$$z = \frac{u + 0.5 - mn/2}{\sqrt{mn(m + n + 1)/12}}, \tag{6}$$

and $\Phi(z)$ is the normal distribution function. As m and n tend to infinity we would expect the absolute error of Equation 5 to tend to zero, but it is unfortunately subject to rather large percentage errors when P is small and m and n are of modest size.

<u>A simplified Edgeworth approximation</u>

Fix and Hodges (1955) determined the first six moments of U and thus obtained the following Edgeworth refinement of Equation 5:

$$P \approx \Phi(z) - \frac{(m^2 + n^2 + mn + m + n)\phi^{(3)}(z)}{20mn(m + n + 1)} + \frac{N\phi^{(5)}(z)}{210m^2n^2(m + n + 1)^2} + \frac{(m^2 + n^2 + mn + m + n)^2\phi^{(7)}(z)}{800m^2n^2(m + n + 1)^2}, \tag{7}$$

where

$$N = 2(m^4 + n^4) + 4mn(m^2 + n^2) + 6m^2n^2 + 4(m^3 + n^3) + 7mn(m + n) + m^2 + n^2 + 2mn - m - n, \tag{8}$$

and $\phi^{(i)}$ = the i^{th} derivative of the normal density function.

If m and n are large and of the same order one would expect from the Edgeworth theory that Equation 7 has an error of order $1/n^3$. Ury (1977) found that Equation 7 gave good results when $m \geq 6$. Despite their accuracy, the complexity of Equations 7 and 8 seem to have limited their usefulness.

We now consider a simplified version of Equations 7 and 8 which is shown in Appendix A to be of the same order of n. Let us replace N by

$$N' = 2(m^2 + n^2 + mn + m + n)^2. \tag{9}$$

Now we write

$$k = \frac{20mn(m + n + 1)}{m^2 + n^2 + mn + m + n}, \tag{10}$$

$$s = z^2, \qquad \text{and} \qquad c = 1 + \frac{T_1}{k} + \frac{T_2}{k^2}, \tag{11}$$

where $$T_1 = s - 3 \text{ and } T_2 = \frac{155s^2 - 416s - 195}{42}. \tag{12}$$

and

$$P = \Phi(zc). \tag{13}$$

An example will be helpful. Suppose that m = 10, n = 11, and that u is observed to be 21. By Equation 6, z = -2.358999, giving by Equation 5 the normal approximation P = .009162. We shall show below that the correct value is P = .0079356. In absolute terms the error does not seem great, but relatively the normal approximation is 15.5% too large. If one thinks of P as a scale for measuring the strength of the evidence against H, a 15% error does not seem trivial.

Now consider the simplified Edgeworth correction. With s = 5.56488 and k = 137.5, Equation 11 gives c = 1 + .018654 + .002884 = 1.021538, so that by Equation 13 P = Φ(-2.409807) = .0079805. The error is brought down from 15.5% to 0.6%.

Since we are using a continuous distribution to approximate an integer-valued one, it is reasonable to use the continuity correction that we made at Equation 6 by adding 0.5 to u. However, investigators have found that this correction actually makes the approximation worse when P is small (Kruskal & Wallis,1952; Lehmann,1975).

In Appendix B we show that a correction for kurtosis is necessary to make the continuity correction helpful. Kruskal and Wallis (1952) and Lehmann (1975) did not correct for kurtosis leading to their paradoxical results as noted above. The present Edgeworth approximations (7) and (13) include a kurtosis correction and consequently benefits from the continuity correction even with small values of P. We also note in Appendix B that Sheppard's correction is not helpful in the present approximation.

Exact values of P with n small

Our simplified Edgeworth approximation, Equation 13, should deal adequately with nearly all applications in which neither sample size is small. Cases will, however, arise with one or both samples small, and one cannot then expect the Edgeworth approach to be accurate.

We now turn to the possibility of providing, in a compact and convenient way, some exact values of P, by giving the integer I defined in Equation 2. Table H of Hodges and Lehmann (1964), for example, gives I for all cases with $u \le 20$ and $m \le n \le 8$. This table requires only one page, but it does not go quite far enough to complement Edgeworth. Lehmann (1975) uses three pages to reach n = 10, even though he gives not exact I values but four-decimal P's. To cover small samples adequately one needs a less direct approach.

