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LAGRANGIAN DISTRIBUTIONS AND CONNECTIONS IN SYMPLECTIC GEOMETRY

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Lagrangian Distributions and Connections
in Symplectic Geometry

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Abstract

We discuss the interplay between lagrangian distributions and connections in symplectic geometry, beginning with the traditional case of symplectic manifolds and then passing to the more general context of poly- and multisymplectic structures on fiber bundles, which is relevant for the covariant hamiltonian formulation of classical field theory. In particular, we generalize Weinstein's tubular neighborhood theorem for symplectic manifolds carrying a (simple) lagrangian foliation to this situation. In all cases, the Bott connection, or an appropriately extended version thereof, plays a central role.

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1 Introduction

In symplectic geometry, lagrangian foliations or, more generally, lagrangian distributions (i.e., lagrangian subbundles of the tangent bundle, which may or may not be involutive) are an important tool. Although unfamiliar from riemannian or even lorentzian geometry, where the notion of a lagrangian subbundle does not arise (except in the rather uninteresting case of two-dimensional lorentzian manifolds), they also appear in the theory of certain pseudo-riemannian manifolds, namely those of zero signature. However, the interplay between lagrangian distributions and connections in symplectic geometry is quite different from, and considerably more interesting than, in pseudo-riemannian geometry. In what follows, we shall show that this interplay admits a completely natural extension to the context of poly- and multisymplectic fiber bundles, whose precise mathematical definition can be found in [12] and which appear naturally in the covariant hamiltonian formulation of classical field theory.

The paper is divided into two parts. The first part (Sect. 2-5) discusses connections which are compatible with a given foliation, whereas the second part (Sect. 6-9) uses them to derive various structure theorems. More specifically, we begin by reviewing some standard issues from symplectic geometry: the construction of the Bott connection in Sect. 2 and the classification of symplectic connections in Sect. 3: here, we also prove an analogous classification theorem for symplectic connections preserving a given lagrangian foliation which – although of independent interest – does not seem to have been explicitly formulated in the literature. Next, we show how to extend this classification to poly-symplectic fiber bundles in Sect. 4 and to multisymplectic fiber bundles in Sect. 5. The second part starts, in Sect. 6, with an exposition of the program to be developed in the remainder of the paper, followed by a study, in Sect. 7, of manifolds equipped with a given foliation by flat affine manifolds (where "flat affine" refers to a given partial connection along the leaves), since this is the situation prevailing in all cases of interest here. The results are applied in Sect. 8 to symplectic manifolds with a lagrangian foliation, allowing us to give a simple proof, as well as a generalization, of Weinstein's tubular neighborhood theorem [26, 29]. Exactly the same technique leads to what we call the structure theorem for polysymplectic and multisymplectic fiber bundles, presented in Sect. 9: it can be viewed as an analogue of the theorem from symplectic geometry that characterizes which symplectic manifolds are cotangent bundles (or "pieces" of cotangent bundles, possibly up to coverings). Finally, in Sect. 10, we present our conclusions.

2 Lagrangian distributions and the Bott connection

In this section, we briefly review the definition of the Bott connection for symplectic manifolds carrying a given lagrangian foliation, noting that a completely analogous concept also exists for pseudo-riemannian manifolds of zero signature (these are the only ones whose tangent spaces admit lagrangian subspaces).

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As a preliminary step, we recall that given a manifold M, a vector bundle V over M and a distribution L on M, a *partial linear connection in V along L* is an $\mathbb{R}$-bilinear map[1]

$$\nabla: \ \Gamma(L) \times \Gamma(V) \ \longrightarrow \ \Gamma(V)$$
$$(X, s) \ \longmapsto \ \nabla_X s \tag{1}$$

which is $\mathfrak{F}(M)$-linear in X and a derivation in s, i.e., satisfies the usual Leibniz rule

$$\nabla_X (fs) \ = \ f \nabla_X s + (X \cdot f) s \qquad \text{for } X \in \Gamma(L), \, f \in \mathfrak{F}(M), \, s \in \Gamma(V) \, . \tag{2}$$

Of course, this gives back the usual definition of a "full" linear connection when $L = TM$ and hence $\Gamma(L)$ is the Lie algebra $\mathfrak{X}(M)$ of all (smooth) vector fields on M. Clearly, the usual definitions of curvature and, in the special case when $V = L$, of torsion also work for partial linear connections if L is supposed to be involutive.

Obviously, partial linear connections can be obtained from "full" ones by *restriction*, that is, by restricting the definition of the covariant derivative of a section from general vector fields on M to vector fields on M along a given vector subbundle L of TM, and conversely, one may ask whether a given partial linear connection admits an *extension* to a "full" one (from which it can be derived by restriction), and if so, how one can classify all possible extensions.

A particularly nice example of a partial linear connection which is not evidently the restriction of a "full" one is the *Bott connection* associated with any involutive distribution on any manifold M: denoting by $L^\perp$ the *annihilator* of L, which by definition is the vector subbundle of the cotangent bundle T^*M of M consisting of 1-forms that vanish on L, this is a partial linear connection ∇^B in $L^\perp$ along L defined by

$$\nabla^B_X \alpha \ = \ \mathbb{L}_X \alpha \qquad \text{for } X \in \Gamma(L), \, \alpha \in \Gamma(L^\perp) \, , \tag{3}$$

where $\mathbb{L}_X$ denotes the Lie derivative (of 1-forms) along X, or more explicitly,

$$(\nabla^B_X \alpha)(Y) \ = \ X \cdot \alpha(Y) - \alpha([X, Y])$$
$$\text{for } X \in \Gamma(L), \, \alpha \in \Gamma(L^\perp), \, Y \in \mathfrak{X}(M) \, . \tag{4}$$

To show that this is really a partial linear connection in $L^\perp$ along L, suppose that X is a vector field on M along L and note that (a) for any section α of $L^\perp$ and any vector field Y on M along L, the expression in eqn (4) vanishes because L is supposed to be involutive, which means that the Lie derivative along X really does map sections α of $L^\perp$ to sections $\mathbb{L}_X \alpha$ of $L^\perp$, and (b) for any function f on M, any section α of $L^\perp$ and any vector field Y on M (not necessarily along L), we have

$$(\mathbb{L}_{fX} \alpha)(Y) \ = \ (fX) \cdot \alpha(Y) - \alpha([fX, Y])$$
$$= \ f(X \cdot \alpha(Y)) - f\alpha([X, Y]) + (Y \cdot f) \alpha(X)$$
$$= \ f(\mathbb{L}_X \alpha)(Y) + (Y \cdot f) \alpha(X) \, ,$$

---

[1]Given a manifold M and a vector bundle W over M, we denote by $\mathfrak{F}(M)$ the algebra of (smooth) functions on M and by $\Gamma(W)$ the space (and $\mathfrak{F}(M)$-module) of (smooth) sections of W. If W is a distribution on M, i.e., a vector subbundle of the tangent bundle TM of M, we shall use the intuitively more appealing expression "vector field on M along W" for a section of W.

and the last term vanishes because α annihilates X, so ∇_X^B is really $\mathfrak{F}(M)$-linear in X. Of course, the Bott connection is *flat*: its curvature vanishes trivially, due to the definition of the Lie bracket of vector fields by means of the formula $[\mathbb{L}_X, \mathbb{L}_Y] = \mathbb{L}_{[X,Y]}$.

Now assume that M is a manifold of even dimension $2n$ and L is an involutive distribution on M which is lagrangian with respect to a given almost symplectic form (i.e., non-degenerate 2-form) ω. Then the "musical isomorphism" [1, p. 166]

$$\omega^\flat : TM \longrightarrow T^*M \quad \text{with inverse} \quad \omega^\sharp : T^*M \longrightarrow TM \tag{5}$$

restricts to an isomorphism

$$\omega^\flat : L \longrightarrow L^\perp \quad \text{with inverse} \quad \omega^\sharp : L^\perp \longrightarrow L \tag{6}$$

which can be used to transfer the Bott connection as defined previously to a partial linear connection in L along L: by abuse of language, it will simply be called the *Bott connection in L*. Explicitly, it is determined by the formula

$$\omega(\nabla_X^B Y, Z) = X \cdot \omega(Y, Z) - \omega(Y, [X, Z]) \quad \text{for } X, Y \in \Gamma(L), \, Z \in \mathfrak{X}(M) \,. \tag{7}$$

This connection has an intuitively appealing interpretation: it is nothing else than a canonical family of ordinary linear connections in the leaves of the foliation generated by L. Moreover, the Bott connection in $L^\perp$ being flat, so is the Bott connection in L. But regarding the latter, we can ask for more: we can ask whether it also has vanishing torsion, since if so, we may conclude that the leaves of the foliation generated by L are flat affine manifolds. Regarding this question, we have the following simple answer.

Theorem 1 *Let M be a manifold equipped with an almost symplectic form ω and let L be any involutive lagrangian distribution on M. Then if ω is closed, the Bott connection ∇^B in L has zero torsion. More generally, the torsion tensor T^B of ∇^B is related to the exterior derivative of ω by the formula*

$$d\omega(X, Y, Z) = \omega(T^B(X, Y), Z) \quad \text{for } X, Y \in \Gamma(L), \, Z \in \mathfrak{X}(M) \,. \tag{8}$$

Proof: Writing out the Cartan formula for the exterior derivative of ω,

$$\begin{aligned} d\omega(X, Y, Z) = {}& X \cdot \omega(Y, Z) - Y \cdot \omega(X, Z) + Z \cdot \omega(X, Y) \\ & - \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X) \,. \end{aligned}$$

and assuming X and Y to be along L, we see that the third of the six terms on the rhs of this equation vanishes since L is supposed to be isotropic, so using the definition of the torsion tensor combined with eqn (7) to give

$$\begin{aligned} \omega(T^B(X, Y), Z) = {}& \omega(\nabla_X^B Y - \nabla_Y^B X - [X, Y], Z) \\ = {}& X \cdot \omega(Y, Z) - \omega(Y, [X, Z]) - Y \cdot \omega(X, Z) + \omega(X, [Y, Z]) - \omega([X, Y], Z) \,, \end{aligned}$$

we arrive at eqn (8), which proves the remaining statements. $\qquad\square$

Of course, this result has been known for a long time; see, e.g., Theorem 7.7 of Ref. [29]. The only difference is that we propose a more systematical and ample use of the term "Bott connection".[2]

3 Symplectic connections

Given a manifold M equipped with a symplectic form ω and an involutive distribution L on M, we can ask the following question: is the Bott connection ∇^B in L the restriction of some torsion-free symplectic connection ∇ on M, and if so, what is the set of such torsion-free symplectic connections?

To gain a better understanding of this question and of its importance for quantization (geometric quantization as well as deformation quantization), let us briefly explain a few well-known facts about symplectic connections. We begin with their definition which – even though it is standard – will be stated explicitly in order to clarify the terminology.

Definition 1 *Let M be a manifold equipped with an almost symplectic form ω. A linear connection ∇ on M is said to be a **symplectic connection** if it preserves ω, i.e., satisfies $\nabla\omega = 0$, or explicitly,*

$$X \cdot \omega(Y,Z) \;=\; \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z) \qquad for\ X,Y,Z \in \mathfrak{X}(M)\ . \tag{9}$$

The same terminology is used for partial linear connections.

In particular, we do not adhere to the convention adopted by some authors who incorporate the condition of being torsion-free into the definition of a symplectic connection.

Regarding the question of whether there exist any torsion-free symplectic connections at all, we begin by noting the following elementary and well known proposition, which can be viewed as an analogue of Theorem 1 for "full" linear connections.

Proposition 1 *Let M be a manifold equipped with an almost symplectic form ω. Then if there exists a torsion-free symplectic connection ∇ on M, ω must be closed. More generally, the torsion tensor T of a symplectic connection ∇ on M is related to the exterior derivative of ω by the formula*

$$d\omega(X,Y,Z) \;=\; \omega(T(X,Y),Z) + \omega(T(Y,Z),X) + \omega(T(Z,X),Y)\ . \tag{10}$$

Proof: This is a special case of Lemma 6 (eqn (65)) in Appendix A. □

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[2]In the case of a lagrangian distribution which is involutive with respect to a pseudo-riemannian metric g of zero signature, the construction is completely analogous, and it can be shown that the Bott connection ∇^B coincides with the restriction of the Levi-Civita connection ∇ if and only if the former has zero torsion; more generally, the difference between the two is proportional to the torsion tensor of ∇^B.

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Conversely, it is well known that on any symplectic manifold, there exist torsion-free symplectic connections [2, 28]. An explicit proof can be found in Sect. 2.1 of Ref. [3]: it is based on modifying a given torsion-free linear connection by adding a judiciously chosen tensor field in order to arrive at a torsion-free symplectic connection. However, the method can be easily generalized so as to start out from an arbitrary linear connection and get what one wants in a single stroke:

Proposition 2   *Let M be a manifold equipped with a symplectic form ω and ∇^0 a general linear connection on M with torsion tensor T^0. Then the formula*

$$
\begin{aligned}
\omega\big(\nabla_X Y, Z\big) &= \omega\big(\nabla^0_X Y, Z\big) + \tfrac{1}{3}\big(\nabla^0_X \omega\big)(Y, Z) + \tfrac{1}{3}\big(\nabla^0_Y \omega\big)(X, Z) \\
&\quad - \tfrac{1}{2}\,\omega\big(T^0(X,Y), Z\big) + \tfrac{1}{6}\,\omega\big(T^0(Z,X), Y\big) + \tfrac{1}{6}\,\omega\big(T^0(Z,Y), X\big)
\end{aligned}
\tag{11}
$$

or equivalently

$$
\begin{aligned}
\omega\big(\nabla_X Y, Z\big) &= \tfrac{1}{6}\,\omega\big(\nabla^0_X Y, Z\big) + \tfrac{1}{6}\,\omega\big(\nabla^0_Y X, Z\big) + \tfrac{1}{6}\,\omega\big(\nabla^0_Z X, Y\big) + \tfrac{1}{6}\,\omega\big(\nabla^0_Z Y, X\big) \\
&\quad + \tfrac{1}{6}\,\omega\big(\nabla^0_X Z, Y\big) + \tfrac{1}{6}\,\omega\big(\nabla^0_Y Z, X\big) + \tfrac{1}{3}\,X\cdot\omega(Y, Z) + \tfrac{1}{3}\,Y\cdot\omega(X, Z) \\
&\quad + \tfrac{1}{2}\,\omega\big([X,Y], Z\big) - \tfrac{1}{6}\,\omega\big([Z,X], Y\big) - \tfrac{1}{6}\,\omega\big([Z,Y], X\big)
\end{aligned}
\tag{12}
$$

defines a torsion-free symplectic connection ∇ on M.

