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GLOBAL PROPERTIES OF A CLASS OF VECTOR FIELDS IN THE PLANE

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Introduction

The present work is concerned with global properties of a class of smooth complex vector fields over the plane. One of the main questions we discuss here is the existence of global solutions to the problem

$$(0.1) \qquad Lu=0, \; du\neq 0 \; \textit{over} \; \mathbb{R}^2;$$

where L is a vector field with no singularities. Even when L has real-valued coefficients it was known for a long time . [W] that problem (0.1) does not admit, in general, solutions. On the other hand it is relatively easy to show, by means of the Poincaré - Bendixson theorem, that the corresponding semi-global problem

(0.2) *Given* $U \subset\subset \mathbb{R}^2$ *find a solution of* $Lu=0$, $du\neq 0$ *over* U

has always solutions when L is real (see, [K] for the original proof of this result).

When L has complex-valued coefficients the situation is much more complicated. We say that L is *locally integrable* if it is possible to cover the plane by open subsets V over each of which we can solve $Lu=0$ with $du\neq 0$. Obvious examples of locally integrable vector fields are the real ones and those with real-analytic coefficients; it is also important to mention that there are examples due to Nirenberg of vector fields that are not locally integrable [Ni].

A natural question that appears is whether local integrability implies the solvability of (0.2). The answer is again negative as it can be seen if one takes the vector field with polynomial coefficients

(0.3) $$L=(y^2-x^2+1+2ixy)\,\partial_x+[2xy+i(x^2-y^2+1)]\partial_y$$

It is easy to see that L is elliptic on $x^2+y^2<1$ and that given u defined in a connected open neighborhood of $x^2+y^2\leq 1$ it necessarily must be constant on $x^2+y^2=1$ and thus everywhere by the maximum principle.

In this work we restrict our study to operators L which are locally integrable and satisfy the following properties:

(0.4) *The set* Σ *where* L *and* $\bar{L}$ *are linearly dependent is non-empty and connected*

(0.5) L *and* $[L,\bar{L}]$ *are linearly independent on* Σ;

(in this conection see [Sj], [T1].)

These operators define our class $\mathcal{L}$ (see section 1) and in section 1 we prove that for operators in this class it is always possible to solve the equation $Lu=0$, $du\neq 0$ in a full neighborhood of Σ; we also prove that there are operators in $\mathcal{L}$ for which problem (0.2) fails to have a solution. We do not know if it is possible to solve problem (0.2) for an operator $L\in\mathcal{L}$ with real-analytic coefficients [notice that the operator L given in (0.3) does not belong to $\mathcal{L}$ for condition (0.5) fails to hold at the points (1,0),(-1,0),(0,1), (0,1)] ←

In section 2 we study the solvability of (0.1) for operators in $\mathcal{L}$ as well as the related question of global reduction (via a global diffeomorphism of $\mathbb{R}^2$) of them to the standard form $\partial_t - it\partial_x$ when $\Sigma \approx \mathbb{R}^1$ or M_δ [see(2.12)] when $\Sigma \approx S^1$. The key argument in our proofs is the use of the conformal structure defined by the operator L restricted to $\mathbb{R}^2 \setminus \Sigma$. As an additional illustration of this technique we show, in the appendix to section 2, that for a hypoelliptic vector field in the plane we can always find a solution to problem (0.1) that is injective.

Finally in section 3 we give descriptions of the ranges $M\mathcal{E}^\infty(\mathbb{R}^2)$, $M\mathcal{E}^\infty(\mathbb{R}^2) \cap \mathcal{E}_c^\infty(\mathbb{R}^2)$, $M_\delta\mathcal{E}^\infty(\mathbb{R}^2)$ and $M_\delta\mathcal{E}^\infty(\mathbb{R}^2) \cap \mathcal{E}_c^\infty(\mathbb{R}^2)$ where $M = \partial_t - it\partial x$ is the so called Mizohata operator and M_δ is its modification given in (2.12). One interesting feature is that, as a consequence of our results, we can prove that $M\mathcal{E}^\infty(\mathbb{R}^2) \cap \mathcal{E}_c^\infty(\mathbb{R}^2)$ is a closed subspace of $\mathcal{E}_c^\infty(\mathbb{R}^2)$ whereas $M_\delta\mathcal{E}^\infty(\mathbb{R}^2) \cap \mathcal{E}_c^\infty(\mathbb{R}^2)$ is not (see theorems 3.3, 3.8 and corollary 3.6).

§1. *The Class of Operators under Study*

In this section we introduce the class $\mathcal{L}$ of complex vector fields globally defined in $\mathbb{R}^2$ which we are going to study in this work.