If both sample sizes are at least as large as u, then the number of rank patterns for which U = u does not depend on m or n but only on u. We denote the number of such patterns by a(u). The function, a(u), has been studied in number theory, because it may be interpreted as the number of ways to partition u into positive integer summands. For example, a(4) = 5 because there are five partitions of four: 4, 3 + 1, 2 + 2, 2 + 1 + 1, and 1 + 1 + 1 + 1. Since a(u) depends on a single argument it is easily tabled. Table 24.5 of Abramowitz and Stegun (1964) gives a(u) up to u = 500. If both sample sizes are at least 31 this table tells us that there are a(31) = 6,842 patterns for which U = 31.

In statistical applications one wants the cumulative

$$A(u) = a(0) + a(1) + ... + a(u). \qquad (14)$$

In Appendix C are given the values of A(u) for $u \le 31$. This simple and brief table provides exact P in the extreme tail, whatever m and n may be. It is just in the extreme tail that exact values are required if the <u>percentage</u> error of P is to be controlled. The relative error of the Edgeworth approximation does not tend to zero in the extreme tail, even as m and n tend to infinity.

Another quantity useful in simplifying exact calculations is

$$B(u) = A(0) + A(1) + ... + A(u). \qquad (15)$$

Appendix C also gives the values of B(u) for $u \leq 15$. If $u \leq m + n$ and $u \leq 2m + 2$ then

$$I = A(u) - B(u - m - 1) - B(u - n - 1), \qquad (16)$$

where B vanishes if its argument is negative.

As an illustration of the use of Equation 16, take the example already considered previously for $m = 10$, $n = 11$, and $u = 21$. Our bounds on u being met, we may calculate $I = A(21) - B(10) - B(9) = 3506 - 423 - 284 = 2799$, or $P = 2799/\binom{21}{10} = .007935563$. This example is at the edge of our method, but as noted above, the simplified approximation is already worth using: its error is less than 0.6%.

<u>Exact values of P with m small, n large</u>

The small table of A and B given in Appendix C provides, in a very easy way, values of P for the majority of applications for which the simplified Edgeworth approximation will not serve. There remains, however, one type of design that is not covered: very small m combined with quite large n. The limitation $u \leq 2m + 2$ makes Equation 16 inadequate in such cases, unless u just happens to be very small. An asymptotic formula is also undependable when one sample size is tiny.

Now it might be argued that this extreme design is a bad one that ought not to be used. Nevertheless, it needs to be considered for three reasons. First, there are applications of the Wilcoxon test in which the sample sizes are not under the experimenter's control, and may turn out to be quite unequal. Second, it is common for the data to have been collected before the statistician has a chance to advise on good design. And finally, there are experiments in which one group size is sharply limited, while we can and should make the other one large.

Our extreme design is just the one for which the exact tables by Fix and Hodges (1955) are ideal. They rest on Euler's formula which under the restriction, $u \leq n$, can be expressed as a series of five simple formulas for m = 1 to 5. All five formulas are given in Appendix C with the formula for m = 5 giving I as the nearest integer to

$$(3u^5 + 120u^4 + 1780u^3 + 12000u^2 + Lu + 30000)/43200, \qquad (19)$$

where L is 36000 for even u and 35325 for odd u.

As m increases, the formulas become more complex and the limitation $u \leq n$ more restrictive. Fortunately, by m = 6 and with large n, the Edgeworth method is reasonably effective.

As an illustration, let m = 5 and n = 26, to take an example just outside the Montreal table (Buckle, et al., 1969). Let u = 26, which is at the limit of the method. Then Equation 19 gives I = 3028.8, which we round to 3029. Thus P = 3029/169911 = 0.0178270. The simplified Edgeworth formula gives P = 0.01788, off by 0.3%.

<u>Accurate p-values for the Wilcoxon</u>

Appendix C gives a concise summary of methods for finding accurate p-values for the U statistic. The first two sections give exact values which are clearly preferable to approximate ones. At (1) is given the exact expression I in terms of A and B. With the aid of the small table of A and B shown at the bottom, this formula deals very easily with applications where neither sample size is too large. There follows at (2) the formulas that provide exact I when $m \leq 5$ and n is large. Then at (3) is the Edgeworth approximation. This is the simplified theoretical series that has error of order $1/n^3$, useful when neither sample size is too small.