Obviously, when ∇^0 is itself torsion-free and symplectic, then $\nabla = \nabla^0$. In passing, we also note that if ∇^0 is torsion-free but not symplectic, then eqn (12) simplifies to

$$
\begin{aligned}
\omega\big(\nabla_X Y, Z\big) &= \tfrac{2}{3}\,\omega\big(\nabla^0_X Y, Z\big) - \tfrac{1}{3}\,\omega\big(\nabla^0_Y X, Z\big) + \tfrac{1}{3}\,\omega\big(\nabla^0_X Z, Y\big) + \tfrac{1}{6}\,\omega\big(\nabla^0_Y Z, X\big) \\
&\quad + \tfrac{1}{3}\,X\cdot\omega(Y, Z) + \tfrac{1}{3}\,Y\cdot\omega(X, Z)\,.
\end{aligned}
\tag{13}
$$

whereas if ∇^0 is symplectic but not torsion-free, then eqn (12) simplifies to

$$
\begin{aligned}
\omega\big(\nabla_X Y, Z\big) &= \tfrac{1}{2}\,\omega\big(\nabla^0_X Y, Z\big) + \tfrac{1}{2}\,\omega\big(\nabla^0_Y X, Z\big) + \tfrac{1}{6}\,\omega\big(\nabla^0_Z X, Y\big) + \tfrac{1}{6}\,\omega\big(\nabla^0_Z Y, X\big) \\
&\quad - \tfrac{1}{6}\,\omega\big(\nabla^0_X Z, Y\big) - \tfrac{1}{6}\,\omega\big(\nabla^0_Y Z, X\big) \\
&\quad + \tfrac{1}{2}\,\omega\big([X,Y], Z\big) - \tfrac{1}{6}\,\omega\big([Z,X], Y\big) - \tfrac{1}{6}\,\omega\big([Z,Y], X\big)\,.
\end{aligned}
\tag{14}
$$

Note the similarity, but also the differences, between these formulas for symplectic manifolds and the definition of the Levi-Civita connection for pseudo-riemannian manifolds. In both cases, existence of torsion-free compatible connections is guaranteed, but in sharp contrast with the pseudo-riemannian case, torsion-free symplectic connections are far from unique: rather, one can show that the set of all such connections constitutes an affine space whose difference vector space can be identified with the space of all totally symmetric tensor fields of rank 3; see, e.g., Refs [2, 28] and, for an explicit proof, Sect. 2.1 of Ref. [3]. This ambiguity has important implications in mathematical physics, being closely related to the famous factor ordering problem of quantum mechanics. More specifically, there is a famous construction of star products in deformation quantization [11], now commonly

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known as the *Fedosov construction*, which uses as one of its essential ingredients a torsion-free symplectic connection on classical phase space, and different choices lead to different factor ordering rules. This ground-breaking contribution to the quantization problem has even led some authors to refer to symplectic manifolds equipped with a fixed torsion-free symplectic connection as *Fedosov manifolds* [14], and it provides compelling motivation for geometers to study the question as to what further restrictions on the choice of torsion-free symplectic connections are implied by introducing additional covariantly constant geometric structures – ideally to the point of singling out a unique representative.

Of course, there are many possible such structures, among which we may mention, as particularly important and interesting examples, Kähler manifolds and hamiltonian G-spaces; an overview can be found in Ref. [3]. Here, we shall study a specific type, given by the choice of a lagrangian distribution. This is the kind of additional structure one meets in geometric quantization [30], and the question of how to construct torsion-free symplectic connections compatible with it has first been investigated by Heß [20,21]. (Somewhat more generally, geometric quantization uses lagrangian vector subbundles of the complexified tangent bundle, called *polarizations*, but we shall in this paper restrict ourselves to real polarizations, for the sake of simplicity.) The main theorem of Heß regarding this question, stated in Ref. [20] and proved in detail in Ref. [21], states that given two involutive lagrangian distributions L_1 and L_2 which are transversal, there is a *unique* symplectic connection preserving both of them: it has come to be known as the *bilagrangian* connection.[3] Another way of looking at this result is in terms of pseudo-riemannian geometry, since in this case the bilagrangian connection, being torsion-free, is simply the Levi-Civita connection associated with the pseudo-riemannian metric g of zero signature that can be constructed naturally from ω together with L_1 and L_2 [9,10] by setting

$$g(X,Y) \;=\; \omega\big((\mathrm{pr}_1 - \mathrm{pr}_2)X, Y\big) \,,$$

where pr_1 and pr_2 is the projection onto L_1 along L_2 and onto L_2 along L_1, respectively.

However, in order to generalize the construction of adequate symplectic connections to the poly- and multisymplectic framework, we must focus on the situation where we are given a single involutive lagrangian distribution L on M, rather than two transversal ones. The question is whether there always exists a torsion-free symplectic connection on M that preserves L and, if so, what is the affine space of all such connections. This is a problem of independent interest even within the traditional context of symplectic geometry, and one that seems to have received little attention so far.

As a first step in this direction, we note the following extension of Proposition 1.

Proposition 3 *Let M be a manifold equipped with an almost symplectic form ω and let L be a lagrangian distribution on M. Then if there exists a torsion-free symplectic connection ∇ on M preserving L, ω must be closed and L must be involutive. In this case, the restriction of any such connection to L coincides with the Bott connection in L.*

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[3]Actually, the theorem of Heß is significantly more general because it applies even when the lagrangian distributions are not involutive or the form ω is not closed, but we shall not go into this here.

Remark 1 The last statement is valid under much less restrictive assumptions on the torsion tensor T of ∇ than stated above: it suffices that $T(X,Y)$ should be along L whenever at least one of its arguments is along L.

Proof: The first statement has been proved in Proposition 1. The second statement follows directly from Lemma 5 in Appendix A. For the third statement, let us assume that ∇ is any symplectic connection on M preserving L with torsion tensor T such that $T(X,Y)$ is along L as soon as X or Y is along L. Then the claim is equivalent to the condition that for all $X, Y \in \Gamma(L)$ and $Z \in \mathfrak{X}(M)$,

$$\omega(\nabla_X^B Y, Z) = \omega(\nabla_X Y, Z),$$

which can be derived by comparing eqn (7) with eqn (9) taking into account that

$$\omega(Y, [X, Z]) = \omega(Y, \nabla_X Z)$$

since L being isotropic and stable under ∇, the expressions $\omega(Y, \nabla_Z X)$ and $\omega(Y, T(X, Z))$ vanish under these assumptions. $\qquad\square$

Conversely, we can use a partition of unity argument to prove that the conditions stated in Proposition 3 (ω is closed and L is involutive) are not only necessary but also sufficient to guarantee existence of torsion-free symplectic connections preserving the distribution L.

Theorem 2 *Let M be a manifold equipped with a symplectic form ω and let L be an involutive lagrangian distribution on M. Then there exist torsion-free symplectic connections ∇ on M preserving L, and the set of all such connections constitutes an affine space whose difference vector space can be identified with the space of all tensor fields of rank 3 on M which (a) are totally symmetric and (b) vanish whenever at least two of their arguments are along L.*

Proof: Concerning existence, we can under the hypotheses of the theorem apply the Darboux theorem to guarantee that locally (i.e., on a sufficiently small open neighborhood of each point of M), there exists a torsion-free flat symplectic connection on M preserving L: it is simply the linear connection on M whose Christoffel symbols vanish identically in these coordinates. (Here, we use a strengthened version of the Darboux theorem which guarantees the existence of a system of local coordinates (q^i, p_i) around each point such that not only ω takes the standard form $dq^i \wedge dp_i$ but also L is generated by the $\partial/\partial p_i$, say; a detailed proof can be found, for example, in [27, Theorem 1.1].) Now using a covering of M by such Darboux coordinate neighborhoods, passing to a locally finite refinement $(U_\alpha)_{\alpha \in A}$, denoting the corresponding family of linear connections by $(\nabla_\alpha)_{\alpha \in A}$ and choosing a partition of unity $(\chi_\alpha)_{\alpha \in A}$ subordinate to the open covering $(U_\alpha)_{\alpha \in A}$, we can define

$$\nabla = \sum_{\alpha \in A} \chi_\alpha \nabla_\alpha.$$

Then it is clear that ∇ preserves ω as well as L and is torsion-free, since this is true for each ∇_α and since the conditions of preserving a given differential form, of preserving a given vector subbundle and of being torsion-free are all local (i.e., behave naturally under restriction to open subsets) as well as affine.[4] Regarding uniqueness, or rather the amount of non-uniqueness, we can write the difference between any linear connection ∇' on M and a fixed torsion-free symplectic connection ∇ on M preserving L in the form

$$\nabla'_X Y = \nabla_X Y + S(X, Y) \, .$$

Moreover, we introduce a (covariant) tensor field ω_S of rank 3 on M which, due to non-degeneracy of ω, carries exactly the same information as S itself, given by

$$\omega_S(X, Y, Z) = \omega(S(X, Y), Z) \, .$$

Then it is clear that ∇' will be torsion-free if and only if S is symmetric, or equivalently, ω_S is symmetric in its first two arguments, that ∇' will be symplectic if and only if S satisfies the identity

$$\omega(S(X, Y), Z) + \omega(Y, S(X, Z)) = 0 \, ,$$

or equivalently, ω_S is symmetric in its last two arguments, and that ∇' will preserve L if and only if $S(X, Y)$ is along L whenever X or Y is along L, or equivalently, ω_S vanishes whenever at least two of its arguments are along L. $\qquad\square$

These considerations show that there are (at least) three rather different methods for proving existence of torsion-free symplectic connections: (a) by modifying a given linear connection through addition of an appropriately chosen tensor field (see Proposition 2), (b) by employing the construction of the bilagrangian connection due to Heß and (c) by a partition of unity argument. For our purposes, however, the first two are not fully adequate since the first provides connections that may not preserve any lagrangian distribution (this is not enough), whereas the second provides connections preserving two transversal lagrangian distributions (this is too much). The problem with the second construction is that the bilagrangian connection associated with two lagrangian distributions is torsion-free if and only if both of them are involutive. Therefore, it must be modified when one wants to deal with situations where one is given a naturally defined involutive lagrangian distribution L which has no distinguished lagrangian complement and which may not even admit any involutive lagrangian complement at all: an important example is provided by cotangent bundles where the lagrangian foliation given by the structure as a vector bundle admits tranversal lagrangian submanifolds (such as the zero section or, more generally, the graph of any closed 1-form) but no natural transversal lagrangian foliation. Such a modification can always be performed by applying the construction of

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[4]The situation with respect to curvature is different because the condition of being flat, although still local, is not affine, so although each ∇_α is flat, this will in general no longer be true for ∇. However, the curvature of ∇ does vanish when evaluated on two vector fields along L, since there ∇ coincides with the Bott connection, which is flat.

Proposition 2 to the bilagrangian connection associated with an arbitrarily chosen la-
grangian complement L' of L, which preserves L' but has non-vanishing torsion (except
when L' is involutive), trading it for what we might call a lagrangian connection, which
no longer preserves L' (except when L' is involutive) but has vanishing torsion. However,
this procedure is somewhat artificial, and as it turns out, it cannot be extended to the
poly- and multisymplectic setting to be discussed in the next two sections – in contrast
to the method based on a partition of unity argument, which extends in a completely
straightforward manner.

4 Polysymplectic connections

We begin by stating the definition of a polysymplectic structure, as given in Ref. [12]. To
this end, we recall first of all that given a fiber bundle P over a manifold M with bundle
projection $\pi : P \longrightarrow M$ and corresponding vertical bundle VP (the kernel of the tangent
map $T\pi : TP \longrightarrow TM$ of π), a vertical vector field on P is a section of VP whereas a ver-
tical r-form on P is a section of the r-th exterior power of the dual bundle V^*P of VP and,
more generally, a (totally covariant) vertical tensor field of rank r on P is a section of the r-
th tensor power of the dual bundle V^*P of VP. Similarly, given an additional auxiliary vec-
tor bundle $\hat{T}$ over M, a vertical r-form on P and, more generally, a (totally covariant) verti-
cal tensor field of rank r on P taking values in $\hat{T}$ – or more precisely, in the pull-back $\pi^*(\hat{T})$
of $\hat{T}$ to P – is a section of the tensor product of the aforementioned exterior power / tensor
power with $\pi^*(\hat{T})$. In what follows, we shall denote the Lie algebra of vertical vector fields
on P by $\mathfrak{X}_V(P)$ and the space of vertical r-forms on P taking values in $\hat{T}$ by $\Omega_V^r(P; \pi^*\hat{T})$.[5]
For such forms, there is a complete Cartan calculus, strictly analogous to the Cartan calcu-
lus for (vector-valued) differential forms; in particular, there is a naturally defined notion
of vertical exterior derivative,

$$
\begin{aligned}
d_V : \quad \Omega_V^r(P; \pi^*\hat{T}) \quad &\longrightarrow \quad \Omega_V^{r+1}(P; \pi^*\hat{T}) \\
\alpha \quad &\longmapsto \quad d_V\alpha
\end{aligned} \quad ,
\tag{15}
$$

and of vertical Lie derivative along a vertical vector field X,

$$
\begin{aligned}
\mathbb{L}_X : \quad \Omega_V^r(P; \pi^*\hat{T}) \quad &\longrightarrow \quad \Omega_V^r(P; \pi^*\hat{T}) \\
\alpha \quad &\longmapsto \quad \mathbb{L}_X\alpha
\end{aligned} \quad ,
\tag{16}
$$

which are defined by exactly the same formulas as in the standard case, namely

$$
\begin{aligned}
(d_V\alpha)(X_0,\ldots,X_r) = &\sum_{i=0}^r (-1)^i X_i \cdot \big(\alpha(X_0,\ldots,\hat{X}_i,\ldots,X_r)\big) \\
&+ \sum_{0\leqslant i<j\leqslant r} (-1)^{i+j} \alpha([X_i,X_j],X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_r)
\end{aligned} \quad ,
\tag{17}
$$

---

[5]Note that speaking of vertical forms or (totally covariant) tensor fields constitutes a certain abuse of
language because these are really equivalence classes of ordinary differential forms or (totally covariant)
tensor fields.

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where $X_0, X_1, \ldots, X_r \in \mathfrak{X}_V(P)$, and

$$(\mathbb{L}_X \alpha)(X_1, \ldots, X_r) = X \cdot \big(\alpha(X_1, \ldots, X_r)\big) - \sum_{i=1}^{r} \alpha(X_1, \ldots, [X, X_i], \ldots, X_r), \quad (18)$$

where $X, X_1, \ldots, X_r \in \mathfrak{X}_V(P)$: this makes sense since VP is an involutive distribution on P. Here and throughout the remainder of this section, the symbol $\cdot$ stands for the directional derivative of sections of $\pi^*(\hat{T})$ along vertical vector fields: this makes sense since upon restriction to each fiber, a vertical vector field is simply an ordinary vector field on the fiber and a section of a vector bundle obtained as the pull-back of a vector bundle over M becomes a function on the fiber taking values in a fixed vector space.