First of all we assume that every L has no singularities, that is, $L(p)=0$, $p \in \mathbb{R}^2$. Let Σ be the subset of the plane consisting of all points p such that $L(p)$ and $\overline{L(p)}$ are linearly dependent. Our next assumption is that

(1.1) $\qquad p \in \Sigma \Rightarrow L(p)$ *and* $[L,\bar{L}](p)$ *are linearly independent.*

This condition implies that Σ is a closed, smooth (embedded) submanifold of dimension 1. We will also require that Σ is connected; thus Σ is diffeomorphic either to $\mathbb{R}^1$ or to the unit circle S^1.

Our last requeriment on L is that " local integrability" holds. This means the following: Given any point $p \in \mathbb{R}^2$ we can find and open neighborhood U of p and a smooth function $Z : U \to \mathbb{C}$ satisfying

(1.2) $$LZ=0, \ dZ \neq 0 \ \textit{over} \ U.$$

At this point we can recall a result of F.Treves [T1] which states that local integrability is equivalent to the existence of an open covering of $\mathbb{R}^2$ by coordinate charts (U,x,t) over each of which L is a multiple of the *Mizohata operator*

$$M = \partial_t - it\partial_x . \tag{1.3}$$

Definition 1.1: *Let* $\Omega \subset \mathbb{R}^2$ *be an open set. A smooth complex vector field* L *is said to be integrable over* Ω *if there is a smooth function* $Z:\Omega \to \mathbb{C}$ *such that* $LZ=0$, $dZ \neq 0$ *over* Ω. *If* L *is integrable over* $\mathbb{R}^2$ *we will say that* L *is globally integrable.*

For instance, the Mizohata operator is globally integrable, for the function

$$W(x,t) = x + it^2/2 \tag{1.4}$$

obviously satisfies $MW=0$, $dW \neq 0$ over $\mathbb{R}^2$.

We now show that Treves' result quoted before can be globalized to a tubular neighborhood of Σ :

Theorem 1.2: *Let* $L \in \mathcal{L}$. *Then we can find a tubular neighborhood* U *of* Σ *and a coordinate system* (x,t) *over* U *such that:*

(i) Σ *is defined by* $t=0$;

(ii) $L|U$ *is a multiple of the Mizohata operator* (1.3). *In particular* L *is integrable over* U.

Proof: First suppose that $\Sigma = \mathbb{R}^1$. By [1] we can cover Σ by open neighborhoods U_j, $j \in \mathbb{Z}$ such that

(i) $U_j \cap U_{j+2} = \phi$

(ii) $U_j \cap U_{j+1}$ is connected,

(iii) There are coordinate functions $(x_j,t_j): U_j \to]-1,1[\times]-1,1[$

such that

$$L \mid U_j = g_j(\partial_{t_j} - it_j\partial_x)$$

with $g_j \neq 0$ on U_j.

We let

$$U = \bigcup_{j\in\mathbb{Z}} U_j$$

and define an involution $\phi : U \to U$ as follows: if $p \in U_j$ we set $\phi(p)$ in such way that $x_j(p) = x_j(\phi(p))$, $t_j(p) = -t_j(\phi(p))$. In order to verify that is well defined we need the following lemma:

<u>*Lemma 1.3:*</u> *Let* $\Omega \subset]-a,a[\times]-b,b[$ *be connected and suppose that* $f: \Omega \to \mathbb{C}$ *is a continuous function satisfying* $Mf=0$ (see(1.3)). *Then there exists a holomorphic function* F *on* $\mathcal{O} = \{x+it^2/2 : (x,t) \in \Omega, t\neq 0\}$ *which is continuous on* $\mathcal{O} \cup \{(x,0) \in \Omega\}$ *and such that* $f(x,t) = F(x+it^2/2)$.

<u>*Proof:*</u> For $x+iy \in \mathcal{O}$ we set

$$F(x+iy) = f(x,\sqrt{2y}).$$

By the chain rule it follows easily that F is holomorphic in $\mathcal{O}$ and continuous in $\mathcal{O} \cup \{(x,0) \in \Omega\}$. Furthermore $x+iy \to f(x,-\sqrt{2y})$ is also holomorphic in $\mathcal{O}$, continuous in $\mathcal{O} \cup \{(x,0) \in \Omega\}$ and agrees with F in $\{(x,0) \in \Omega\}$. Thus it coincides with F everywhere and the lemma is proved.