Evaluation of the Present Approach

Ury (1977) reported very accurate p-values for Equation 7, proposed by Fix and Hodges (1955), which we designate FH_2. The simplified version of FH_2 given by Equations 9 to 13 is designated, FH_s. Perhaps the most popular methods in current use are the normal approximation, z, and its continuity corrected version z_{cc}. Table 1 presents p-values from each of these four procedures as well as the exact p-values.

The exact values were computed by a FORTRAN routine using Dineen and Blakesley's (1973) algorithm. The method generates the sampling distribution using a recurrence formula. The algorithm starts with the known distribution for $m = 1$ and proceeds to generate the full distribution for each larger value of m until the desired distribution is obtained. The method is quite accurate giving results correct to nine decimal figures. However, it can be quite slow as m and n increase. Specific u values were chosen to produce exact probabilities in the lower tail of the distribution near the nominal levels: .0001, .001, .01, .025, .05, .10, .15, .20, .30, and .40.

Table 2 presents the results from Table 1 in the form of error percentages. Positive errors indicate conservative values. From Tables 1 and 2 it is clear that z and z_{cc} give poor estimates of p-values for the cases considered except in isolated instances. Both

FH_2 and FH_s give good estimates of exact p-values with FH_2 being slightly more accurate especially for very low p-values ($p < .001$) There are a few cases where the accuracy of FH_s is not very good. For example, with $m = 6$, $n = 45$ and $u = 21$, Table 1 shows the exact $P = .0001024$ while FH_s gives a value of .0001454 for an error of 42.00%. The value of $m = 6$ is too large for either of the exact methods given in Appendix C. However, most users will probably not require great accuracy with p values less than .001. If we consider only those cases for which exact solutions are not readily available and for which the exact $P \geq .001$ then the worst case found for FH_s is $m = 6$, $n = 6$, $u = 0$ where the exact $P = .001082$, FH_s value is .001169 and the error is 8.06%. Since in this case both z and z_{cc} have errors in excess of 80%, the present approach using FH_s seems to represent a considerable improvement over these alternatives.

Tables 1 and 2 about here

The present approach does not allow for ties. An algorithm which does handle ties is provided by Mehta et al. (1984).

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Table 1

Some Exact and Approximate Probabilities of the U Distribution

m	n	u	P(U ≤ u)	Z_{cc}	Z	FH_2	FH_s
3	5	0	.01786	.01844	.01267	.01502	.01510
		1	.03571	.03682	.02632	.03617	.03612
		2	.07143	.06802	.05053	.07205	.07206
		3	.12500	.11652	.08986	.12563	.12572
		4	.19643	.18555	.14836	.19786	.19799
		5	.28571	.27549	.22803	.28729	.28742
		6	.39286	.38280	.32736	.39001	.39009
3	15	0	.001225	.004576	.003843	.001764	.002149
		2	.004902	.008909	.007578	.005686	.005917
		3	.008578	.012195	.010440	.009016	.009167
		6	.02819	.02901	.02531	.02747	.02747
		8	.05025	.04860	.04292	.04953	.04953
		11	.10172	.09626	.08654	.10211	.10214
		13	.15074	.14316	.13020	.15190	.15194
		15	.21324	.20347	.18713	.21369	.21373
		17	.28676	.27681	.25733	.28645	.28647
		20	.41176	.40635	.38355	.41119	.41120
3	45	1	.0001156	.0024690	.0023103	.0003785	.0007931
		5	.0009251	.0041369	.0038841	.0015314	.0018963
		15	.01006	.01339	.01267	.01039	.01051
		22	.02648	.02764	.02632	.02600	.02601
		28	.05001	.04835	.04625	.04921	.04921
		36	.09875	.09336	.08986	.09875	.09877