Definition 2 *A **polypresymplectic fiber bundle** is a fiber bundle P over an n-dimensional manifold M equipped with a vertical $(k+1)$-form*

$$\hat{\omega} \in \Omega_V^{k+1}(P; \pi^*(\hat{T}))$$

*of constant rank on the total space P taking values in (the pull-back to P of) a fixed $\hat{n}$-dimensional vector bundle $\hat{T}$ over the same manifold M, called the **polypresymplectic form along the fibers** of P, or simply the **polypresymplectic form**, and said to be of rank N, such that $\hat{\omega}$ is vertically closed,[6]*

$$d_V \hat{\omega} = 0, \quad (19)$$

*and such that at every point p of P, $\hat{\omega}_p$ is a polypresymplectic form of rank N on the vertical space $V_p P$: this means that there exists a subspace L_p of $V_p P$ of codimension N, called the **polylagrangian subspace**,[7] such that the "musical map"*

$$\hat{\omega}_p^\flat : V_p P \longrightarrow \textstyle\bigwedge^k V_p^* P \otimes \hat{T}_{\pi(p)}$$

given by contraction of $\hat{\omega}_p$ in its first argument, when restricted to L_p, yields a linear isomorphism

$$L_p / \ker \hat{\omega}_p \cong \textstyle\bigwedge^k L_p^\perp \otimes \hat{T}_{\pi(p)}$$

*where $L_p^\perp$ is the annihilator of L_p in $V_p^* P$. Moreover, it is assumed that the kernels $\ker \hat{\omega}_p$ as well as the polylagrangian subspaces L_p at the different points of P fit together smoothly into distributions $\ker \hat{\omega}$ and L on P: the latter is called the **polylagrangian distribution** of $\hat{\omega}$. If $\hat{\omega}$ is non-degenerate, we say that P is a **polysymplectic fiber bundle** and $\hat{\omega}$ is a **polysymplectic form along the fibers** of P, or simply a **polysymplectic form**. If M reduces to a point, we speak of a **poly(pre)symplectic manifold**. The case of main interest is when $\hat{\omega}$ is a 2-form, i.e., $k = 1$.*

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[6]As in the symplectic case, the possible absence of the integrability condition $d_V \hat{\omega} = 0$ will be indicated by adding the term "almost".

[7]The terminology, as well as the justification for using the definite article, stems from the fact that this subspace, if it exists, is more than just lagrangian (i.e., maximal isotropic) and that, as soon as either $\hat{n} > 1$ or else $\hat{n} = 1$ but then $N > k > 1$, or in other words, except when $\hat{\omega}$ is an ordinary two-form or a volume form, it is necessarily unique.

Thus the characteristic feature of a polysymplectic fiber bundle P with a polysymplectic form $\hat{\omega}$ is the existence of a special subbundle L of its vertical bundle VP which is not only lagrangian (in particular, isotropic) but has the even stronger property that the "musical vector bundle homomorphism" $\hat{\omega}^\flat : VP \longrightarrow \bigwedge^k V^*P \otimes \pi^*(\hat{T})$, when restricted to L, provides a vector bundle isomorphism

$$\hat{\omega}^\flat : L \xrightarrow{\cong} \bigwedge^k L^\perp \otimes \pi^*(\hat{T}) . \tag{20}$$

As has been proved in Ref. [12], as soon as $\hat{n} > 2$, L is necessarily involutive.

The following example provides what may be considered the "standard model" of a polysymplectic fiber bundle:

Example 1 Let E be an arbitrary fiber bundle over an n-dimensional manifold M, with projection $\pi_E : E \longrightarrow M$, and let $\hat{T}$ be a fixed $\hat{n}$-dimensional vector bundle over the same manifold M. Consider the bundle

$$P = \bigwedge^k V^*E \otimes \pi_E^*(\hat{T}) \tag{21}$$

of vertical k-forms on E taking values in the pull-back of $\hat{T}$ to E, with projections $\pi^k : P \longrightarrow E$ and $\pi = \pi_E \circ \pi^k : P \longrightarrow M$. Using the tangent map $T\pi^k : TP \longrightarrow TE$ of π^k and its restriction $V\pi^k : VP \longrightarrow VE$ to the vertical bundles, we define the **canonical k-form** on P, which is a vertical k-form $\hat{\theta}$ on P taking values in $\pi^*(\hat{T})$, by

$$\hat{\theta}_\alpha(v_1, \ldots, v_k) = \alpha\big(V_\alpha\pi^k \cdot v_1, \ldots, V_\alpha\pi^k \cdot v_k\big)$$
$$\text{for } \alpha \in P \text{ and } v_1, \ldots, v_k \in V_\alpha P . \tag{22}$$

Then $\hat{\omega} = -d_V\hat{\theta}$ is a polysymplectic $(k+1)$-form, with polylagrangian distribution $L = \ker(T\pi^k)$ (the vertical bundle for the projection to E), contained in $VP = \ker(T\pi)$ (the vertical bundle for the projection to M).

When $k = 1$ and $\hat{n} = 1$ (with the understanding that the auxiliary vector bundle $\hat{T}$ is the trivial real line bundle $M \times \mathbb{R}$), we have the "standard model" of a symplectic fiber bundle. In particular, when, in addition, $n = 0$ (i.e., the base manifold M is reduced to a single point), we recover the cotangent bundle of the single fiber, which is an arbitrary manifold, as the "standard model" of a symplectic manifold. On the other hand, when $k = 1$ and $\hat{n} = n - 1$ (with the understanding that the auxiliary vector bundle $\hat{T}$ is the bundle $\bigwedge^{n-1} T^*M$ of $(n-1)$-forms on M), P can be identified with the twisted dual $\vec{J}^\circledast E$ of the linearized jet bundle $\vec{J}E$ of E (which is the difference vector bundle of the usual jet bundle JE of E), because

$$\vec{J}E \cong \pi_E^*(T^*M) \otimes VE$$

implying

$$\vec{J}^*E \cong V^*E \otimes \pi_E^*(TM)$$

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for the common dual $\vec{J}^{\bullet}E$ and

$$\vec{J}^{\circledast}E \;\cong\; V^{\bullet}E \otimes \pi_E^{\bullet}(TM) \otimes \pi_E^{\bullet}\big(\textstyle\bigwedge^n T^{\bullet}M\big)$$

for the twisted dual $\vec{J}^{\circledast}E = \vec{J}^{\bullet}E \otimes \pi_E^{\bullet}\big(\bigwedge^n T^{\bullet}M\big)$, so we get a canonical isomorphism

$$\vec{J}^{\circledast}E \;\cong\; V^{\bullet}E \otimes \pi_E^{\bullet}\big(\textstyle\bigwedge^{n-1} T^{\bullet}M\big) \tag{23}$$

of vector bundles over E. This bundle plays an important role in the covariant hamiltonian formalism of classical field theory [4, 13, 16, 17].

As a first application of the isomorphism (20) beyond those discussed in Ref. [12], we show that, just as in the symplectic case, it allows us to construct a polysymplectic version of the Bott connection. The idea is simple: start with the Bott connection in $L^{\perp}$ as defined in eqns (3) and (4) (with $\mathfrak{X}(M)$ replaced by $\mathfrak{X}_V(P)$, $\Omega^1(M)$ replaced by $\Omega_V^1(P)$ and the common Lie derivative replaced by the vertical Lie derivative introduced at the beginning of this section) and take the tensor product of its k-th exterior power with the trivial partial linear connection in $\pi^{\bullet}(\hat{T})$ along L, to obtain a partial linear connection ∇^B in $\bigwedge^k L^{\perp} \otimes \pi^{\bullet}(\hat{T})$ along L, which we call the Bott connection in $\bigwedge^k L^{\perp} \otimes \pi^{\bullet}(\hat{T})$: it is still given by a suitable restriction of the (vertical) Lie derivative of (vertical) forms; explicitly,

$$(\nabla_X^B \alpha)(Y_1, \ldots, Y_k) \;=\; X \cdot \big(\alpha(Y_1, \ldots, Y_k)\big) - \sum_{i=1}^{k} \alpha(Y_1, \ldots, [X, Y_i], \ldots, Y_k) \tag{24}$$
$$\text{for } X \in \Gamma(L),\ \alpha \in \Gamma(\textstyle\bigwedge^k L^{\perp} \otimes \pi^{\bullet}(\hat{T})),\ Y_1, \ldots, Y_k \in \mathfrak{X}_V(P) \,.$$

Now using the isomorphism (20), we can transfer it to a partial linear connection ∇^B in L along L and arrive at

Definition 3 *Let P be an almost polysymplectic fiber bundle over a manifold M with almost polysymplectic form $\hat{\omega}$ and involutive polylagrangian distribution L. Then there exists a naturally defined partial linear connection ∇^B in L along L which we call the* **polysymplectic Bott connection**; *explicitly, it is determined by the formula*

$$\hat{\omega}\big(\nabla_X^B Y, Z_1, \ldots, Z_k\big) \;=\; X \cdot \big(\hat{\omega}(Y, Z_1, \ldots, Z_k)\big) - \sum_{i=1}^{k} \hat{\omega}(Y, Z_1, \ldots, [X, Z_i], \ldots, Z_k) \tag{25}$$
$$\text{for } X, Y \in \Gamma(L),\ Z_1, \ldots, Z_k \in \mathfrak{X}_V(P) \,.$$

As in the symplectic case, the polysymplectic Bott connection is flat, and for its torsion we have the following analogue of Theorem 1:

Proposition 4 *Let P be an almost polysymplectic fiber bundle over a manifold M with almost polysymplectic form $\hat{\omega}$ and involutive polylagrangian distribution L. Then if $\hat{\omega}$ is vertically closed, the polysymplectic Bott connection ∇^B has zero torsion. More generally,*

the torsion tensor T^B of ∇^B is related to the vertical exterior derivative of $\hat{\omega}$ by the formula

$$d_V\hat{\omega}(X,Y,Z_1,\ldots,Z_k) = \hat{\omega}(T^B(X,Y),Z_1,\ldots,Z_k)$$
$$\text{for } X,Y \in \Gamma(L),\ Z_1,\ldots,Z_k \in \mathfrak{X}_V(P)\ . \tag{26}$$

In particular, this implies that in a polysymplectic fiber bundle, the leaves of the poly-lagrangian foliation are flat affine manifolds.

Proof: The only property that remains to be checked is eqn (26): this is a simple calculation using the fact that $\hat{\omega}(X,Y,\ldots) = 0$ when $X,Y \in \Gamma(L)$ since L is isotropic:

$$\hat{\omega}(\nabla^B_X Y - \nabla^B_Y X - [X,Y], Z_1,\ldots,Z_k)$$

$$= X \cdot (\hat{\omega}(Y,Z_1,\ldots,Z_k)) - Y \cdot (\hat{\omega}(X,Z_1,\ldots,Z_k))$$

$$- \sum_{i=1}^k (-1)^i\, Z_i \cdot (\hat{\omega}(X,Y,Z_1,\ldots,\hat{Z}_i,\ldots,Z_k)$$

$$- \hat{\omega}([X,Y],Z_1,\ldots,Z_k)$$

$$- \sum_{i=1}^k (-1)^i\, \hat{\omega}([X,Z_i],Y,Z_1,\ldots,\hat{Z}_i,\ldots,Z_k)$$

$$+ \sum_{i=1}^k (-1)^i\, \hat{\omega}([Y,Z_i],X,Z_1,\ldots,\hat{Z}_i,\ldots,Z_k)$$

$$+ \sum_{1\leqslant i<j\leqslant k} (-1)^{i+j}\, \hat{\omega}([Z_i,Z_j],X,Y,Z_1,\ldots,\hat{Z}_i,\ldots,\hat{Z}_j,\ldots,Z_k)$$

$$= d_V\hat{\omega}(X,Y,Z_1,\ldots,Z_k)\ .$$

$$\square$$

Now we turn to polysymplectic connections. By analogy with the symplectic case, the definition of the concept is more or less obvious, except that we cannot expect to obtain anything beyond partial linear connections along the vertical bundle.

Definition 4 *A poly(pre)symplectic connection on an almost poly(pre)symplectic fiber bundle P over a manifold M with almost poly(pre)symplectic form $\hat{\omega}$ and poly-lagrangian distribution L is a partial linear connection ∇ in the vertical bundle VP of P along VP itself which preserves both $\hat{\omega}$ and L; in particular, it satisfies $\nabla\hat{\omega} = 0$, or explicitly,*

$$X \cdot (\hat{\omega}(X_0,\ldots,X_k)) = \sum_{i=0}^k \hat{\omega}(X_0,\ldots,\nabla_X X_i,\ldots,X_k) \tag{27}$$
$$\text{for } X,X_0,\ldots,X_k \in \mathfrak{X}_V(P)\ .$$

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Note that as soon as the polylagrangian distribution L is unique, the invariance of $\hat{\omega}$ under parallel transport with respect to ∇ already implies that of L. As mentioned before,[7] the only exceptions to this situation can occur when $\hat{\omega}$ is an ordinary two-form or a volume form: in these cases, invariance of L becomes a separate condition.

As in the symplectic case, the existence of torsion-free poly(pre)symplectic connections imposes certain constraints.

Proposition 5 *Let P be an almost poly(pre)symplectic fiber bundle over a manifold M with almost poly(pre)symplectic form $\hat{\omega}$ and polylagrangian distribution L. Then if there exists a torsion-free poly(pre)symplectic connection ∇ on P, $\hat{\omega}$ must be vertically closed and L must be involutive. More generally, the torsion tensor T of a poly(pre)symplectic connection ∇ over P is related to the exterior derivative of $\hat{\omega}$ by the formula*

$$d_V \hat{\omega}(X_0, \ldots, X_r) \;=\; -\sum_{0 \leqslant i < j \leqslant r} (-1)^{i+j}\, \hat{\omega}(T(X_i, X_j), X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_r) \, . \quad (28)$$

Finally, if $\hat{\omega}$ is non-degenerate, the restriction of any torsion-free polysymplectic connection to L coincides with the polysymplectic Bott connection in L.

Remark 2 As in the symplectic case, the last statement is valid under the same less restrictive assumptions on the torsion tensor T of ∇ as before: it suffices that $T(X,Y)$ should be along L as soon as X or Y is along L.

Proof: The first two statements follow directly from Lemma 6 (eqn (66)) and Lemma 5 in Appendix A. For the third statement, let us assume that $\hat{\omega}$ is non-degenerate and ∇ is any polysymplectic connection on P with torsion tensor T such that $T(X,Y)$ is along L as soon as X or Y is along L. Then the claim is equivalent to the condition that for all $X, Y \in \Gamma(L)$ and $Z_1, \ldots, Z_k \in \mathfrak{X}_V(P)$,

$$\hat{\omega}(\nabla_X^B Y, Z_1, \ldots, Z_k) \;=\; \hat{\omega}(\nabla_X Y, Z_1, \ldots, Z_k)$$

which can be derived by comparing eqn (25) with the corresponding expression from eqn (27) taking into account that, for $1 \leqslant i \leqslant k$,

$$\hat{\omega}(Y, Z_1, \ldots, [X, Z_i], \ldots, Z_k) \;=\; \hat{\omega}(Y, Z_1, \ldots, \nabla_X Z_i, \ldots, Z_k)$$

since L being isotropic and stable under ∇, the expressions $\hat{\omega}(Y, Z_1, \ldots, \nabla_{Z_i} X, \ldots, Z_k)$ and $\hat{\omega}(Y, Z_1, \ldots, T(X, Z_i), \ldots, Z_k)$ vanish under these assumptions. $\qquad\square$

Conversely, we can use a partition of unity argument to prove that the conditions stated in Proposition 5 ($\hat{\omega}$ is vertically closed and L is involutive) are not only necessary but also sufficient to guarantee existence of torsion-free polysymplectic connections.