We continue with the proof of theorem 1.2 By lemma 1.3 we have

$$x_{j+1}(p)+it_{j+1}(p)^2/2= F(x_j(p)+it_j(p)^2/2).$$

$p\in U_j\cap U_{j+1}$, where F is a holomorphic function on $\mho_j=\{x_j(p)+it_j(p)^2/2:\ p\in U_j\cap U_{j+1},\ t_j(p)\neq 0\}$ and continuous on $\mho_j\cup\{(x_j(p),0):\ p\in U_j\cap U_{j+1}\}$. Hence $p,q\in U_j\cap U_{j+1}$, $p\neq q$, $x_j(p)=x_j(q)$, $t_j(p)=-t_j(q)$ imply $x_{j+1}(p)=x_{j+1}(q)$, $t_{j+1}(p)=-t_{j+1}(q)$. This proves that ϕ is well-defined.

It follows from the definition of ϕ that ϕ_*L is a multiple of L. Furthermore Σ divides U in two components U_1 and U_2 in such a way that ϕ interchanges U_1 and U_2 and is the identity on Σ.

We now make the assumption that

(iv) U_1 *is simply connected.*

Since L is elliptic on U_1 it defines a structure of Riemann surface over the latter and then, by uniformization, there is a biholomorphism $Z_1: U_1\to H$; here H is the half-plane in $\mathbb{C}$ defined by $\mathrm{Im}\,\zeta>0$ with the canonical complex structure and U_1 has the complex structure just introduced. Obviously

$$LZ_1=0,\quad dZ_1\neq 0\quad \text{on } U_1$$

and standard arguments about the regularity up to the boundary of the Riemann mapping (see, for instance, [N]) show that Z_1 extends, in a C^∞ manner, as a map $U_1\cup\Sigma\to H\cup\mathbb{R}^1$. Furthermore, the same argument shows that the trace $Z_1|\Sigma$ has non-vanishing derivative. Hence the function

$$Z: U \longrightarrow \mathbb{C}$$

defined by $Z(p)=Z_1(p)$, $p\in U_1\cup\Sigma$, $Z(p)=Z_1(\phi(p))$, $p\in U_2$ is $\mathcal{C}^\infty$ and satisfies

$$LZ=0,\ dZ\neq 0 \ \text{ on } \ U.$$

We now set

$$x(p)=\operatorname{Re}Z(p);\ t(p)=\begin{cases}(2\operatorname{Im}Z(p))^{1/2}, & p\in U_1\cap\Sigma\\ -(2\operatorname{Im}Z(p))^{1/2}, & p\in U_2.\end{cases}$$

We claim that (x,t) is a diffeomorphism satisfying the required properties. First of all it is clear that $p\longrightarrow(x(p),t(p))$ is one-to-one; also it is easy to see that dx and dt are linearly independent outside Σ. Next, in a local chart U_j, we have

$$\partial_{t_j}Z-it_j\partial_x Z=0.$$

Consequently $\operatorname{Im}z(x_j,0)=0$, $\partial_{t_j}\operatorname{Im}Z(x_j,0)\equiv 0$. Moreover $\partial^2_{t_j}\operatorname{Im}Z(x_j,0)$ never vanishes for $Z_1|\Sigma$ has non-vanishing derivative. Thus, in a neighborhood of Σ in U_j

$$\operatorname{Im}Z(x_j,t_j)=t_j^2\,h(x_j,t_j)$$

with $h>0$. This shows that $p\longrightarrow t(p)$ is a $\mathcal{C}^\infty$ map and that dx and dt are linearly independent on Σ, which concludes the proof that $p \longrightarrow (x(p),t(p))$ is a diffeomorphism. Finally, the push-forward of L by this diffeomorphism is equal to

$$L(x)\partial_x + L(t)\partial_t = L(t)\,(\partial_t - it\partial_x)$$

since $LZ = L(x+it^2/2) = 0$.

The proof in the case $\Sigma \cong S^1$ is analogous. The only difference is that we cannot take U_1 simply-connected. But in this case we can assume $\Pi_1(U_1) \cong \mathbb{Z}$ which implies that U_1 is conformally equivalent to an annulus $\{\sigma < |\zeta| < 1\}$ where $\{|\zeta|=1\}$ corresponds to Σ. With this remark the proof becomes the same as the one just described. Q.E.D.

It follows from the proof that the neighborhood U can be taken in such a way so as to contain one simply-connected component of $\mathbb{R}^2 \setminus \Sigma$.

Next we show that we cannot achieve a more global result than the one we have obtained. Let (x,t) denote the coordinates in $\mathbb{R}^2$ and consider the vector field

$$L = \partial_t - i(t+\psi(t,x))\partial_x\,, \tag{1.5}$$

where the function ψ will be described in what follows.