		42	.15102	.14348	.13872	.15191	.15194
		46	.19444	.18555	.17991	.19518	.19520*
		54	.29949	.28989	.28265	.29889	.29890*
		61	.40489	.39915	.39095	.40415	.40416*
4	5	0	.007937	.009982	.007153	.006737	.006929
		1	.01587	.01867	.01374	.01595	.01599
		2	.03175	.03310	.02502	.03208	.03206
		3	.05556	.05567	.04321	.05750	.05748
		4	.09524	.08895	.07082	.09423	.09426
		5	.14286	.13517	.11034	.14359	.14365
		6	.20635	.19563	.16359	.20582	.20590
		7	.27778	.27015	.23122	.27994	.28002
		8	.36508	.35665	.31210	.36373	.36379
4	15	0	.0002580	.0015889	.0013499	.0003079	.0005316
		2	.001032	.002980	.002555	.001224	.001434
		7	.009804	.012224	.010724	.009896	.009962
		10	.02425	.02559	.02275	.02415	.02415
		13	.05005	.04947	.04457	.05023	.05023
		16	.09236	.08851	.08076	.09236	.09237*
		19	.15351	.14686	.13567	.15352	.15354*
		21	.20537	.19766	.18406	.20532	.20534*
		24	.29825	.29116	.27425	.29821	.29823*
		27	.40480	.40129	.38209	.40517	.40517*
4	45	5	$.0^{4}8496$	.0010160	$.0^{3}95539$	$.0^{4}9349$	$.0^{3}27534$
		13	.0009156	.0026080	.0024645	.0010547	.0012390
		28	.01018	.01236	.01179	.01018	.01024

		36	.02409	.02538	.02432	.02400	.02401
		44	.04889	.04831	.04651	.04892	.04892
		54	.10161	.09744	.09433	.10159	.10160*
		60	.14693	.14070	.13666	.14678	.14679*
		66	.20278	.19542	.19042	.20263	.20263*
		75	.30468	.29824	.29194	.30478	.30478*
		82	.39586	.39210	.38510	.39607	.39607*
6	6	0	.001082	.002537	.001974	.000976	.001169
		3	.007576	.010120	.008155	.007864	.007937
		5	.02056	.02266	.01869	.02106	.02106
		7	.04654	.04635	.03908	.04681	.04680
		9	.08983	.08674	.07477	.09031	.09032
		11	.15476	.14898	.13117	.15541	.15544
		12	.19697	.18924	.16833	.19648	.19652
		14	.29437	.28759	.26092	.29441	.29444*
		16	.40909	.40509	.37439	.40867	.40869*
6	12	1	.0001077	.0006163	.0005226	.0001006	.0001847
		5	.001023	.002141	.001846	.001063	.001157
		11	.009104	.010877	.009604	.009203	.009234
		15	.02645	.02743	.02460	.02652	.02651*
		18	.05123	.05060	.04591	.05121	.05121*
		22	.10644	.10304	.09489	.10646	.10647*
		24	.14539	.14072	.13053	.14539	.14541*
		26	.19225	.18680	.17449	.19228	.19229*
		30	.30823	.30323	.28707	.30822	.30823*
		33	.41004	.40744	.38936	.41012	.41013*

6	45	21	.0001024	.0004530	.0004299	.0000948	.0001454*
		35	.0009602	.0018135	.0017305	.0009715	.0010333*
		57	.01030	.01173	.01129	.01033	.01035*
		68	.02508	.02594	.02507	.02508	.02509*
		78	.04975	.04929	.04782	.04973	.04973*
		90	.09931	.09663	.09416	.09930	.09930*
		98	.14700	.14297	.13969	.14670	.14700*
		105	.19897	.19422	.19023	.19896	.19897*
		116	.29869	.29430	.28929	.29870	.29870*
		126	.40429	.40187	.39623	.40431	.40431*
8	8	0	$.0^{4}7770$	$.0^{3}46955$	$.0^{3}38877$	$.0^{4}5590$	$.0^{3}12670$
		4	.0009324	.0019380	.0016379	.0009473	.0010341
		10	.01033	.01197	.01043	.01038	.01040
		13	.02494	.02601	.02300	.02505	.02505
		16	.05245	.05178	.04645	.05244	.05244
		19	.09744	.09463	.08608	.09758	.09758*
		21	.13932	.13507	.12400	.13944	.13945*
		23	.19114	.18601	.17228	.19131	.19132*
		26	.28687	.28176	.26431	.28688	.28689*
		29	.39922	.39645	.37636	.39933	.39933*
8	40	36	.0001074	.0003171	.0003104	.0001029	.0001244*
		53	.001010	.001608	.001538	.001014	.001042*
		77	.01017	.01124	.01083	.01018	.01019*
		89	.02491	.02557	.02476	.02492	.02492*
		100	.05023	.04988	.04847	.05023	.05023*
		113	.10123	.09915	.09676	.10123	.10123*