Theorem 3 *Let P be a poly(pre)symplectic fiber bundle over a manifold M with poly(pre)symplectic form $\hat{\omega}$ and involutive polylagrangian distribution L. Then there exist*

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torsion-free poly(pre)symplectic connections ∇ on P, and the set of all such connections constitutes an affine space whose difference vector space can, for non-degenerate $\hat{\omega}$, be identified with the space of all vertical tensor fields of rank $k+2$ on P taking values in the auxiliary vector bundle $\pi^*(\hat{T})$ which (a) have symmetry corresponding to the irreducible representation of the permutation group S_{k+2} given by the Young pattern
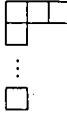
(with k boxes in the first column) and (b) vanish whenever at least two of their arguments are along L.

Proof: Concerning existence, we can under the hypotheses of the theorem apply the polysymplectic Darboux theorem (see [12, Theorem 10]) to guarantee that locally (i.e., on a sufficiently small open neighborhood of each point of P), there exists a torsion-free flat polysymplectic connection on P: it is simply the partial linear connection in VP along VP whose Christoffel symbols vanish identically in these coordinates. Now using a covering of P by such Darboux coordinate neighborhoods, passing to a locally finite refinement $(U_\alpha)_{\alpha \in A}$, denoting the corresponding family of partial linear connections by $(\nabla_\alpha)_{\alpha \in A}$ and choosing a partition of unity $(\chi_\alpha)_{\alpha \in A}$ subordinate to the open covering $(U_\alpha)_{\alpha \in A}$, we can define

$$\nabla = \sum_{\alpha \in A} \chi_\alpha \nabla_\alpha \,.$$

Then it is clear that ∇ is a partial linear connection in VP along VP, preserves $\hat{\omega}$ as well as L and is torsion-free, since this is true for each ∇_α and since the conditions of preserving a given differential form, of preserving a given vector subbundle and of being torsion-free are all local (i.e., behave naturally under restriction to open subsets) as well as affine.[8] Regarding uniqueness, or rather the amount of non-uniqueness, we can write the difference between any partial linear connection ∇' in VP along VP and a fixed torsion-free polysymplectic connection ∇ on P in the form

$$\nabla'_X Y = \nabla_X Y + S(X,Y) \,.$$

Moreover, we introduce a (totally covariant) vertical tensor field $\hat{\omega}_S$ of rank $k+2$ on P taking values in $\hat{T}$ which, for non-degenerate $\hat{\omega}$, carries exactly the same information as S itself, given by

$$\hat{\omega}_S(X,Y,Z_1,\ldots,Z_k) = \hat{\omega}(S(X,Y),Z_1,\ldots,Z_k) \,.$$

Obviously, $\hat{\omega}_S$ is totally antisymmetric in its last k arguments and it is clear that ∇' will be torsion-free if and only if S is symmetric, or equivalently, $\hat{\omega}_S$ is symmetric in its first

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[8]With respect to curvature, the same comment as in Footnote 4 applies.

two arguments, that ∇' will preserve $\hat{\omega}$ if and only if S satisfies the identity

$$\hat{\omega}(S(X,Y),Z_1,\ldots,Z_k) + \sum_{i=1}^{k}\hat{\omega}(Y,Z_1,\ldots,Z_{i-1},S(X,Z_i),Z_{i+1},\ldots,Z_k) \;=\; 0 \;,$$

or equivalently, $\hat{\omega}_S$ satisfies the cyclic identity

$$\hat{\omega}_S(X,Y,Z_1,\ldots,Z_k) - \sum_{i=1}^{k}\hat{\omega}_S(X,Z_i,Z_1,\ldots,Z_{i-1},Y,Z_{i+1},\ldots,Z_k) \;=\; 0 \;,$$

and that ∇' will preserve L if and only if $S(X,Y)$ is along L whenever X or Y are along L, or equivalently, $\hat{\omega}_S$ vanishes whenever at least two of its arguments are along L. Finally, it is well known that, together with symmetry in the first two arguments and antisymmetry in the last k arguments, this cyclic identity identifies the tensor $\hat{\omega}_S$ as belonging to the irreducible representation of the permutation group S_{k+2} given by the Young pattern stated in the theorem; see, e.g., [19, p. 249]. □

5 Multisymplectic connections

We begin by stating the definition of a multisymplectic structure, as given in Ref. [12]. To this end, we recall first of all that given a fiber bundle P over a manifold M with bundle projection $\pi : P \longrightarrow M$, an r-form on P is said to be $(r-s)$-horizontal, where $0 \leqslant s \leqslant r$, if its contraction with more than s vertical vectors vanishes; we shall in what follows denote the bundle of such forms by $\bigwedge_s^r T^*P$.

Definition 5 *A **multipresymplectic fiber bundle** is a fiber bundle P over an n-dimensional manifold M equipped with a $(k+1-r)$-horizontal $(k+1)$-form*

$$\omega \in \Gamma\big(\textstyle\bigwedge_r^{k+1} T^*P\big)$$

*of constant rank on the total space P, where $1 \leqslant r \leqslant k+1$ and $k+1-r \leqslant n$, called the **multipresymplectic form** and said to be of **rank** N and **horizontality degree** $k+1-r$, such that ω is closed,[9]*

$$d\omega \;=\; 0 \;. \tag{29}$$

*and such that at every point p of P, ω_p is a multipresymplectic form of rank N and horizontality degree $k+1-r$ on the tangent space T_pP: this means that there exists a subspace L_p of T_pP contained in V_pP and of codimension N there, called the **multilagrangian***

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[9]Once again, the possible absence of the integrability condition $d\omega = 0$ will be indicated by adding the term "almost".

subspace,[10] *such that the "musical map"*

$$\omega_p^\flat: \ T_p P \ \longrightarrow \ \textstyle\bigwedge^k T_p^* P$$

or

$$\omega_p^\flat: \ V_p P \ \longrightarrow \ \textstyle\bigwedge_{r-1}^k T_p^* P$$

given by contraction of ω_p in its first argument, when restricted to L_p, yields a linear isomorphism

$$L_p / \ker \omega_p \ \cong \ \textstyle\bigwedge^k L_p^\perp \cap \bigwedge_{r-1}^k T_p^* P \ .$$

*Moreover, it is assumed that the kernels $\ker \omega_p$ as well as the multilagrangian subspaces L_p at the different points of P fit together smoothly into distributions $\ker \omega$ and L on P: the latter is called the **multilagrangian distribution** of ω. If ω is non-degenerate, we say that P is a **multisymplectic fiber bundle** and ω is a **multisymplectic form**. If M reduces to a point, we speak of a **multi(pre)symplectic manifold**. The case of main interest is when ω is an $(n-1)$-horizontal $(n+1)$-form, i.e., $k = n$, $r = 2$.*

Again, the characteristic feature of a multisymplectic fiber bundle P with a multi-symplectic form ω is the existence of a special subbundle L of its tangent bundle TP, contained in its vertical bundle VP, which is not only lagrangian (in particular, isotropic) but has the even stronger property that the "musical vector bundle homomorphism" $\omega^\flat : TP \longrightarrow \bigwedge^k T^* P$ or $\omega^\flat : VP \longrightarrow \bigwedge_{r-1}^k T^* P$, when restricted to L, provides a vector bundle isomorphism

$$\omega^\flat: \ L \ \xrightarrow{\ \cong\ } \ \textstyle\bigwedge^k L^\perp \cap \bigwedge_{r-1}^k T^* P \ . \tag{30}$$

As has been proved in Ref. [12], as soon as $\binom{n}{k+1-r} > 2$, L is necessarily involutive.

Again, the following example provides a "standard model" of a multisymplectic fiber bundle:

Example 2   Let E be an arbitrary fiber bundle over an n-dimensional manifold M, with projection $\pi_E : E \longrightarrow M$. Consider the bundle

$$P \ = \ \textstyle\bigwedge_{r-1}^k T^* E \tag{31}$$

of $(k + 1 - r)$-horizontal k-forms on E, where $1 \leqslant r \leqslant k + 1$ and $k + 1 - r \leqslant n$, with projections $\pi_{r-1}^k : P \longrightarrow E$ and $\pi = \pi_E \circ \pi_{r-1}^k : P \longrightarrow M$. Using the tangent map $T\pi_{r-1}^k : TP \longrightarrow TE$ of π_{r-1}^k, we define the **canonical k-form** on P, which is a $(k + 1 - r)$-horizontal k-form θ on P, by

$$\begin{aligned} \theta_\alpha(v_1, \dots, v_k) \ &= \ \alpha\big(T_\alpha \pi_{r-1}^k \cdot v_1, \dots, T_\alpha \pi_{r-1}^k \cdot v_k\big) \\ &\text{for } \alpha \in P \text{ and } v_1, \dots, v_k \in T_\alpha P \ . \end{aligned} \tag{32}$$

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[10]The terminology, as well as the justification for using the definite article, stems from the fact that this subspace, if it exists, is more than just lagrangian (in particular, isotropic) and, as soon as either $r < k+1$ or else $r = k+1$ and then $n = 0$, $N > k > 1$ and also either $k+1-r < n$ or else $k+1-r = n$ and then $N + n > k > n + 1$, is necessarily unique.

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Then $\omega = -d\theta$ is a multisymplectic $(k+1)$-form, with multilagrangian distribution $L = \ker(T\pi_{-1}^k)$ (the vertical bundle for the projection to E), contained in $VP = \ker(T\pi)$ (the vertical bundle for the projection to M).

When $k = n$ and $r = 2$, we have the "standard model" of a multisymplectic fiber bundle since in this case, as is explicitly demonstrated in the literature [4, 16, 17], P can be identified with the twisted dual $J^\circledast E$ of the jet bundle JE of E, i.e., we have a canonical isomorphism

$$J^\circledast E \cong \textstyle\bigwedge_1^n T^* E \tag{33}$$

of vector bundles over E. This bundle plays a central role in the covariant hamiltonian formalism of classical field theory [4, 13, 16, 17].

Again, as a first application of the isomorphism (30) beyond those discussed in Ref. [12], we show that it allows us to construct a multisymplectic version of the Bott connection. Namely, consider the k-th exterior power of the Bott connection in $L^\perp$, as defined in eqns (3) and (4) (with $\mathfrak{X}(M)$ replaced by $\mathfrak{X}(P)$), which is a partial linear connection ∇^B in $\bigwedge^k L^\perp$ along L: it is still given by a suitable restriction of the Lie derivative of forms; explicitly,

$$(\nabla_X^B \alpha)(Y_1, \ldots, Y_k) = X \cdot \big(\alpha(Y_1, \ldots, Y_k)\big) - \sum_{i=1}^{k} \alpha(Y_1, \ldots, [X, Y_i], \ldots, Y_k) \tag{34}$$
$$\text{for } X \in \Gamma(L),\ \alpha \in \Gamma(\textstyle\bigwedge^k L^\perp),\ Y_1, \ldots, Y_k \in \mathfrak{X}(P) \, .$$

Now noting that it preserves $\bigwedge^k L^\perp \cap \bigwedge_{r-1}^k T^* P$ (the expression in eqn (34) vanishes if at least r of the vector fields $Y_1, \ldots, Y_k$ are vertical, since $L \subset VP$ and VP is involutive), we can use the isomorphism (30) to transfer it to a partial linear connection ∇^B in L along L and arrive at

Definition 6 *Let P be an almost multisymplectic fiber bundle over a manifold M with almost multisymplectic form ω and involutive multilagrangian distribution L. Then there exists a naturally defined partial linear connection ∇^B in L along L which we call the* **multisymplectic Bott connection***; explicitly, it is determined by the formula*

$$\omega\big(\nabla_X^B Y, Z_1, \ldots, Z_k\big) = X \cdot \big(\omega(Y, Z_1, \ldots, Z_k)\big) - \sum_{i=1}^{k} \omega(Y, Z_1, \ldots, [X, Z_i], \ldots, Z_k) \tag{35}$$
$$\textit{for } X, Y \in \Gamma(L),\ Z_1, \ldots, Z_k \in \mathfrak{X}(P) \, .$$

As in the symplectic case, the multisymplectic Bott connection is flat, and for its torsion we have the following analogue of Theorem 1.

Proposition 6 *Let P be an almost multisymplectic fiber bundle over a manifold M with almost multisymplectic form ω and involutive multilagrangian distribution L. Then*

if ω is closed, the multisymplectic Bott connection ∇^B has zero torsion. More generally, the torsion tensor T^B of ∇^B is related to the exterior derivative of ω by the formula

$$
\begin{aligned}
d\omega\left(X, Y, Z_1, \ldots, Z_k\right) &= \omega(T^B(X, Y), Z_1, \ldots, Z_k) \\
&\text{for } X, Y \in \Gamma(L), \, Z_1, \ldots, Z_k \in \mathfrak{X}(P) \, .
\end{aligned}
\tag{36}
$$

In particular, this implies that in a multisymplectic fiber bundle, the leaves of the multilagrangian foliation are flat affine manifolds.

Proof: The proof is identical to the proof of the Proposition 4 and will therefore not be repeated here. □

Now we turn to multisymplectic connections, whose definition is analogous to the ones given previously, the main difference being that these are full connections and not just partial ones.

Definition 7 *A **multi(pre)symplectic connection** on an almost multi(pre)-symplectic fiber bundle P over a manifold M with almost multi(pre)symplectic form ω and multilagrangian distribution L is a linear connection ∇ in P which preserves both ω and L, as well as the vertical bundle VP of P; in particular, it satisfies $\nabla\omega = 0$, or explicitly,*

$$
\begin{aligned}
X \cdot \left(\omega(X_0, \ldots, X_k)\right) &= \sum_{i=0}^{k} \omega(X_0, \ldots, \nabla_X X_i, \ldots, X_k) \\
&\text{for } X, X_0, \ldots, X_k \in \mathfrak{X}(P) \, .
\end{aligned}
\tag{37}
$$

From the point of view of fiber bundle theory, the requirement that the vertical bundle should be invariant under parallel transport with respect to ∇ is a natural consistency condition: it is necessary in order that parallel transport maps points in the same fiber to points in the same fiber. Concerning invariance of the multilagrangian distribution, we note as before that as soon as L is unique, the invariance of ω under parallel transport with respect to ∇ already implies that of L. In the few exceptional cases where this uniqueness does not prevail,[10] invariance of L becomes a separate condition.

Finally, we note that the existence of torsion-free multisymplectic connections imposes the same kind of constraints as before and that when these constraints are satisfied, such connections can be completely classified. We just give the statements and omit the proofs since these are obtained by almost literally repeating those of Proposition 5 and Theorem 3 above.

Proposition 7 *Let P be an almost multi(pre)symplectic fiber bundle over a manifold M with almost multi(pre)symplectic form ω and multilagrangian distribution L. Then if there exists a torsion-free multi(pre)symplectic connection ∇ on P, ω must be closed and L must be involutive. More generally, the torsion tensor T of a multi(pre)symplectic*

connection ∇ on P is related to the exterior derivative of ω by the formula

$$d\omega(X_0, \ldots, X_k) = -\sum_{0 \leqslant i < j \leqslant k} (-1)^{i+j}\, \omega(T(X_i, X_j), X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k)$$
$$for\ X_0, \ldots, X_k \in \mathfrak{X}(P)\ . \tag{38}$$

Finally, if ω is non-degenerate, the restriction of any torsion-free multisymplectic connection to L coincides with the multisymplectic Bott connection in L.