Let $\{D_j\}$ be a sequence of closed discs contained in the half-plane

$$H^+ = \{(x,t) : t>0\} \tag{1.6}$$

converging to the point $(0,a)$, where $a>0$. We suppose that the discs D_j are pairwise disjoint and then take $\psi \in \mathcal{C}_c^\infty(H^+)$, $\psi\geq 0$, and $\operatorname{supp}\psi \subset \bigcup_j D_j$, $\psi > 0$ on $\overset{\circ}{D}_j$.

We observe that the operator L given in (1.5) belongs to our class . Notice that, in this case, Σ is the line $t=0$ and that local integrability follows outside Σ by ellipticity and on Σ because L is the Mizohata operator in a neighborhood of Σ.

Theorem 1.4: Let Ω be an open, connected neighborhood of the origin in $\mathbb{R}^2$, symmetric with respect to Σ and containing the point $(0,a)$ If $Z : \Omega \longrightarrow \mathbb{C}$ is a $\mathcal{C}^1$ function satisfying

$$LZ=0 \quad \text{on} \quad \Omega \tag{1.7}$$

it follows that $dZ=0$ *at* $(0,a)$.

Proof: Let $\Omega^- =\{(x,t)\ \Omega: t<0\}$. We can find a holomorphic function f on $\{x+it^2/2: (x,t) \in \bar{\Omega}\}$, continuous up to $t=0$ such that

$$Z(x,t)= f(x+it^2/2) \tag{1.8}$$

over Ω^- [see the proof of lemma 1.2]. If we assume that all discs D_j are contained in Ω it follows, by analytic continuation, that (1.8) holds on $\Omega\setminus\bigcup_j D_j$.

Hence, for each j,

$$0= \int_{\partial D_j} f(x+it^2/2)\ d(x+it^2/2)=$$

$$= \int_{\partial D_j} Z(x,t)\, d(x+it^2/2) = \iint_{D_j} MZ(x,t)\, dxdt =$$

$$= i \iint_{D_j} \psi(x,t) \frac{\partial Z}{\partial x}(x,t)\, dxdt$$

by Green's theorem.

Thus, for each j, there exist $p_j, q_j \in D_j$ such that

$\mathrm{Re}\,\frac{\partial Z}{\partial x}(p_j) = \mathrm{Im}\,\frac{\partial Z}{\partial x}(q_j) = 0;$

This implies $\frac{\partial Z}{\partial x}(0,a)=0$ since $p_j,\ q_j \to (0,a)$.

But $LZ=0$ implies that $\frac{\partial Z}{\partial t}$ also vanishes at $(0,a)$.

The proof is complete.

Reasoning like in [Ni] we can construct another function ψ such that the operator L defined in (1.5)belongs to $\mathcal{L}$ and has the following property:

(1.9) *There is an open, connected neighborhood* Ω *of the origin on* $\mathbb{R}^2$, *symmetric with respect to* Σ *such that given any* $\mathcal{C}^1$ *function* $Z: \Omega \to \mathbb{C}$ *satisfying* (1.7) *it follows that* Z *is constant.*

The details are left to the reader.

§2. *Global Integrability; Global Reduction To The Mizohata Operator*

In section 1 we have proved that for an operator L in the class $\mathcal{L}$ we can, in a full neighborhood of its characteristic set, reduce it to the Mizohata operator. In this section we investigate conditions on L in order to conclude that this kind of reduction in valid in the whole $\mathbb{R}^2$. At the same time we study the problem of global integrability for L.

An operator $L \in \mathcal{L}$ will be said to be of *type I* if $\Sigma \cong \mathbb{R}^1$ In this case $\mathbb{R}^2 \setminus \Sigma$ has two connected components Ω_1, Ω_2, each of them simply-connected. Since L is elliptic on $\mathbb{R}^2 \setminus \Sigma$ it defines, over the latter, a structure of Riemmann surface [this argument has already appeared in the proof of theorem 1.2] and consequently, by uniformization, there are biholomorphisms $Z_j: \Omega_j \to H$, $j=1,2$ (it is easy to conclude that Ω_j cannot be conformally equivalent to $\mathbb{C}$). Here H is as in the proof of theorem 1.2. The functions Z_j are $\mathcal{C}^\infty$ up to Σ and the traces $\phi_j = Z_j|\Sigma$, $j=1,2$, have non-vanishing derivatives.