	122	.15292	.14977	.14657	.15292	.15292*
	129	.20301	.19940	.19556	.20301	.20302*
	140	.29811	.29479	.29003	.29812	.29812*
	150	.39829	.39635	.39103	.39830	.39830*
1212	12	.0001006	.0002960	.0002660	.0000964	.0001163
	20	.0009149	.0014728	.0013401	.0009189	.0009458
	32	.01024	.01129	.01046	.01026	.01027*
	38	.02593	.02655	.02482	.02595	.02595*
	43	.05027	.04994	.04703	.05028	.05028*
	49	.09890	.09697	.09210	.09893	.09893*
	53	.14567	.14274	.13633	.14570	.14570*
	57	.20476	.20125	.19324	.20478	.20479*
	62	.29494	.29168	.28185	.29496	.29496*
	67	.39937	.39751	.38642	.39939	.39939*
1240	77	$.0^{4}9847$	$.0^{3}20835$	$.0^{3}19996$	$.0^{4}9703$	$.0^{3}10365$*
	102	.001034	.001412	.001363	.001035	.001044*
	134	.01026	.01097	.01066	.01026	.01027*
	150	.02553	.02596	.02531	.02553	.02553*
	164	.05078	.05053	.04941	.05078	.05078*
	180	.09952	.09813	.09627	.09952	.09952*
	191	.14819	.14609	.14362	.14819	.14819*
	200	.19793	.19548	.19249	.19793	.19793*
	215	.29959	.29733	.29358	.29960	.29960*
	228	.40266	.40138	.39719	.40266	.40266*
2525	127	.0001010	.0001656	.0001596	.0001006	.0001027*
	156	.001001	.001236	.001197	.001001	.001004*

	193	.01000	.01047	.01021	.01000	.01000*
	211	.02471	.02502	.02445	.02471	.02471*
	227	.04970	.04955	.04856	.04970	.04970*
	246	.10112	.10017	.09848	.10112	.10112*
	258	.14879	.14738	.14515	.14879	.14879*
	268	.19827	.19663	.19395	.19828	.19828*
	285	.30169	.30018	.29682	.30169	.30170*
	299	.40130	.40043	.39668	.40130	.40130*
2545	267	.0001012	.0001497	.0001462	.0001010	.0001020*
	314	.001002	.001184	.001160	.001002	.001003*
	373	.009891	.010263	.010097	.009891	.009892*
	402	.02469	.02493	.02458	.02469	.02469*
	428	.05038	.05025	.04962	.05038	.05038*
	457	.09979	.09905	.09799	.09979	.09979*
	477	.14987	.14874	.14732	.14987	.14987*
	493	.20016	.19885	.19714	.20016	.20016*
	519	.30029	.29908	.29695	.30029	.30029*
	541	.39914	.39844	.39607	.39914	.39914*

*Exact probabilities are <u>not</u> available from either method given in the Appendix.

Table 2

Error Percentages of Approximations to the U Distribution

m	n	u	P(U ≤ u)	Z_{cc}	Z	FH_2	FH_3
3	5	0	.01786	3.29	-29.03	-15.87	-15.43
		1	.03571	3.09	-26.31	1.28	1.14
		2	.07143	-4.77	-29.26	.87	.88
		3	.12500	-6.78	-28.12	.50	.58
		4	.19643	-5.54	-24.47	.73	.80
		5	.28571	-3.58	-20.19	.55	.60
		6	.39286	-2.56	-16.67	-.73	-.70
3	15	0	.001225	273.39	213.58	43.90	75.38
		2	.004902	81.74	54.59	16.00	20.71
		3	.008578	42.16	21.70	5.10	6.86
		6	.02819	2.93	-10.22	-2.54	-2.52
		8	.05025	-3.27	-14.58	-1.41	-1.42
		11	.10172	-5.36	-14.92	.39	.42
		13	.15074	-5.03	-13.63	.77	.80
		15	.21324	-4.58	-12.24	.21	.23
		17	.28676	-3.47	-10.26	-.11	-.10
		20	.41176	-1.31	-6.85	-.14	-.14
3	45	1	.0001156	2035.15	1897.98	227.30	585.88
		5	.0009251	347.20	319.87	65.55	104.99
		15	.01006	33.08	25.98	3.29	4.48
		22	.02648	4.39	-.62	-1.82	-1.78
		28	.05001	-3.32	-7.53	-1.61	-1.60
		36	.09875	-5.46	-9.01	-.01	.02