Proof: Analogous to that of Proposition 5. $\qquad\square$

Theorem 4 *Let P be a multi(pre)symplectic fiber bundle over a manifold M with multi(pre)symplectic form ω and involutive multilagrangian distribution L. Then there exist torsion-free multi(pre)symplectic connections ∇ on P, and the set of all such connections constitutes an affine space whose difference vector space can, for non-degenerate ω, be identified with the space of all tensor fields of rank $k+2$ on P which (a) have symmetry corresponding to the irreducible representation of the permutation group S_{k+2} given by the Young pattern*
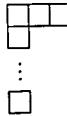
(with k boxes in the first column) and (b) vanish whenever at least $r+1$ of their arguments are vertical or at least two of their arguments are along L.

Proof: Analogous to that of Theorem 3. $\qquad\square$

6 Structure Theorems

In the previous two sections, we have presented " standard models" for polysymplectic and multisymplectic fiber bundles: they are certain bundles of forms built over a given fiber bundle, much in the same way as the cotangent bundle of a given manifold is the "standard model" of a symplectic manifold. But of course we may wonder whether there are other interesting examples, based on other methods. In particular, a natural question to ask is whether there exists a polysymplectic or multisymplectic analogue not only of the cotangent bundle construction, but also of the coadjoint orbit construction of symplectic geometry.

An alternative approach consists in looking at the converse question, which in the context of symplectic geometry can be stated as follows: How can we characterize, among all

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symplectic manifolds, those which (up to a symplectomorphism) are cotangent bundles? As it turns out, this issue is solved by Weinstein's tubular neighborhood theorem.

As an initial step, we mention two conditions that are obviously necessary: for a symplectic manifold P to be symplectomorphic to the cotangent bundle T^*Q of some other manifold Q, it must be exact (the de Rham cohomology class of its symplectic form must vanish), and it must admit a lagrangian foliation, whose leaves are of course the cotangent spaces. Note that each of these conditions already excludes most of the interesting coadjoint orbits, such as Souriau's 2-sphere and, more generally, all (co)adjoint orbits of compact semisimple Lie groups, which are also Kähler manifolds. But there are at least two other aspects that turn out to be important.

The first aspect is that P admits not only a lagrangian foliation but also lots of submanifolds complementary to it: these submanifolds, which may or may not be lagrangian, are the graphs of 1-forms.[11] (As is well known, such a graph is a lagrangian submanifold if and only if the corresponding 1-form is closed.) Note, however, that even though there are many such complementary submanifolds (there are even many of them passing through each point of P), they are isolated, i.e., there is no canonical way to make them come in families that would form a second foliation complementary to the first one. But at any rate, they are natural candidates for a manifold Q satisfying $P \cong T^*Q$.

The second aspect is that the lagrangian foliation is not arbitrary but is simple, i.e., the quotient space of leaves can be given the structure of a manifold such that the canonical projection becomes a surjective submersion. Again, this quotient space is a natural candidate for a manifold Q satisfying $P \cong T^*Q$.

Weinstein's tubular neighborhood theorem deals with the converse question: Suppose that P is a symplectic manifold, with symplectic form ω, which admits a simple lagrangian foliation $\mathcal{F}$ (i.e., a lagrangian foliation whose leaves are the connected components of the level sets of a surjective submersion from P onto some other manifold), and let Q be any submanifold of P complementary to $\mathcal{F}$.[12] In its original version [29], the theorem states that if Q is lagrangian, then there is a tubular neighborhood of Q in P which is symplectomorphic to a neighborhood of the zero section of the cotangent bundle T^*Q of Q. This result is easily generalized to the case when Q is not lagrangian: it is enough to substitute the standard symplectic form $-d\theta$ on T^*Q by a modified symplectic form $-d\theta + \tau^*\omega_Q$ where τ is the canonical projection of T^*Q to Q and ω_Q is the restriction of ω to Q; see [8]. A global version of this result was given by Thompson [26], under the hypothesis that the leaves of $\mathcal{F}$ are simply connected and geodesically complete: in

---

[11]We use the term "complementary" as a stronger version of the term "transversal": given two submanifolds X_1 and X_2 of a manifold X and a point x in their intersection, we say that they are transversal at x if $T_xX = T_xX_1 + T_xX_2$ and are complementary at x if $T_xX_1 \oplus T_xX_2 = T_xX$. In the literature, a submanifold complementary to the leaves of a foliation is often called a "cross section" of that foliation – a term inspired by fiber bundle theory when the submanifold is the graph of some map.

[12]Note that the condition that Q should be complementary to $\mathcal{F}$ is only local: it does not guarantee that the intersection of Q with every leaf of $\mathcal{F}$ reduces to a single point. All it implies is that this intersection must be discrete and hence at most countable, but it can contain many distinct points or, at the other extreme, even be empty.

this case, the manifold P becomes an affine fiber bundle over the quotient manifold $P/\mathcal{F}$ whose difference vector bundle is its cotangent bundle $T^*(P/\mathcal{F})$, and therefore there exist submanifolds Q of P complementary to $\mathcal{F}$ which satisfy $Q \cong P/\mathcal{F}$, i.e., which meet every leaf of $\mathcal{F}$ in precisely one point.[13] Furthermore, when we choose one such submanifold Q, we get a global symplectomorphism from P onto T^*Q that takes ω to $-d\theta$ or, more generally, to $-d\theta + \tau^*\omega_Q$, as before. The problem with the approach of [26] is that it is not intrinsic, since the structure of P as an affine bundle over $P/\mathcal{F}$ and the meaning of geodesic completeness of the leaves seem to depend on the choice of additional ingredients (the author uses an auxiliary riemannian metric, or rather its Levi-Civita connection).

In the remainder of the paper, we shall not only give a much more transparent proof of all these theorems, but we shall also show that this allows us to generalize them, without any additional effort, to the setting of polysymplectic and multisymplectic geometry, where they become natural structure theorems since these geometries come with an in-built lagrangian foliation, right from the start. The main natural ingredients used in our proofs, whose importance in this context seems to have been underestimated in the past, are (a) the Bott connection and (b) the concept of Euler vector field.

7   Simple foliations by flat affine manifolds

The main technical tool, which in what follows will be employed in various different contexts and which therefore deserves to be treated separately, in order to avoid unnecessary repetitions, is the notion of a simple foliation of a manifold by flat affine submanifolds.

Initially, suppose that P is any manifold. According to the Frobenius theorem, a foliation $\mathcal{F}$ of P corresponds to an involutive distribution L on P, such that for every point p in P,

$$L_p = T_p \mathcal{F}_p , \qquad (39)$$

where $\mathcal{F}_p$ is the leaf of $\mathcal{F}$ passing through p. Such a foliation is called *simple* if its leaves are the connected components of the level sets of a surjective submersion $\pi : P \longrightarrow \bar{P}$, that is, for $p \in P$ and $\bar{p} \in \bar{P}$ with $\bar{p} = \pi(p)$, we have

$$\mathcal{F}_p = \text{connected component of } \pi^{-1}(\bar{p}) \text{ containing } p . \qquad (40)$$

In particular, a fiber bundle with connected fibers is a simple foliation. It is also obvious that the leaves of a simple foliation are closed embedded (and not just immersed) submanifolds. Finally, given any surjective submersion $\pi : P \longrightarrow \bar{P}$, we can always decompose the projection π into the composition of two projections,

$$P \longrightarrow P/\mathcal{F} \longrightarrow \bar{P} , \qquad (41)$$

---

[13]This follows from the fact that an affine fiber bundle always admits global sections (which is easy to prove using partitions of unity).

where the first is a surjective submersion with connected fibers and the second is a local diffeomorphism.

Given an arbitrary surjective submersion $\pi : P \longrightarrow \bar{P}$, there are two special types of vector fields on the manifold P: vertical vector fields and, more generally, projectable vector fields:

Definition 8   *Let P and $\bar{P}$ be manifolds and $\pi : P \longrightarrow \bar{P}$ be a surjective submersion. A vector field X on P is said to be **vertical** (with respect to π) if for any $p \in P$, $T_p\pi \cdot X(p) = 0$, and is said to be **projectable** (with respect to π) if for any $p_1, p_2 \in P$ with $\pi(p_1) = \pi(p_2)$, $T_{p_1}\pi \cdot X(p_1) = T_{p_2}\pi \cdot X(p_2)$.*

If X is a projectable vector field on P, then it is clear that for any $\bar{p} \in \bar{P}$, there exists a unique vector $\bar{X}(\bar{p}) \in T_{\bar{p}}\bar{P}$ such that $T_p\pi \cdot X(p) = \bar{X}(\bar{p})$ for all $p \in P$ with $\pi(p) = \bar{p}$, and using local charts for P and $\bar{P}$ in which the submersion π is represented by a constant projection, we can check that since X is smooth, so is $\bar{X}$. Thus we can characterize a projectable vector field as a vector field X on P which can be pushed forward by π to a (unique) vector field $\bar{X}$ on $\bar{P}$, to which it is π-related,[14] and a vertical vector field as a projectable vector field which, when pushed forward by π, gives zero. This implies immediately that in the Lie algebra $\mathfrak{X}(P)$ of vector fields on P, the projectable vector fields form a Lie subalgebra $\mathfrak{X}_P(P)$ and the vertical vector fields form an ideal $\mathfrak{X}_V(P)$ within this Lie subalgebra, i.e.

$$Y, Z \text{ projectable} \implies [Y, Z] \text{ projectable} , \tag{42}$$

$$X \text{ vertical}, Y \text{ projectable} \implies [X, Y] \text{ vertical} . \tag{43}$$

Finally, we have

Lemma 1   *Let P and $\bar{P}$ be manifolds and $\pi : P \longrightarrow \bar{P}$ be a surjective submersion. Then every vector field $\bar{X}$ on $\bar{P}$ is the push-forward of some projectable vector field X on P by π.*

Proof: Using local charts of P and $\bar{P}$ where the submersion π is represented by a constant projection, we see that every point p of P has an open neighborhood U_p on which we can construct a vector field X_p that projects to $\bar{X}\big|_{\pi(U_p)}$. Choosing a locally finite refinement $(U_i)_{i\in I}$ of the open covering $(U_p)_{p\in P}$ of P and a subordinate partition of unity $(\chi_i)_{i\in I}$, we can define a vector field X on P by

$$X = \sum_{i\in I} \chi_i X_{p(i)}$$

and verify that its projects to $\bar{X}$. $\qquad\qquad\square$

---

[14]Recall that given any smooth map $f : M \longrightarrow N$ between manifolds M and N, two vector fields X on M and Y on N are said to be f-related if for every point m of M, we have $T_m f \cdot X(m) = Y(f(m))$. An elementary but important theorem, used constantly, states that if X_1 on M and Y_1 on N are f-related and X_2 on M and Y_2 on N are also f-related, then their Lie brackets, $[X_1, X_2]$ on M and $[Y_1, Y_2]$ on N, are f-related.

Thus we obtain the following exact sequence of Lie algebras:

$$0 \longrightarrow \mathfrak{X}_V(P) \longrightarrow \mathfrak{X}_P(P) \longrightarrow \mathfrak{X}(\bar{P}) \longrightarrow 0 . \tag{44}$$

For later use, we note the following corollary:

Lemma 2   *Let P and $\bar{P}$ be manifolds and $\pi : P \longrightarrow \bar{P}$ be a surjective submersion. Then for every tangent vector $u \in T_pP$, there is a projectable vector field X on P such that $X(p) = u$.*

Now we turn to the main subject of this section: the study of manifolds P equipped with a simple foliation $\mathcal{F}$ whose leaves are flat affine submanifolds of P. This means that the involutive distribution L tangent to $\mathcal{F}$, according to eqn (39), is endowed with a partial linear connection ∇ with vanishing curvature and torsion. In this case, we can define two special types of fields on P, both of which are vertical (i.e., along $\mathcal{F}$, or L) that play an important role: the covariant constant vector fields and the Euler vector fields:

Definition 9   *Let P be a manifold equipped with a simple foliation $\mathcal{F}$ with involutive tangent distribution L, and let ∇ be a partial linear connection in L along L with vanishing curvature and torsion. We say that a vector field X tangent to $\mathcal{F}$ is **covariantly constant** along the leaves of $\mathcal{F}$, or simply covariantly constant, if for any vector field Z tangent to $\mathcal{F}$, we have*

$$\nabla_Z X = 0 . \tag{45}$$

*We also say that a vector field Σ tangent to $\mathcal{F}$ is an **Euler vector field** if for any vector field Z tangent to $\mathcal{F}$, we have*

$$\nabla_Z \Sigma = Z . \tag{46}$$

The standard situation where these types of vector fields can be defined naturally is on the total space of a vector bundle: in this case, there is a preferred Euler vector field, namely, the one that vanishes on the zero section. However, the same construction also works for affine bundles – although in this case, we lose uniqueness of the Euler vector field, since the notion of the zero section has disappeared. In general, Definition 9 implies immediately that the sum of a covariantly constant vector field and an Euler vector field is an Euler vector field, and conversely, the difference between two Euler vector fields is a covariantly constant vector field, so the Euler vector fields constitute an affine space whose difference vector space is the space of covariantly constant vector fields. It is also clear that both types of vector fields are uniquely determined by their value at a single point of each leaf, and using local coordinate systems adapted to the surjective submersion $P \longrightarrow P/\mathcal{F}$ in which the Christoffel symbols of the connection ∇ vanish identically, we can prove that both always exist, at least locally.

Completing the "menu" of ingredients, suppose now that Q is a submanifold of P complementary to $\mathcal{F}$, i.e., for every point q of Q, we have

$$T_qP = T_qQ \oplus T_q\mathcal{F}_q = T_qQ \oplus L_q . \tag{47}$$

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Note that this condition of complementarity does not necessarily imply that Q must intersect all leaves. However, considering again the surjective submersion $\pi : P \longrightarrow P/\mathcal{F}$, it does imply that every point of Q has an open neighborhood in P whose intersection with Q is a local section of π and, hence, that $\pi(Q)$ is open in $P/\mathcal{F}$. Moreover, it also implies that the inclusion of Q in P, followed by the projection π, as a map

$$Q \longrightarrow P/\mathcal{F} \tag{48}$$

is a local diffeomorphism onto its image, which is an open submanifold of $P/\mathcal{F}$. Thus, replacing P by its open submanifold $\pi^{-1}(\pi(Q))$ and $\mathcal{F}$ by restriction to this submanifold, we can assume without loss of generality that Q intersects all leaves, i.e., that the map (48) is surjective.