We select two distinct points $\bar{p}, \bar{q} \in \Sigma$ and suppose that

$$\phi_j(\bar{p}) = 0, \ \phi_j(\bar{q}) = 1, \ \phi_j(\infty) = \infty, \ j=1,2. \tag{2.1}$$

Notice that

$$\phi_j(\Sigma) =]-b_j, \infty[\ \text{where} \ b_j \in [-\infty, 0[. \tag{2.2}$$

Theorem 2.1 L *is globally integrable if and only if there are* *holomorphic functions* F_j *on* H, $j=1,2$, *such that*

(i) F_j *is* $\mathcal{C}^\infty$ *up to the region of the real axis defined by* $b_j < x < \infty$, $j=1,2$;

(ii) F_j *has non-vanishing derivative (even at the boundary points),* $j=1,2$;

(iii) $F_1 \circ \phi_1 = F_2 \circ \phi_2$ *over* Σ.

Proof: If there is $Z \in \mathcal{C}^\infty(\mathbb{R}^2)$ satisfying $LZ=0$, $dZ \neq 0$ then $F_j = Z \circ Z_j^{-1}$, $j=1,2$ satisfy (i), (ii) and (iii). For the converse it suffices to notice that

$$Z = \begin{cases} F_j \circ Z_j & \text{on } \Omega_j,\ j=1,2 \\ F_1 \circ \phi_1 & \text{on } \Sigma \end{cases}$$

is a $\mathcal{C}^\infty$ map which satisfies $LZ=0$, $dZ \neq 0$ on $\mathbb{R}^2$.

Theorem 2.2 *The two following properties are equivalent:*

(2.3) $b_1 = b_2 = -\infty$, $\phi_1 = \phi_2$ *over* Σ;

(2.4) *There is a global diffeomorphism* $\chi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ *sending* Σ *onto* $\{(X,T): T=0\}$ *and such that* $\chi_* L$ *is a multiple of the Mizohata operator* (1.3) *in the coordinates* (X,T).

Proof: First assume the validity of (2.4) *and consider*

$$Z = W \circ \chi,\ W(X,T) = X + iT^2/2$$

(see(1.4)), which obviously satisfies $LZ=0$, $dZ\neq 0$ on $\mathbb{R}^2$. Furthemore $Z|\Omega_j$ is a biholomorphism from Ω_j onto H with respect to the complex structures induced by L and $\bar{\partial}$ respectively. Then

$$Z_j(p)= aZ(p)+b, \quad p\in\Omega_j$$

where $a= [\bar{Z}(q)-\bar{Z}(p)]^{-1}$, $b= -\bar{Z}(p).a$, which shows that (2.3) holds.

Conversely, let us assume the validity of (2.3) and introduce $Z: \mathbb{R}^2\longrightarrow \mathbb{C}$ by the formula

$$(2.4) \qquad Z(p)=\begin{cases} Z_j(p) & p\in\Omega_j \\ \phi_1(p) & p\in\Sigma \end{cases}.$$

Condition (2.3) implies the continuity of Z. But $LZ=0$ in $\mathbb{R}^2$ and then Z is, in fact, a C^∞ map.

Proceeding as in the proof of Theorem 1.2 one can show that

(2.5) $\mathrm{Im}Z$ *vanishes precisely to second order at* Σ.

This property will then imply that the map

$$(2.6) \qquad \chi(p)=(X(p),\ T(p)),\ \begin{cases} X(p)= \mathrm{Re}Z(p) \\ T(p)= \pm(2\mathrm{Im}Z(p))^{1/2} \end{cases}.$$

is a global diffeomorphism from $\mathbb{R}^2$ onto itself (here the "+" or "−" sign is taken according to wether p belongs to Ω_1 or Ω_2 respectively). Finally, since

$$Z = X + iT^2/2$$

we have

$$L = (LT)\,(\partial_T - iT\partial_X).$$

The proof is complete.

Next we give examples of operators of type I that are globally integrable but cannot be globally reduced to the Mizohata operator.

Our first example is the operator

$$L = \partial_t - ite^t\partial_x \tag{2.7}$$

which is globally integrable since the function

$$Z(x,t) = x + i(t-1)e^t \tag{2.8}$$

has non-vanishing differential and satisfies $LZ=0$. With the notation introduced at the begining of this section we choose

$$\Omega_1 = H^+ = \{(x,t): t > 0\}$$

$$\Omega_2 = H^- = \{(x,t): t < 0\},$$

and $\bar{p} = (0,0)$, $\bar{q} = (1,0)$.

Then

$$Z_1(x,t) = x + i\,[(t-1)e^t + 1]$$

$$Z_2(x,t)= \frac{1}{e^{\pi}-1}\{e^{\pi[x+i((t-1)e^t+1]}-1\}.$$

Notice that $b_1=-\infty$, $b_2=-(e^{\pi}-1)^{-1}$ and thus property (2.3) does not hold true in this case. We also remark that, with the notation of theorem 2.1, we have

$$F_1(z)= z, \quad F_2(z)= \frac{1}{\pi}\log(e^{\pi}-1)z+1]$$

and that this last function is $\mathcal{C}^{\infty}$ up to the real axis only in the region $x > b_2$.