		42	.15102	-4.99	-8.14	.59	.61
		46	.19444	-4.57	-7.47	.38	.39*
		54	.29949	-3.20	-5.62	-.20	-.20*
		61	.40489	-1.42	-3.44	-.18	-.18*
4	5	0	.00794	25.78	-9.87	-15.11	-12.70
		1	.01587	17.61	-13.42	.51	.75
		2	.03175	4.25	-21.18	1.06	.98
		3	.05556	.21	-22.23	3.49	3.47
		4	.09524	-6.60	-25.64	-1.06	-1.03
		5	.14286	-5.38	-22.77	.51	.56
		6	.20635	-5.19	-20.72	-.26	-.22
		7	.27778	-2.75	-16.76	.78	.81
		8	.36508	-2.31	-14.51	-.37	-.35
4	15	0	.0002580	515.85	423.22	19.35	106.05*
		2	.001032	188.74	147.59	18.62	38.93
		7	.009804	24.69	9.39	.94	1.61
		10	.02425	5.51	-6.19	-.43	-.41
		13	.05005	-1.16	-10.96	.36	.35
		16	.09236	-4.17	-12.57	-.01	.01*
		19	.15351	-4.33	-11.62	.01	.02*
		21	.20537	-3.75	-10.37	-.02	-.01*
		24	.29825	-2.38	-8.04	-.01	-.01*
		27	.40480	-.87	-5.61	.09	.09*
4	45	5	$.0^{4}8496$	1095.97	1024.58	10.05	224.10
		13	.0009156	184.83	169.15	15.19	35.32
		28	.01018	21.44	15.81	-.01	.56

		36	.02409	5.35	.94	-.36	-.32
		44	.04889	-1.17	-4.86	.07	.07
		54	.10161	-4.11	-7.16	-.02	-.01*
		60	.14693	-4.24	-6.99	-.10	-.09*
		66	.20278	-3.63	-6.10	-.08	-.07*
		75	.30468	-2.11	-4.18	.03	.03*
		82	.39586	-.95	-2.72	.05	.05*
6	6	0	.001082	134.46	82.39	-9.84	8.06
		3	.007576	33.59	7.64	3.80	4.76
		5	.02056	10.22	-9.12	2.41	2.42
		7	.04654	-.41	-16.01	.58	.56
		9	.08983	-3.43	-16.76	.54	.55
		11	.15476	-3.74	-15.25	.42	.44
		12	.19697	-3.92	-14.54	-.25	-.23
		14	.29437	-2.31	-11.36	.01	.02*
		16	.40909	-.98	-8.48	-.10	-.10*
6	12	1	.0001077	472.06	385.11	-6.64	71.41
		5	.001023	109.20	80.32	3.87	13.05
		11	.009104	19.48	5.50	1.10	1.43
		15	.02645	3.70	-6.99	.25	.25*
		18	.05123	-1.22	-10.38	-.03	-.03*
		22	.10644	-3.19	-10.85	.01	.02*
		24	.14539	-3.21	-10.22	.00	.01*
		26	.19225	-2.84	-9.24	.01	.02*
		30	.30823	-1.62	-6.86	-.00	.00*
		33	.41004	-.64	-5.04	.02	.02*

6	45	21	.0001024	342.42	319.82	-7.40	42.00*
		35	.0009602	88.86	80.21	1.17	7.61*
		57	.01030	13.93	9.66	.28	.46*
		68	.02508	3.42	-.04	.01	.02*
		78	.04975	-.92	-3.88	-.04	-.04*
		90	.09931	-2.70	-5.19	-.01	-.01*
		98	.14700	-2.74	-4.97	-.00	.00*
		105	.19897	-2.38	-4.39	-.00	.00*
		116	.29869	-1.47	-3.15	.00	.00*
		126	.40429	-.60	-1.99	.00	.00*
8	8	0	$.0^{4}7770$	504.31	400.34	-28.05	63.06
		4	.0009324	107.85	75.67	1.59	10.91
		10	.01033	15.87	.94	.40	.62
		13	.02494	4.30	-7.79	.43	.42
		16	.05245	-1.27	-11.44	-.01	-.01
		19	.09744	-2.88	-11.65	.14	.15*
		21	.13932	-3.04	-11.00	.09	.10*
		23	.19114	-2.68	-9.87	.09	.09*
		26	.28687	-1.78	-7.87	.00	.01*
		29	.39922	-.70	-5.73	.03	.03*
8	40	36	.0001074	195.30	180.64	-4.20	15.88*
		53	.001010	59.27	52.29	.41	3.23*
		77	.01017	10.46	6.51	.12	.21*
		89	.02491	2.64	-.62	.02	.02*
		100	.05023	-.71	-3.51	-.01	-.01*
		113	.10123	-2.05	-4.41	-.00	-.00*