With these preliminaries out of the way, we want to show how to build, using the geodesic flow with respect to the connection ∇ that radially emanates from Q, a canonical local diffeomorphism, denoted by $\exp_Q$ and adequately called the *exponential*, between the vector bundle $L|_Q$ and the manifold P. More precisely, if for $q \in Q$ and $u_q \in L_q$, the geodesic in $\mathcal{F}_q$ with initial position q and initial velocity u_q is (momentarily) denoted by $F(.\,; u_q)$, the map

$$\exp_Q : \mathrm{Dom}(\exp_Q) \longrightarrow P \tag{49}$$

with domain given by

$$\mathrm{Dom}(\exp_Q) \;=\; \bigcup_{q \in Q} \{\, u_q \in L_q \mid F(1; u_q) \text{ exists} \,\} \tag{50}$$

is defined by

$$\exp_Q(u_q) \;=\; F(1; u_q) \,. \tag{51}$$

This allows us to immediately get rid of the symbol F for the geodesic flow, which is anything but self-explanatory, since the geodesic in $\mathcal{F}_q$ with initial position q and initial velocity u_q is the curve given by $s \mapsto \exp_Q(s u_q)$, i.e., we have

$$\exp_Q(s u_q)\Big|_{s=0} \;=\; q \quad , \quad \frac{d}{ds}\,\exp_Q(s u_q)\Big|_{s=0} \;=\; u_q \tag{52}$$

$$\frac{D}{ds}\frac{d}{ds}\,\exp_Q(s u_q) \;=\; 0 \tag{53}$$

Obviously, the domain $\mathrm{Dom}(\exp_Q)$ of the exponential is a tubular neighborhood of Q in $L|_Q$, and by the fundamental theorem about the dependence of solutions of differential equations on the initial conditions and on parameters, the map $\exp_Q$ is differentiable (i.e., smooth) and induces the identity on Q.

Lemma 3   *Under the hypotheses stated above, the exponential (49) is a local diffeomorphism onto its image, which is an open submanifold of P.*

Proof: As $L|_Q$ and P have the same dimension, it suffices to prove that for all vectors u_q in the domain of the exponential, its tangent map

$$T_{u_q} \exp_Q : T_{u_q}(L|_Q) \longrightarrow T_{\exp_Q(u_q)} P$$

is injective. When u_q is the zero vector, this is obvious, since for all $q \in Q$, we have natural direct decompositions of the tangent spaces to $L|_Q$ and to P at q in a " vertical part" and a " horizontal part",

$$T_q(L|_Q) \;=\; L_q \ominus T_q Q \;\;,\;\;\; T_q P \;=\; L_q \oplus T_q Q$$

with respect to which the tangent map

$$T_q \exp_Q : T_q(L|_Q) \longrightarrow T_q P$$

is simply the identity. Thus let us consider the general case where $u_q \in L_q$ is any vector in the domain of $\exp_Q$ and $v_{u_q} \in T_{u_q}(L|_Q)$ is a tangent vector to the total space of the vector bundle $L|_Q$ over Q. Suppose that $T_{u_q} \exp_Q \cdot v_{u_q} = 0$. Then applying the tangent functor to the commutative diagram

$$
\begin{array}{ccc}
L|_Q & \xrightarrow{\;\exp_Q\;} & P \\
\downarrow & & \downarrow \\
Q & \longrightarrow & P/\mathcal{F}
\end{array}
$$

where the lower horizontal arrow is the local diffeomorphism (48), we conclude that v_{u_q} must be vertical and, as $V_{u_q}(L|_Q) \cong L_q$, can be identified with a vector $v_q \in L_q$; more explicitly, $v_{u_q} \in V_{u_q}(L|_Q)$ is the tangent vector

$$v_{u_q} \;=\; \frac{d}{dt}\,(u_q + t v_q)\Big|_{t=0}\,,$$

and hence $T_{u_q} \exp_Q \cdot v_{u_q}$ is the tangent vector

$$T_{u_q} \exp_Q \cdot v_{u_q} \;=\; \frac{d}{dt}\, \exp_Q(u_q + t v_q)\Big|_{t=0}\,.$$

This shows that $T_{u_q} \exp_Q \cdot v_{u_q}$ is the value, at $s = 1$, of a Jacobi field along the geodesic $s \mapsto \exp_Q(s u_q)$, defined as the variation of the following one-parameter family of geodesics:

$$(s,t) \mapsto \exp_Q\big(s(u_q + t v_q)\big)\,,$$

where t is the family parameter and s is the geodesic parameter (for fixed t). Explicitly, the value of this Jacobi field at the point $\exp_Q(s u_q)$ is

$$\frac{d}{dt}\, \exp_Q\big(s(u_q + t v_q)\big)\Big|_{t=0}\,,$$

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showing that if $T_{u_q} \exp_Q \cdot v_{u_q} = 0$, it must vanish at $s = 0$ and at $s = 1$, i.e., q and $p = \exp_Q(u_q)$ would be conjugate points along the geodesic $s \mapsto \exp_Q(su_q)$. But the condition that the connection ∇ has vanishing curvature and torsion excludes the existence of conjugate points along any geodesic, because the differential equation for a Jacobi field, written in components for an autoparallel frame along that geodesic, reduces to an equation of the form $d^2 X^i / ds^2 = 0$, whose solutions have exactly one zero – no more, no less. $\qquad\square$

In particular, it follows that the exponential (49) provides a diffeomorphism

$$\exp_Q : U_0 \longrightarrow U \tag{54}$$

of a convex neighborhood U_0 of Q in the vector bundle $L\big|_Q$ with a tubular neighborhood U of Q in P.[15] By construction, this diffeomorphism is affine.

In the case of geodesic completeness, we can prove an even stronger claim:

Lemma 4   *Under the hypotheses stated above, and if P is geodesically complete with respect to ∇, the exponential (49) defines a covering*

$$\exp_Q : L\big|_Q \longrightarrow P \tag{55}$$

which, for every point q of Q, induces a universal covering

$$\exp_q : L_q \longrightarrow \mathcal{F}_q \tag{56}$$

of the leaf $\mathcal{F}_q$ by the fiber L_q. In particular, if all leaves $\mathcal{F}_q$ are simply connected, the exponential provides a global affine diffeomorphism between $L\big|_Q$ and P.

Proof: Under the hypothesis of geodesic completeness, the domain of the exponential $\exp_Q$ is the entire vector bundle $L\big|_Q$. Moreover, the hypothesis that the connection ∇ should have vanishing curvature and torsion implies that for every point q of Q, the restriction $\exp_q$ of $\exp_Q$ to the fiber L_q is an affine map from the vector space L_q, equipped with the trivial linear connection, to the leaf $\mathcal{F}_q$, equipped with the linear connection $\nabla_q = \nabla\big|_{\mathcal{F}_q}$. (For a much more general statement, see, for example, [23, Chapter 6, Theorem 7.1, p. 257].) So, the lemma follows from a general theorem, stated in more detail and proved in Appendix B, according to which every affine map from a connected, simply connected and geodesically complete affine manifold M to a connected affine manifold M', if it is a local diffeomorphism, it is even a covering; in particular, it is automatically surjective (and M' is automatically geodesically complete). $\qquad\square$

Another result which we shall need in what follows concerns differential forms on foliated manifolds:

---

[15] A neighborhood of the zero section of a vector bundle is called convex if its intersection with each fiber is a convex neighborhood of the origin in that fiber.

Proposition 8 *Let P be a manifold equipped with a simple foliation $\mathcal{F}$ with involutive tangent distribution L, and let α be a k-form on P. Then α is the pull-back of a k-form α_Q on the quotient manifold $Q = P/\mathcal{F}$ by the projection $\pi : P \longrightarrow Q$, $\alpha = \pi^*\alpha_Q$, if and only if, for all vector fields X along $\mathcal{F}$, $X \in \Gamma(L)$, we have $i_X\alpha = 0$ (horizontality condition) and $\mathbb{L}_X\alpha = 0$ (condition of constancy along the leaves).*

Proof: First, observe that for α to be the pull-back of a k-form α_Q on the quotient manifold Q, we must have

$$\alpha(p)(u_1,\ldots,u_k) \;=\; \alpha_Q(\pi(p))\big(T_p\pi \cdot u_1,\ldots,T_p\pi \cdot u_k\big) \tag{57}$$
$$\text{for } p \in P, \; u_1,\ldots,u_k \in T_pP$$

Thus if any of the vectors $u_1,\ldots,u_k$ belongs to L_p, this expression vanishes, so for any $X \in \Gamma(L)$, we must have $i_X\alpha = 0$ and therefore also

$$\mathbb{L}_X\alpha \;=\; \big(di_X + i_X d\big)\,\pi^*\alpha_Q \;=\; i_X\pi^*(d\alpha_Q) \;=\; 0 \,.$$

Conversely, it is clear that if we use eqn (57) to define α_Q in terms of α, we must ensure (i) that for fixed $p \in P$, the expression on the lhs of this equation does not depend on the representatives $u_i \in T_pP$ of the vectors $T_p\pi \cdot u_i \in T_{\pi(p)}Q$, which is guaranteed by the condition of horizontality ($i_X\alpha = 0$ for $X \in \Gamma(L)$), and (ii) that the expression on the lhs of this equation does not depend on the representative $p \in P$ of the point $\pi(p) \in Q$: this can be derived from the condition of constancy along the leaves ($\mathbb{L}_X\alpha = 0$ for $X \in \Gamma(L)$), as follows: Let p and p' be two points of P such that $\pi(p) = \pi(p')$: this means that they belong to the same leaf F, and therefore there is a curve γ entirely contained in the leaf F with $\gamma(0) = p$ and $\gamma(1) = p'$; in particular, we have $\dot\gamma(s) \in L_{\gamma(s)}$ for $0 \leqslant s \leqslant 1$, and we can further assume that $\dot\gamma(s) > 0$ for $0 \leqslant s \leqslant 1$. Using a partition $(s_\alpha)_{\alpha=1,\ldots,r}$ of the interval $[0,1]$ $(0 = s_0 < s_1 < \ldots < s_r < s_{r+1} = 1)$ and a finite family $(U_\alpha)_{\alpha=0,\ldots,r}$ of chart domains U_α for P where π is represented by a constant projection onto some subspace, together with some smooth cutoff function of compact support on P that is 1 on an open neighborhood of the image of the curve γ, it becomes evident that we can find a vector field X on P along $\mathcal{F}$, $X \in \Gamma(L)$, which extends $\dot\gamma$, i.e., such that $\dot\gamma(s) = X(\gamma(s))$ for $0 \leqslant s \leqslant 1$. But this means that γ is an integral curve of X and, more than this, that close to $p = \gamma(0)$, the flow F_X of X is defined at least up to $s = 1$, so that there exist open neighborhoods U_0 of $p = \gamma(0)$ and U_1 of $p' = \gamma(1)$ such that the flow for time 1 establishes a diffeomorphism $F_X(1,.) : U_0 \longrightarrow U_1$ which preserves the leaves of $\mathcal{F}$, since X is tangent to $\mathcal{F}$, i.e., we have $\pi|_{U_1} = F_X(1,.) \circ \pi|_{U_0}$. Now, $\mathbb{L}_X\alpha = 0$ implies $F_X(1,.)^*(\alpha|_{U_1}) = \alpha|_{U_0}$, so using $p' = F_X(1,p)$ and setting $u_i' = T_pF_X(1,.) \cdot u_i$ $(1 \leqslant i \leqslant k)$, we obtain $T_{p'}\pi \cdot u_i' = T_p\pi \cdot u_i$ $(1 \leqslant i \leqslant k)$ and

$$
\begin{aligned}
\alpha_{p'}(u_1',\ldots,u_k') &= \alpha_{F_X(1,p)}\big(T_pF_X(1,.) \cdot u_1,\ldots,T_pF_X(1,.) \cdot u_k\big)\\
&= \big(F_X(1,.)^*(\alpha|_{U_1})\big)_p(u_1,\ldots,u_k)\\
&= \alpha_p(u_1,\ldots,u_k) \,.
\end{aligned}
$$

$\square$

8 Foliated symplectic manifolds

In this section, we consider the geometry of a foliated symplectic manifold, or more precisely, of a symplectic manifold P, with symplectic form ω, that comes equipped with a simple lagrangian foliation $\mathcal{F}$. Of course, lagrangian foliations may exist or not, and they can be simple or not: a classical example of a symplectic manifold which does not admit any lagrangian foliation is the sphere S^2 ("no-hair theorem"), while a classical example of a lagrangian foliation which is regular but not simple is the irrational flow on the torus $\mathbb{T}^2$ (in both cases, the symplectic form is the standard volume form). But in the case of a simple lagrangian foliation, the quotient space $Q = P/\mathcal{F}$ admits a unique manifold structure such that the canonical projection π from P to Q is a surjective submersion. Note that with this convention, Q is a quotient manifold of P, but nothing guarantees "a priori" that it can be realized as a submanifold of P, so the existence of an embedding of Q into P as a closed submanifold is an additional condition that must be imposed separately or deduced from other additional assumptions.[16] In any case, the hypothesis that the foliation $\mathcal{F}$ is lagrangian provides a canonical partial linear connection in L along L with vanishing curvature and torsion, namely the Bott connection ∇^B introduced earlier: it implies that the leaves of $\mathcal{F}$ are flat affine manifolds and is the crucial ingredient in the proof of the following statement:

Theorem 5 *Let P be a symplectic manifold, with symplectic form ω, equipped with an involutive lagrangian distribution L. Suppose that the corresponding foliation $\mathcal{F}$ is simple, writing its leaves as the level sets of a surjective submersion $\pi_P : P \longrightarrow Q$, and suppose finally that the quotient manifold $Q = P/\mathcal{F}$ can be realized as a closed embedded submanifold of P. Under these circumstances, consider the musical isomorphism $\omega^\sharp : L\big|_Q^\perp \longrightarrow L\big|_Q$ (see eqn (6)), together with the isomorphism $L\big|_Q^\perp \cong T^*Q$ (which arises from the direct decomposition (47)), and combined with the exponential $\exp_Q$ as defined in Sect. 7. Then we have the following:*

- *The composition of these isomorphisms provides a diffeomorphism $\phi : V \longrightarrow U$ of a tubular neighborhood U of Q in P with a convex neighborhood V of the zero section of the cotangent bundle T^*Q of Q.*

- *If the leaves of $\mathcal{F}$ are geodesically complete with respect to the Bott connection, the composition of these isomorphisms provides a covering of P by the cotangent bundle of Q, $\phi : T^*Q \longrightarrow P$.*

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[16]In general, there may be topological obstructions to the existence of an embedding of Q into P. Such obstructions are of global nature, since the submersion theorem (or local slice theorem) states that locally, there is always such an embedding. As an example of a set of additional conditions that guarantees its global existence, we mention the hypotheses in the third item of Theorem 5 below — namely that the leaves are geodesically complete with respect to the Bott connection and simply connected, as these ensure that P is an affine bundle over Q, and affine bundles always admit global sections.

- If the leaves of $\mathcal{F}$ are geodesically complete with respect to the Bott connection and simply connected, the composition of these isomorphisms provides a diffeomorphism of P with the cotangent bundle of Q, $\phi : T^*Q \longrightarrow P$.

Furthermore, ϕ preserves fibers, mapping T_q^*Q onto $\mathcal{F}_q$ (or, in the first case, $V \cap T_q^*Q$ onto $U \cap \mathcal{F}_q$), and defining

$$\theta = -i_\Sigma \phi^* \omega , \qquad (58)$$

we have that $\phi^* \omega + d\theta$ is the pull-back of a closed 2-form ω_Q on Q by the projection τ of T^*Q to Q:

$$\phi^* \omega + d\theta = \tau^* \omega_Q . \qquad (59)$$

Finally, the cohomology class $[\omega_Q] \in H^2(Q)$ of ω_Q does not depend on the embedding employed and thus is an invariant of the foliation $\mathcal{F}$.