Next we set

$$L= \partial_t-ite^{-t^2}\partial x \tag{2.9}$$

The function

$$Z(x,t)= x+i(1-e^{-t^2})/2 \tag{2.10}$$

shows that this vector field is also globally integrable. With the same notation as before we have

$$Z_1(x,t)= Z_2(x,t)= \frac{1}{e^{2\pi}-1}\{e^{2\pi[x+i(1-e^{-t^2})/2]}-1\} ;$$

in the case $\phi_1= \phi_2$ and $b_1=b_2=-(e^{2\pi}-1)^{-1}$ and thus the vector field given in (2.9) also cannot be globally reduced to the Mizohata operator.

An operator L will be called of *typeII* if its characteristi set Σ is diffeomorphic to the unit . cle. In this situation $\mathbb{R}^2 \setminus \Sigma$ has two connected components, one of them, Ω_1, bounded and simply-connected and the other, Ω_2, unbounded and with fundamental group isomorphic to the group of the integers. As before, by uniformization, we can determine two maps: The first one is a biholomorphism $Z_1:\Omega_1 \longrightarrow D$, where D is the unit disc in $\mathbb{C}$, with respect to the complex structures induced by L and $\mathbb{C}$ respectively; The second one is a biholomorphism $Z_2:\Omega_2 \longrightarrow A(\sigma)$, where $A(\sigma)$ is the annulus $\{ \sigma<|z| < 1 \}$ and $\sigma \in [0,1[$ is intrinsically associated to L ([S]). We will denote such a number by $\sigma(L)$. In this case the maps Z_j are $\mathcal{C}^\infty$ up to Σ and the traces $\phi_j = Z_j|\Sigma$ map Σ diffeomorphically onto S^1.

We select three points p_1,p_2,p_3 in Σ and suppose that

$$\phi_j(p_1)= 1, \quad \phi_j(p_2)= i, \quad \phi_j(p_3)= -1, \quad j= 1,2.$$

For global integrability the result is similar to the one stated in theorem 2.1 for operators of type I.

<u>*Theorem 2.3*</u> *Let* $L \in \mathcal{L}$ *be of type II. Then* L *is globally integrable if and only if there are holomorphic functions* $F_1: D \longrightarrow \mathbb{C}$, $F_2: A(\sigma(L)) \longrightarrow \mathbb{C}$ *satisfying:*

(i) F_j *are smooth up to* S^1;

(ii) F_j *have non-vanishing derivatives* even at S^1;

(iii) $F_1 \circ \phi_1 = F_2 \circ \phi_2$ *over* Σ.

The proof, being similar to that of theorem 2.1, will be ommited.

We now present an example of an operator of type II. Let $\delta \in [0,1[$ and let

$$g_\delta :]0,\infty[\longrightarrow \mathbb{R}^1$$

be a smooth function satisfying the following properties:

$$(2.11)\qquad \begin{cases} g_\delta(r) = r^{-1} , & r \in]0,1/3[, \\ g_\delta(r) = 1-r ; & r \in]2/3,4/3[, \\ g_\delta(r) = 0 \text{ if and only if } r=1, & \\ \int_1^\infty g_\delta(s)ds = \log\delta . & \end{cases}$$

We define

$$(2.12)\qquad M_\delta = \frac{1}{2}\cdot e^{i\theta}[\partial_r + i g_\delta(r)\partial_\theta]$$

where (r,θ) are polar coordinates in the plane. The presence of the factor $e^{i\theta}/2$ is due to the fact that,with this definition, M_δ is the Cauchy Riemann operator when $r<1/3$; in particular M_δ is well defined at the origin.

If we write

$$(2.13)\qquad Z_\delta(r,\theta) = e^{i[\theta - i \int_1^r g_\delta(s)ds]}$$

it follows immediately that Z_δ defines a smooth function over $\mathbb{R}^2$ and that

$$M_\delta Z_\delta = 0, \quad dZ_\delta \neq 0 \quad \text{over} \quad \mathbb{R}^2.$$

Thus M_δ is globally integrable. [notice that Z_δ is the identity over S^1]

We leave to the reader the verification that M_δ is operator of type II with $\Sigma = S^1$ and $\sigma(M_\delta) = \delta$.