	122	.15292	-2.05	-4.15	.00	.00*
	129	.20301	-1.78	-3.67	.00	.00*
	140	.29811	-1.11	-2.71	.00	.00*
	150	.39829	-.49	-1.82	.00	.00*
1212	12	.0001006	194.30	164.45	-4.19	15.58
	20	.0009149	60.98	46.48	.44	3.38
	32	.01024	10.19	2.11	.18	.25*
	38	.02593	2.38	-4.28	.07	.07*
	43	.05027	-.65	-6.43	.04	.04*
	49	.09890	-1.96	-6.87	.02	.03*
	53	.14567	-2.01	-6.41	.02	.02*
	57	.20476	-1.71	-5.62	.01	.02*
	62	.29494	-1.10	-4.44	.01	.01*
	67	.39937	-.47	-3.24	.00	.00*
1240	77	$.0^{4}9847$	111.59	103.07	-1.46	5.26*
	102	.001034	36.60	31.82	.10	1.03*
	134	.01026	6.95	3.93	.05	.07*
	150	.02553	1.70	-.84	.01	.01*
	164	.05078	-.50	-2.70	.00	.00*
	180	.09952	-1.39	-3.27	.00	.00*
	191	.14819	-1.42	-3.09	.00	.00*
	200	.19793	-1.24	-2.75	.00	.00*
	215	.29959	-.76	-2.01	.00	.00*
	228	.40266	-.32	-1.36	.00	.00*
2525	127	.0001010	63.93	57.94	-.39	1.60*
	156	.001001	23.49	19.59	.03	.33*

	193	.01000	4.72	2.06	.02	.03*
	211	.02471	1.23	-1.05	.01	.01*
	227	.04970	-.30	-2.28	.00	.00*
	246	.10112	-.94	-2.61	.00	.00*
	258	.14879	-.95	-2.44	.00	.00*
	268	.19827	-.83	-2.18	.00	.00*
	285	.30169	-.50	-1.62	.00	.00*
	299	.40130	-.22	-1.15	.00	.00*
25 45	267	.0001012	47.87	44.41	-.19	.80*
	314	.001002	18.20	15.81	.01	.17*
	373	.009891	3.77	2.09	.01	.01*
	402	.02469	.97	-.46	.00	.00*
	428	.05038	-.25	-1.51	.00	.00*
	457	.09979	-.74	-1.81	.00	.00*
	477	.14987	-.75	-1.70	.00	.00*
	493	.20016	-.66	-1.51	.00	.00*
	519	.30029	-.40	-1.11	.00	.00*
	541	.39914	-.18	-.77	-.00	-.00*

*Exact probabilities are not available from either method given in the Appendix.

Appendix A

Equivalence Up to Terms of Order $1/n^3$ of Equations 7 and 13.

N and N' differ by terms of order n^3, but since either will be divided by a term of order n^6, the substitution will change Equation 7 by an amount of order $1/n^3$. Since this is the magnitude of the error of Equation 7, we may reasonably say that an equivalent approximation results when N is replaced by N'.

Note that k is of the order n. An easy check shows that Equation 7 with N replaced by N', may be written in the form

$$P = \Phi(z) - \frac{\Phi^{(3)}(z)}{k} + \frac{80\Phi^{(5)}(z)}{21k^2} + \frac{\Phi^{(7)}(z)}{2k^2} .$$

If we let $s = z^2$, the Chebychev-Hermite polynomials permit us to write

$$\Phi^{(3)}(z) = -z\Phi(z)(s - 3)$$

$$\Phi^{(5)}(z) = -z\Phi(z)(s^2 - 10s + 15),$$

and $$\Phi^{(7)}(z) = -z\Phi(z)(s^3 - 21 s^2 + 105s - 105).$$

Combining these equations gives after simplification

$$P = \Phi(z) + z\Phi(z)\left[\frac{s - 3}{k} + \frac{-21s^3 + 281s^2 - 605s - 195}{42k^2}\right].$$

Clearly this would be easier to use than Equation 7, and the advantage is still greater if we put Equation 13 in its Cornish-Fisher version given by Equations 11 and 12. Taylor expansion about z now shows that Equation 13 agrees with the above equations except for terms of order $1/n^3$. Thus Equation 13 is equivalent to Equation 7 up to terms of order $1/n^3$.