Remark 3 The last statement ensures that ϕ is "almost" a symplectomorphism: $\phi^* \omega$ differs from the standard symplectic form of the cotangent bundle only by the pull-back of a closed 2-form on the base. If Q is a lagrangian submanifold of P, then $\omega_Q = 0$ and ϕ will be a symplectomorphism. In this special case, the first statement of the above theorem, which is of local nature (with respect to the structure of P along the leaves of the foliation $\mathcal{F}$), is known as Weinstein's symplectic tubular neighborhood theorem, established in [29]. The third statement has first been proved in [26]. Here, besides establishing also the second statement, we give a more direct proof for all three of them, avoiding the use of additional and artificial ingredients (such as the auxiliary riemannian metric employed in [26]): this will also allow us to formulate and prove an extension of this theorem to the case of polysymplectic and multisymplectic geometry, treated in the next section.

Proof: In view of Lemmas 3 and 4, we just need to prove the final part, contained in eqns (58) and (59). To simplify the presentation, we consider only the first and third statement, where ϕ is a diffeomorphism and hence can be used to identify V with U and T^*Q with P, respectively. (The second statement, where ϕ is just a local diffeomorphism, can be treated similarly, taking into account that in this case, the Euler vector field Σ may fail to be globally defined on P, but it can be replaced by a family of Euler vector fields locally defined on P, which leads to a family of local formulas of the same type as eqns (58) and (59).) Therefore, we suppress the reference to the pull-back by ϕ. The argument will be based on Proposition 8, according to which it is sufficient to show that for every vertical vector field X, we have $i_X(\omega + d\theta) = 0$ (horizontality condition) and $\mathbb{L}_X(\omega + d\theta) = 0$ (condition of constancy along the leaves). Since ω is closed, the second of these conditions follows directly from the first:

$$i_X(\omega + d\theta) = 0 \implies \mathbb{L}_X(\omega + d\theta) = (di_X + i_X d)(\omega + d\theta) = d(i_X(\omega + d\theta)) = 0 .$$

To prove the first, we must show that for every vector field X along $\mathcal{F}$ and every vector field Y, we have $(\omega + d\theta)(X, Y) = 0$, and due to Lemma 2, we may do so assuming,

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without loss of generality, that Y is projectable. Now using the definitions of the Bott connection and of the Euler vector field, we have

$$\omega(X,Y) \;=\; \omega(\nabla^B_X \Sigma, Y) \;=\; X \cdot \omega(\Sigma, Y) - \omega(\Sigma, [X,Y]) \;.$$

Since X is vertical and Y is projectable, $[X,Y]$ is also vertical (see eqn (43)), and since L is lagrangian, the second term vanishes, so we get

$$\omega(X,Y) \;=\; X \cdot \omega(\Sigma, Y) \;.$$

Using the definition of θ, eqn (58), together with the fact that this implies that θ vanishes on vertical vector fields, we have

$$\begin{aligned}
(\omega + d\theta)(X,Y) &= \omega(X,Y) + X \cdot \theta(Y) - Y \cdot \theta(X) - \theta([X,Y]) \\
&= X \cdot \omega(\Sigma, Y) + X \cdot \theta(Y) - Y \cdot \theta(X) - \theta([X,Y]) \\
&= -X \cdot \theta(Y) + X \cdot \theta(Y) - Y \cdot \theta(X) - \theta([X,Y]) \\
&= 0 \;.
\end{aligned}$$

Finally, we must address the issue of uniqueness, or rather the amount of non-uniqueness, of the decomposition (59), generated by the fact that there are different Euler vector fields, corresponding to different choices of the embedding of the quotient manifold $P/\mathcal{F}$ into P. Thus let Σ_1 and Σ_2 be two Euler vector fields, and define $\theta_1 = -i_{\Sigma_1}\omega$ and $\theta_2 = -i_{\Sigma_2}\omega$. Then for every vertical vector field X,

$$i_X(\theta_1 - \theta_2) \;=\; \omega(X, \Sigma_2 - \Sigma_1) \;=\; 0 \;,$$

since L is lagrangian, while for every projectable vector field Y,

$$\begin{aligned}
\mathbb{L}_X(\theta_1 - \theta_2)(Y) &= X \cdot ((\theta_1 - \theta_2)(Y)) - (\theta_1 - \theta_2)([X,Y]) \\
&= X \cdot \omega(\Sigma_2 - \Sigma_1, Y) - \omega(\Sigma_2 - \Sigma_1, [X,Y]) \\
&= \omega(\nabla^B_X(\Sigma_2 - \Sigma_1), Y) \\
&= 0 \;,
\end{aligned}$$

where we have used the definition of the Bott connection and the fact that $\Sigma_2 - \Sigma_1$ is covariantly constant. According to Proposition 8, it follows that there is a 1-form θ_Q on Q such that $\theta_1 - \theta_2 = \pi^*\theta_Q$, implying

$$\omega + d\theta_1 \;=\; \pi^*\omega_Q^{(1)} \;\;,\;\; \omega + d\theta_2 \;=\; \pi^*\omega_Q^{(2)} \;,$$

with

$$\theta_1 - \theta_2 \;=\; \pi^*\theta_Q \;\;,\;\; \omega_Q^{(1)} - \omega_Q^{(2)} \;=\; d\theta_Q \;.$$

In particular, the cohomology class of ω_Q does not depend on the choice of embedding. $\square$

9 Structure of polysymplectic and multisymplectic fiber bundles

In analogy with the symplectic case, we can now formulate our main theorem about the structure of polysymplectic and multisymplectic fiber bundles. The additional ingredient, as compared to the symplectic case, comes from the fact that the underlying manifold P is now the total space of a fiber bundle over some other manifold M,[17] with bundle projection denoted by $\pi : P \longrightarrow M$, and that the distribution L is vertical with respect to this projection. Roughly speaking, this implies that the submanifold of P representing the quotient space $P/\mathcal{F}$, which is now denoted by E, should be the total space of a fiber bundle on M, whose projection will be denoted by $\pi_E : E \longrightarrow M$, as in the examples in Sects 4 and 5. Thus, the condition that E is a submanifold of P complementary to $\mathcal{F}$ and, at the same time, to the fibers of the projection π, leads us to replace eqn (47) by the condition that for every point e of E, we have

$$T_e P = T_e E \oplus L_e \qquad \text{and} \qquad V_e P = V_e E \oplus L_e , \qquad (60)$$

where $V_e P$ denotes the vertical space with respect to the projection π and $V_e E$ denotes the vertical space with respect to the projection π_E.

In the case of polysymplectic fiber bundles, we have

Theorem 6 *Let P be a polysymplectic fiber bundle over a manifold M, with projection $\pi : P \longrightarrow M$, polysymplectic form $\hat{\omega}$ and involutive polylagrangian distribution L. Suppose that the corresponding foliation $\mathcal{F}$ is simple, writing its leaves as the level sets of a surjective submersion $\pi_P : P \longrightarrow E$, that π induces a surjective submersion $\pi_E : E \longrightarrow M$ so that $\pi = \pi_E \circ \pi_P$, and finally that the quotient manifold $E = P/\mathcal{F}$ (a) can be realized as a closed embedded submanifold of P and (b) is the total space of a fiber bundle over M with respect to the projection π_E. Under these circumstances, consider the musical isomorphism $\hat{\omega}^\sharp : \bigwedge^k L\big|_E^\perp \otimes \pi_E^*(\hat{T}) \longrightarrow L\big|_E$ (see eqn (20)), together with the isomorphism $L\big|_E^\perp \cong V^*E$ (which arises from the direct decomposition (60)), and combined with the exponential $\exp_E$ as defined in Sect. 7. Then we have the following:*

- *The composition of these isomorphisms provides a diffeomorphism $\phi : V \longrightarrow U$ of a tubular neighborhood U of E in P with a convex neighborhood V of the zero section of the model vector bundle $\bigwedge^k V^*E \otimes \pi_E^*(\hat{T})$ of Example 1.*

- *If the leaves of $\mathcal{F}$ are geodesically complete with respect to the Bott connection, the composition of these isomorphisms provides a covering of P by the model vector bundle, $\phi : \bigwedge^k V^*E \otimes \pi_E^*(\hat{T}) \longrightarrow P$.*

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[17]In applications to physics, the base manifold M is space-time. In classical mechanics, this reduces to a copy of the real line (time axis) which is usually suppressed, but it reappears immediately when one considers non-autonomous systems, passing from symplectic manifolds to contact manifolds and then, using Cartan's trick of adding yet another copy of the real line (energy axis), back to symplectic manifolds.

- *If the leaves of $\mathcal{F}$ are geodesically complete with respect to the Bott connection and simply connected, the composition of these isomorphisms provides a diffeomorphism of P with the model vector bundle, $\phi : \bigwedge^k V^* E \otimes \pi_E^*(\hat{T}) \longrightarrow P$.*

Furthermore, ϕ preserves fibers, mapping $\bigwedge^k V_e^* E \otimes \hat{T}_{\pi_E(e)}$ onto $\mathcal{F}_e$ (or, in the first case, $V \cap \bigwedge^k V_e^* E \otimes \hat{T}_{\pi_E(e)}$ onto $U \cap \mathcal{F}_e$), and defining

$$\hat{\theta} = -i_{\Sigma}\phi^*\hat{\omega} , \tag{61}$$

we have that $\phi^*\hat{\omega} + d_V\hat{\theta}$ is the pull-back of a vertically closed k-form $\hat{\omega}_E$ on E by the projection π^k of $\bigwedge^k V^* E \otimes \pi^*(\hat{T})$ to E:

$$\phi^*\hat{\omega} + d_V\hat{\theta} = (\pi^k)^*\hat{\omega}_E . \tag{62}$$

Finally, the cohomology class $[\hat{\omega}_E] \in H^k(E)$ of $\hat{\omega}_E$ does not depend on the embedding employed and thus is an invariant of the foliation $\mathcal{F}$.

Proof: The proof is completely analogous to the proof of Theorem 5 for foliated symplectic manifolds, and the calculations to verify the formula (61) are carried out with a vector field X along $\mathcal{F}$ and k projectable vector fields $Y_1, \ldots, Y_k$, all of them vertical with respect to the projection π to M. $\qquad\square$

Turning to the case of multisymplectic fiber bundles, we have

Theorem 7 *Let P be a multisymplectic fiber bundle over a manifold M, with projection $\pi : P \longrightarrow M$, multisymplectic form ω and involutive multilagrangian distribution L. Suppose that the corresponding foliation $\mathcal{F}$ is simple, writing its leaves as the level sets of a surjective submersion $\pi_P : P \longrightarrow E$, that π induces a surjective submersion $\pi_E : E \longrightarrow M$ so that $\pi = \pi_E \circ \pi_P$, and finally that the quotient manifold $E = P/\mathcal{F}$ (a) can be realized as a closed embedded submanifold of P and (b) is the total space of a fiber bundle over M with respect to the projection π_E. Under these circumstances, consider the musical isomorphism $\omega^\flat : \bigwedge_{r-1}^k L\big|_E^\perp \longrightarrow L\big|_E$ (see eqn (30)), together with the isomorphism $L\big|_E^\perp \cong T^*E$ (which arises from the direct decomposition (60)), and combined with the exponential $\exp_E$ as defined in Sect. 7. Then we have the following:*

- *The composition of these isomorphisms provides a diffeomorphism $\phi : V \longrightarrow U$ of a tubular neighborhood U of E in P with a convex neighborhood V of the zero section of the model vector bundle $\bigwedge_{r-1}^k T^*E$ of Example 2.*

- *If the leaves of $\mathcal{F}$ are geodesically complete with respect to the Bott connection, the composition of these isomorphisms provides a covering of P by the model vector bundle, $\phi : \bigwedge_{r-1}^k T^*E \longrightarrow P$.*

- *If the leaves of $\mathcal{F}$ are geodesically complete with respect to the Bott connection and simply connected, the composition of these isomorphisms provides a diffeomorphism of P with the model vector bundle, $\phi : \bigwedge_{r-1}^k T^*E \longrightarrow P$.*

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Furthermore, ϕ preserves fibers, mapping $\bigwedge_{r-1}^{k} T_{e}^{*} E$ onto $\mathcal{F}_{e}$ (or, in the first case, $V \cap \bigwedge_{r-1}^{k} T_{e}^{*} E$ onto $U \cap \mathcal{F}_{e}$), and defining

$$\theta = -i_{\Sigma} \phi^{*} \omega , \tag{63}$$

we have that $\phi^{*}\omega + d\theta$ is the pull-back of a closed k-form ω_{E} on E by the projection π_{r-1}^{k} of $\bigwedge_{r-1}^{k} T^{*} E$ to E:

$$\phi^{*}\omega + d\theta = (\pi_{r-1}^{k})^{*}\omega_{E} . \tag{64}$$

Finally, the cohomology class $[\omega_{E}] \in H^{k}(E)$ of ω_{E} does not depend on the embedding employed and thus is an invariant of the foliation $\mathcal{F}$.

Proof: Analogous to that of Theorem 6, eliminating only the condition of verticality of the projectable vector fields relative to the projection over M. □

10   Conclusions

The main new results reported in this paper are the theorems on existence of torsion-free polysymplectic and multisymplectic connections and their complete classification (Theorems 3 and 4), together with the structure theorems on polysymplectic and multi-symplectic fiber bundles (Theorems 6 and 7) which show that, under certain mild additional assumptions, these are exhausted by the well-known standard examples of bundles of forms (Examples 1 and 2). All these generalize corresponding theorems of symplectic geometry which we have decided to include not only for the sake of completeness (given that most of them do not appear to have been stated explicitly in the existing literature, at least not in their full generality), but also because our proofs use different techniques. For example, we could not find an explicit statement of the theorem on the existence and classification of torsion-free symplectic connections that preserve a single lagrangian foliation, included here as Theorem 2. (What one can find easily are classification theorems for torsion-free symplectic connections which either are subject to no further constraints or else are required to preserve two transversal lagrangian foliations: as is well known, the latter case leads to a unique answer, namely the bilagrangian connection first constructed by Heß. This situation is somewhat surprising since after all, the case of a single lagrangian foliation *is* very important: it is the situation one encounters when dealing with cotangent bundles! Indeed, a cotangent bundle is a symplectic manifold carrying a distinguished lagrangian foliation but no natural candidate for a second one that would be transversal to it: all one finds are single lagrangian submanifolds transversal to it, namely the zero section or, more generally, the graph of any closed 1-form on the base manifold.) Similarly, the global versions of Weinstein's tubular neighborhood theorem do not seem to have been formulated in their full generality, and the existing proofs use rather artificial additional ingredients which, as we show, are really unnecessary.

Regarding the extension from symplectic to polysymplectic and multisymplectic geometry, one of the central concepts is the Bott connection: it is a partial linear connection in and along the corresponding polylagrangian or multilagrangian distribution L and is a natural geometric object at least when L is uniquely determined and involutive, which is the generic case [12]. Since this connection is both torsion-free and flat, it implies that, just as in symplectic geometry, the leaves of the corresponding foliation are flat affine manifolds – a fact that imposes severe restrictions on the underlying geometry. *The upshot is that polysymplectic and multisymplectic geometry is analogous not to the geometry of general symplectic manifolds but rather to that of foliated symplectic manifolds, and that is why there is no generic polysymplectic or multisymplectic analogue of the coadjoint orbit construction, since typically such orbits do not admit lagrangian foliations.*

Such observations, when applied to classical field theory, support a general picture concerning the role of position variables and momentum variables in physics.