<u>*Theorem 2.4*</u> *For an operator* L *of type II the following statements are equivalent:*

(2.14) $\phi_1 = \phi_2$ *over* Σ;

(2.15) *There is a global diffeomorphism* $\chi: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *such that* $\chi(\Omega_1) = D, \chi(\Omega_2) = \mathbb{C} \setminus \bar{D}$, $\chi(\Sigma) = S^1$ *and such that* $\chi_* L$ *is a multiple of* $M_{\sigma(L)}$.

<u>*Proof:*</u> Let us first prove that (2.15) implies (2.14).

The mapping $Z_1 \circ (Z_{\sigma(L)} \circ \chi)^{-1}$ is an automorphism of D [with the standard complex structure]. Similarly $Z_2 \circ (Z_{\sigma(L)} \circ \chi)^{-1}$ is an automorphism of $A(\sigma(L))$ and hence both these maps are linear fractional transformations, which we call T_1 and T_2 respectively. Then $\phi_j = T_j \circ \chi$ on Σ, $j=1,2$ which implies $T_1(\chi(p_k)) = T_2(\chi(p_k))$, $k = 1,2,3$. Thus $T_1 = T_2$ and consequently (2.14)holds.

We will now prove that (2.14)implies (2.15)

As in the proof of theorem 2.1 the map

$$Z(p)=\begin{cases} Z_j(p), & p\in\Omega_j,\ j=1,2 \\ \phi_1(p), & p\in\dot{\Sigma} \end{cases} \tag{2.16}$$

is of class $\mathcal{C}^\infty$, satisfies $LZ=0$ and has non-vanishing differential. Writing

$$Z(p)= R(p)e^{i\theta(p)}, \quad p\in\mathbb{R}^2 \tag{2.17}$$

and making use of the coordinates (x,t) provided by theorem 1.2 one can derive, from the relation $MZ=0$, the following identities

$$R= 1, \quad R_t= 0, \quad R_{tt}= -\theta_x \quad \text{over } \Sigma. \tag{2.18}$$

Next we consider the transformation

$$f(\eta)= e^{\int_1^\eta g_{\sigma(L)}(s)ds}, \quad \eta >0. \tag{2.19}$$

This map defines, by restriction, two diffeomorphisms

$$f_1: [0,1]\simeq [0,1], \quad f_2:[1,\infty[\simeq]\sigma(L),1].$$

We are now ready to define the diffeomorphism χ. Let

$$\chi(p)= (r(p)\cos\theta(p),\ r(p)\sin\theta(p)), \quad p\in\mathbb{R}^2$$

where θ is as in (2.17) and

$$r(p)=\begin{cases} f_1^{-1}(R(p)), & p\in\Omega_1 \\ f_2^{-1}(R(p)), & p\in\Omega_2. \end{cases}$$

Using (2.9) we get

$$r(p)=\begin{cases} 1-\ (-2\log R(p))^{1/2} & p\in \Omega_1\cap V \\ 1+\ (-2\log R(p))^{1/2} & p\in \Omega_2\cap V, \end{cases}$$

where V is a tubular neighborhood of Σ and then (2.18) shows that r is $\mathcal{C}^\infty$ near Σ. Also $r=R$ in a neighborhood of $Z^{-1}\{0\}$. It is now easy to conclude that χ is a diffeomorphism with the required properties.

We now present a result for the operator M_δ which is the analogue of the one stated in lemma 1.3 for the Mizohata operator:

Lemma 2.5 *Let* $\beta\in]0,\infty]$ *and let* Ω *be the ball in* $\mathbb{R}^2$ *defined by* $r<1+\beta$. *If* u *is a continuous function on* Ω *satisfying* $M_\delta u=0$ *then there exists a continuous function* F *on* $\bar{D}$, *holomorphic in* D *such that* $u=F\circ Z_\delta$ *over* Ω.

The proof will be left to the reader.

Corollary 2.5 *Let* L *be an operator of type II satisfying* (2.14) *and let* $u\in\mathcal{C}^0(\mathbb{R}^2)$ *be a solution of* $Lu=0$. *Then* u *is bounded.*

Proof: Combine Theorem 2.4 and Lemma 2.5.

Appendix to section 2: Hypoelliptic vector fields. —

In most of the proofs of section 2 we have used, in an essential way, the classification of simply-connected Riemann surfaces. As an additional illustration of this approch we will show in this appendix that for a hypoelliptic vector field L one has, in a simply-connected domain Ω, a global solution to the problem

(A.1) $LZ=0$, $dZ\neq 0$, Z *one-to-one in* Ω.