Appendix B

Continuity and Sheppard's Corrections

The main Edgeworth correction, as shown at Equation 11, adds to c the kurtosis correction, T_1/k, which is of order $1/n$. Kruskal & Wallis (1952) and Lehmann (1975) do not correct for kurtosis, thereby committing by omission an error of order $1/n$. Inspection of Equation 6 shows that the continuity correction is of order $1/n^{3/2}$. They are therefore in the position of debating whether to make a small error after having swallowed a large one.

Approaching the question more precisely, let us note from Equation 12 that when $z < -\sqrt{3}$ the correction T_1/k is positive. From this it follows that the kurtosis error tends to increase a small P. Since the continuity correction also increases P, it is only to be expected that it will be harmful when P is small and not kurtosis-corrected. This is just what Kruskal, Wallis and Lehmann found. We, however, are in Equation 7 making corrections as small as $1/n^2$, and thus for us the continuity correction of order $1/n^{3/2}$ is essential.

Our analysis also throws light on the desirability of a Sheppard correction. The continuous Edgeworth "distribution" should of course be fitted using its own "moments." But the moments used earlier are those of the discrete Wilcoxon distribution, which may be thought of as obtained from the Edgeworth by grouping into unit intervals. In principle, therefore, we should apply Sheppard's corrections for grouping before using the Wilcoxon moments to fit the Edgeworth approximation.

The most important Sheppard correction calls for subtracting 1/12 from the Wilcoxon variance. A glance at Equation 4 shows that this changes the standard deviation, and hence P, by a term of order $1/n^3$. But we are ignoring terms of this order so there is no strong argument for the Sheppard correction in this problem.

Appendix C

Significance Probability, P, of the Mann-Whiney U Test

Sample sizes $m \le n$. Let u be the number of times an item of one sample outranks an item of the other, with the samples arranged so that $u \le mn/2$.

$P = P_H(U \le u) = I/\binom{m+n}{m}$ Use 2P for a two-sided test.

(1) Small n. If $u \le 2m + 2$ and $u \le m + n$, then

$$I = A(u) - B(u - m - 1) - B(u - n - 1)$$

where B vanishes for negative arguments; A & B are tabled below.

(2) Small m, large n. If $u \le n$, then I is the nearest integer:

to $u + 1$, if $m = 1$;

to $(u^2 + 4u + 4)/4$, if $m = 2$;

to $(2u^3 + 21u^2 + 66u + 60)/72$, if $m = 3$;

to $(u^4 + 22u^3 + 166u^2 + Lu + 500)/576$, if $m = 4$,

where L is 504 for even u and 486 for odd u;

to $(3u^5 + 120u^4 + 1780u^3 + 2000u^2 + Lu + 30000)/43200$, if $m = 5$,

where L is 36000 for even u and 35325 for odd u.

(3) Let $z = \dfrac{(u + 0.5 - 0.5mn)}{\sqrt{mn(m + n + 1)/12}}$, $s = z^2$,

and $k = \dfrac{20mn(m+n+1)}{m^2+n^2+mn+m+n}$. If m and n are large of the same order, then $P = \Phi(zc)$, where Φ is the normal probability and

$$c = 1 + \frac{1}{k}\left\{s - 3 + \frac{1}{42k}[-195 - s(416 - 155s)]\right\}.$$

u	A(u)	B(u)	u	A(u)	B(u)	u	A(u)	u	A(u)
0	1	1	8	67	187	16	915	24	7338
1	2	3	9	97	284	17	1212	25	9296
2	4	7	10	139	423	18	1597	26	11732
3	7	13	11	195	618	19	2087	27	14742
4	12	26	12	272	890	20	2714	28	18460
5	19	45	13	373	1263	21	3506	29	23025
6	30	75	14	508	1771	22	4508	30	28629
7	45	120	15	684	2455	23	5763	31	35471

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