In the usual hamiltonian formulation of classical mechanics, these variables are essentially treated on an equal footing: they can be thought of as ingredients of local coordinate systems in a symplectic manifold, called phase space, and transformations between such local coordinate systems, called canonical transformations, are symmetries of the theory – a point of view that has been triumphant in the mathematical treatment of completely integrable systems, whose solution is achieved through a judiciously chosen canonical transformation to so-called action-angle variables. As a result, many have been led to believe that there is a general "democracy" between position and momentum variables.

However, it is well known that this "democracy" is lost upon quantization: in contrast to what happens in classical mechanics, canonical transformations mixing position and momentum variables are no longer symmetries of quantum mechanics, since they cannot be implemented by unitary operators in the Hilbert space of states.

What is much less known is that this loss of symmetry is by no means a specific feature of going to the quantum world, simply because the same thing happens in (relativistic) field theory: here too, this "democracy" just disappears!

The central reason seems to be that, already at the classical level, relativity is built on fundamental new principles of physics that require a clear-cut distinction between the two types of variables. Perhaps the most important of them all is space-time locality, which postulates that events localized in space-like separated regions of space-time cannot exert any direct influence on each other: obviously, this principle refers to space-like separation in space-time and not in momentum space! Therefore, it is not a defect but rather a virtue of polysymplectic and multisymplectic geometry, whose proposal is to provide the correct mathematical framework for the hamiltonian formulation of (relativistic) classical field theory, that they incorporate, from the very beginning, a clear geometrical distinction between position and momentum variables, in terms of a given distribution describing the "collection of all momentum directions". This characterization is as it should be: coordinate and frame independent, as well as intrinsically defined and unique; its mere existence being in sharp contrast to the situation in symplectic geometry, where specifying

a lagrangian distribution is a matter of choice. Thus in (relativistic) field theory, the lack of "democracy" in the sense described before is *not* a quantum effect, but rather the result of physical principles which already prevail at the classical level.

Appendix A: Auxiliary formulas

In the course of this paper, we have repeatedly made use of the following two elementary facts.

Lemma 5   *Let M be a manifold and let L be a distribution on M. Then if there exists a torsion-free linear connection ∇ on M preserving L, or more generally, if there exist an involutive distribution V on M containing L and a torsion-free partial linear connection ∇ in V along V preserving L, L must be involutive.*

Proof: This follows simply by looking at the definition of the torsion tensor of ∇,

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

which implies that if ∇ is torsion-free and preserves L, then when X and Y are along L, so must be $[X, Y]$. $\square$

Lemma 6   *Given a manifold M and a linear connection ∇ on M with torsion tensor T, we have for any differential form α of degree r and any $r + 1$ vector fields $X_0, \ldots, X_r$ on M*

$$\sum_{i=0}^{r} (-1)^i \, (\nabla_{X_i} \alpha)(X_0, \ldots, \hat{X}_i, \ldots, X_r)$$

$$= d\alpha(X_0, \ldots, X_r) + \sum_{0 \leqslant i < j \leqslant r} (-1)^{i+j} \, \alpha(T(X_i, X_j), X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_r) \, . \tag{65}$$

Similarly, given a fiber bundle P over a manifold M, with vertical bundle VP, and a partial linear connection ∇ in VP along VP with torsion tensor T, we have for any vertical differential form α of degree r and any $r + 1$ vertical vector fields $X_0, \ldots, X_r$ on P

$$\sum_{i=0}^{r} (-1)^i \, (\nabla_{X_i} \alpha)(X_0, \ldots, \hat{X}_i, \ldots, X_r)$$

$$= d_V \alpha(X_0, \ldots, X_r) + \sum_{0 \leqslant i < j \leqslant r} (-1)^{i+j} \, \alpha(T(X_i, X_j), X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_r) \, . \tag{66}$$

Proof: Both statements follow from the same elementary calculation:

$$\sum_{i=0}^{r} (-1)^i \, (\nabla_{X_i}\alpha)(X_0,\ldots,\hat{X}_i,\ldots,X_r)$$

$$= \sum_{i=0}^{r} (-1)^i \, X_i \cdot \alpha(X_0,\ldots,\hat{X}_i,\ldots,X_r)$$

$$- \sum_{i=0}^{r}\sum_{j=0}^{i-1} (-1)^i \, \alpha\big(X_0,\ldots,\nabla_{X_i}X_j,\ldots,\hat{X}_i,\ldots,X_r\big)$$

$$- \sum_{i=0}^{r}\sum_{j=i+1}^{r} (-1)^i \, \alpha\big(X_0,\ldots,\hat{X}_i,\ldots,\nabla_{X_i}X_j,\ldots,X_r\big)$$

$$= \sum_{i=0}^{r} (-1)^i \, X_i \cdot \alpha(X_0,\ldots,\hat{X}_i,\ldots,X_r)$$

$$+ \sum_{0\leqslant i<j\leqslant r} (-1)^{i+j} \, \alpha\big(\nabla_{X_i}X_j - \nabla_{X_j}X_i, X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_r\big)$$

$$= d\alpha(X_0,\ldots,X_r) \quad \text{or} \quad d_V\alpha(X_0,\ldots,X_r)$$

$$+ \sum_{0\leqslant i<j\leqslant r} (-1)^{i+j} \, \alpha\big(\nabla_{X_i}X_j - \nabla_{X_j}X_i - [X_i,X_j], X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_r\big)$$

$\square$

Appendix B: Affine manifolds and maps

We recall some concepts and facts about affine manifolds and affine maps between them, following [23].

Definition 10 *An **affine manifold** is a manifold equipped with a linear connection ∇. A (smooth) map $f : M \longrightarrow M'$ between affine manifolds is called an **affine map** if its tangent map $Tf : TM \longrightarrow TM'$ preserves parallel transport, i.e., for any curve γ in M from x to y with image curve $\gamma' = f \circ \gamma$ in M' from $f(x)$ to $f(y)$, the following diagram commutes:*

$$\begin{array}{ccc}
T_xM & \xrightarrow{\ T_xf\ } & T_{f(x)}M' \\
{\scriptstyle U_\gamma^\nabla(x,y)}\downarrow & & \downarrow{\scriptstyle U_{\gamma'}^{\nabla'}(f(x),f(y))} \\
T_yM & \xrightarrow[\ T_yf\ ]{} & T_{f(y)}M'
\end{array}$$

Obviously, an affine map takes geodesics into geodesics and therefore commutes with the exponential, in the sense that

$$f\big(\exp_x(u)\big) \;=\; \exp_{f(x)}\big(T_x f \cdot u\big) \qquad \text{for } x \in M,\, u \in \mathrm{Dom}(\exp_x) \subset T_x M$$

(see [23, Chapter 6, Proposition 1.1, p. 225]).

An important property of riemannian manifolds that extends to affine manifolds is the existence of convex geodesic balls around each point. First, we say that an open neighborhood U_x of a point x in an affine manifold M is a *normal neighborhood* of x if there is an open neighborhood U_x^0 of the origin in $T_x M$ contained in the domain $\mathrm{Dom}(\exp_x)$ of the exponential $\exp_x$ such that the latter restricts to a diffeomorphism $\exp_x : U_x^0 \longrightarrow U_x$. Second, a *geodesic ball* around a point x of M is a normal neighborhood B_x of x obtained as the inverse image of an open ball in $T_x M$ around the origin, of radius ρ, say, where ρ is sufficiently small, with respect to some (arbitrarily chosen) scalar product in $T_x M$. Now it can be shown [23, Chapter 3, Theorem 8.7, p. 149] that geodesic balls B_x of sufficiently small radius ρ have two additional useful properties: (a) B_x is geodesically convex (i.e., any two points of B_x can be connected by a geodesic entirely contained in B_x) and (b) B_x is a normal neighborhood not only of x but of any of its points. Whenever this is the case, B_x will be called a *convex geodesic ball*.

Theorem 8 *Let M and M' be connected affine manifolds and let $f : M \longrightarrow M'$ be an affine map. Suppose that M is simply connected and geodesically complete and that f is a local diffeomorphism. Then f is a covering (in particular, it is surjective), establishing M as the universal covering manifold of M', and M' is also geodesically complete.*

Remark 4 The "riemannian version" of this theorem (which assumes that M and M' are riemannian manifolds and f is isometric) is well known and can be found in many textbooks, but the proofs given usually make use of the Hopf-Rinow theorem and therefore do not extend to the present situation, where we do not have metrics (in the topological sense). An alternative approach can be found in [22, Chapter 10, Theorem 18, p. 167], and the proof presented below is an adaptation of that to the affine case.

Proof: We begin by showing that f is onto. Considering that M' is connected and f is a local diffeomorphism, so that its image $f(M)$ is necessarily an open submanifold of M', it suffices to show that $f(M)$ is also closed. Thus let $x' \in M'$ be a point in the closure of $f(M)$ and let B' be a convex geodesic ball in M' around x'. Then there exist a point $y' \in B' \cap f(M)$ and, due to the fact that B' is a normal neighborhood of y' as well, a tangent vector $u' \in T_{y'} M'$ such that $\exp_{y'}(u') = x'$. Choose $y \in M$ such that $f(y) = y'$ and, using that f is local diffeomorphism, $u \in T_y M$ such that $T_y f \cdot u = u'$. Set $x = \exp_y(u)$. Then since f is affine, we have $f(x) = x'$. The argument also shows that M' is geodesically complete. Finally, to show that f is a covering, note that the inverse image $f^{-1}(x')$ of a point $x' \in M'$ under the local diffeomorphism f is a discrete subset of M, and we can always choose a scalar product on $T_{x'} M'$ and, for every $x \in f^{-1}(x')$, a scalar product on $T_x M$ such that $T_x f : T_x M \longrightarrow T_{x'} M'$ is isometric; then the inverse

image under f of a convex geodesic ball around x', of sufficiently small radius, will be the disjoint union, parametrized by $x \in f^{-1}(x')$, of the convex geodesic balls around x, of the same radius. □

References

[1] R. Abraham & J.E. Marsden, *Foundations of Mechanics*, 2nd edition, Benjamin-Cummings, Reading 1978.

[2] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz & D. Sternheimer, *Quantum Mechanics as a Deformation of Classical Mechanics*, Lett. Math. Phys. **1** (1977) 521-530; *Deformation Theory and Quantization*, Part I, Ann. Phys. **111** (1978) 61-110.

[3] P. Bieliavsky, M. Cahen, S. Gutt, J. Rawnsley & L. Schwachhofer, *Symplectic Connections*, Int. J. Geom. Meth. Mod. Phys. **3** (2006) 375-420.

[4] J.F. Cariñena, M. Crampin & L.A. Ibort: *On the Multisymplectic Formalism for First Order Field Theories*, Diff. Geom. Appl. **1** (1991) 345-374.

[5] M. De Wilde & P.B.A. Lecomte, *Star-Products on Cotangent Bundles*, Lett. Math. Phys. **7** (1983) 235-241.

[6] M. De Wilde & P.B.A. Lecomte, *Existence of Star-Products on Exact Symplectic Manifolds*, Ann. Inst. Four. **35** (1985) 117-143.

[7] M. De Wilde & P.B.A. Lecomte, *Existence of Star-Products and of Formal Deformations of the Poisson Lie Algebra of Arbitrary Symplectic Manifolds*, Lett. Math. Phys. **7** (1983) 487-496.

[8] J. Dittmann & G. Rudolph, *Canonical Realizations of Lie Algebras Associated with Foliated Coadjoint Orbits*, Ann. Inst. Henri Poincaré **43** (1985) 251-267.

[9] F. Etayo & R. Santamaria, *The Canonical Connection of a Bi-Lagrangian Manifold*, J. Phys. A: Math. Gen. **34** (2001) 981-987.

[10] F. Etayo, R. Santamaria & U.R. Tras, *The Geometry of a Bi-Lagrangian Manifold*, Diff. Geom. Appl. **24** (2006) 33-59.

[11] B.V. Fedosov, *A Simple Geometric Construction of Deformation Quantization*, J. Diff. Geom. **40** (1994) 213-238.

[12] M. Forger & L. Gomes, *Multisymplectic and Polysymplectic Structures on Fiber Bundles*, arXiv:0708.1586.

[13] M. Forger & S. Romero, *Covariant Poisson Brackets in Geometric Field Theory*, Commun. Math. Phys. **256** (2005) 375-410.

[14] I. Gelfand, V. Retakh & M. Shubin, *Fedosov Manifolds*, Adv. Math. **136** (1998) 104-140.

[15] W.H. Greub, S. Halperin & R. Vanstone, *Connections, Curvature and Cohomology*, Vol. 2, Academic Press, New York 1973.

[16] M.J. Gotay, *A Multisymplectic Framework for Classical Field Theory and the Calculus of Variations I. Covariant Hamiltonian Formalism*, in: M. Francaviglia (ed.), Mechanics, Analysis and Geometry: 200 Years After Lagrange, North Holland, Amsterdam 1991, pp. 203-235.

[17] M.J. Gotay, J. Isenberg, J.E. Marsden & R. Montgomery, *Momentum Maps and Classical Relativistic Fields. Part I: Covariant Field Theory*, arXiv:physics/9801019.

[18] V. Guillemin & S. Sternberg, *Geometric Asymptotics*, Mathematical Surveys **14**, American Mathematical Society, Providence 1977.

[19] M. Hamermesh, *Group Theory*, Addison-Wesley, Reading 1962.

[20] H. Heß, *Connections on Symplectic Manifolds and Geometric Quantization*, in: Differential Geometric Methods in Theoretical Physics (Proceedings of the Conferences held at Aix-en-Provence, France, and Salamanca, Spain, 1979), pp. 153-166, eds: P.L. García, A. Pérez-Rendón & J.M. Souriau, Lecture Notes in Mathematics **836**, Springer, Berlin 1980.

[21] H. Heß, *Symplectic Connections in Geometric Quantization and Factor Orderings*, PhD Thesis, Freie Universität Berlin, Berlin 1981.

[22] N.J. Hicks, *Notes on Differential Geometry*, Nostrand, Toronto 1965.

[23] S. Kobayashi & K. Nomizu, *Foundations of Differential Geometry*, Vol. 1, Interscience, New York 1963.

[24] M. Kontsevich, *Operads and Motives in Deformation Quantization*, Lett. Math. Phys. **48** (1999) 35-72.

[25] I. Moerdijk & M. Mrčun, *Introduction to Foliations and Lie Groupoids*, Cambridge University Press, Cambridge 2003.

[26] G. Thompson, *Symplectic Manifolds with Lagrangian Fibration*, Lett. Math. Phys. **12** (1986) 241-248.

[27] I. Vaisman, *Basics of Lagrangian Foliations*, Publicacions Matemàtiques **33** (1989) 559-575.

[28] J. Vey, *Crochet de Poisson sur une Variété Symplectique*, Comment. Math. Helv. **50** (1975) 421-454.

[29] A. Weinstein, *Symplectic Manifolds and Their Lagrangian Submanifolds*, Adv. Math. **6** (1971) 329-346.

[30] N. Woodhouse, *Geometric Quantization*, Oxford University Press, Oxford 1991.

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RT-MAP-1201 – Michael Forger and Daniel V. Paulino
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