We recall that a vector field L defined in Ω is called *hypoelliptic* if $\operatorname{singsupp} u = \operatorname{singsupp} Lu$ for any $u\in D'(\Omega)$; notice that the class of hypoelliptic vector fields over $\mathbb{R}^2$ is disjoint from the class $\mathcal{L}$ introduced in section 1.

Proposition A.1 *Let* L *be a vector field over an open subset* $\Omega \subseteq \mathbb{R}^2$. *The following statements are equivalent:*

(i) L *is hypoelliptic*

(ii) *There exist an open covering* $\{U\}$ *of* Ω *and for each* U *a* $\mathcal{C}^\infty$ *map* $Z_U\colon U\to\mathbb{C}$ *such that* $LZ_U=0$, $dZ_U\neq 0$; Z_U *one-to-one in* U.

We sketch the proof. If L is hypoelliptic then it satisfies condition (P) (see [H] and [T_2]) which implies, due to theorem 3.2 in [T3], the existence of a system $\{Z_U\colon U\to\mathbb{C}\}$ of $\mathcal{C}^\infty$ maps satisfying $LZ_U=0$, $dZ_U\neq 0$ over U (see also proposition 2.1

in [T1]). Here $\{U\}$ forms an open covering of Ω. We can assume that each open set U is small enough in order to be the domain of a coordinate system (x,t) in which we can write

$$Z_U(x,t) = x + i\Phi_U(x,t),$$

where Φ_U is a $\mathcal{E}^\infty$ real-valued function. The hypoellipticity of L is then equivalent to the fact that

$$t \to \Phi_U(x,t)$$

is one-to-one for each fixed x([H],[T2]) which is also equivalent to the injectivity of Z_U.

We now apply this result. By a theorem of Baouendi - Treves [BT] the maps Z_U and Z_V are holomorphically related in an overlap $U \cap V$ and then the system $\{Z_U\}$ defines, over Ω a structure of Riemann surface $(\Omega, \mathfrak{M})$ whose Cauchy-Riemann bundle is generated by L. Let us point out that when L is not elliptic the identity map $(\Omega, \mathfrak{M}) \to \Omega$ is not smooth [here the latter has the $\mathcal{E}^\infty$ structure induced by $\mathbb{R}^2$]. In any case, when Ω is simply-connected we can, by uniformization, assert the existence of a smooth map $Z: \Omega \to \mathbb{C}$ satisfying (A.1).

§3. *The Global Range*

In this final section we study the global range for the Mizohata operator (1.3) and its generalization given in (2.12).

We start with the operator M given in (1.3). By standard arguments (see, for instance, [T3, p.23]) one can prove that for an arbitrary $f\in\mathcal{E}^\infty(\mathbb{R}^2)$ there exists $u\in\mathcal{E}^\infty(\mathbb{R}^2)$ such that $Mu-f$ vanishes to infinite order at $t=0$. Let us call $f_1=Mu-f$. The change of variables $t\to s=t^2/2$ converts the equation $Mv=f_1$ into the equation

$$(3.1)\qquad (\partial_x+i\partial_s)v= if_1(x,\sqrt{2s})/\sqrt{2s},\quad s>0.$$

Since the right-hand side of this equation can be extended as a $\mathcal{E}^\infty$ function on $\mathbb{R}^2$ we can find a $\mathcal{E}^\infty$ solution of (3.1) in the whole plane $\mathbb{R}^2$. Going back to the original variables (x,t) we reach the following conclusion

Proposition 3.1 *Given* $f\in\mathcal{E}^\infty(\mathbb{R}^2)$ *there exists* $u\in\mathcal{E}^\infty(\mathbb{R}^2)$ *such that* $Mu=f$ *on* H^+.

We are still using the notation

$$H^+=\{(x,t): t>0\},$$

$$H^-=\{(x,t): t<0\}$$

and we will denote by $\mathcal{H}$ the subspace of $\mathcal{E}^\infty(\overline{H^+})$ consisting of all functions that are holomorphic in the variable $x+it\in H^+$.

<u>Theorem 3.2</u> *The following statements are equivalent for a function* $f \in \mathcal{E}^\infty(\mathbb{R}^2)$:

(i) *There exists* $u \in \mathcal{D}'(\mathbb{R}^2)$ *such that* $Mu = f$;

(ii) *There exists* $u \in \mathcal{E}^\infty(\mathbb{R}^2)$ *such that* $Mu = f$;

(iii) *There exist* $u^+ \in \mathcal{E}^\infty(\overline{H}^+)$, $u^- \in \mathcal{E}^\infty(\overline{H}^-)$ *such that* $Mu^\pm = f|H^\pm$ *and such that* $u^+ - u^- = F$ *on* $t=0$ *for some* $F \in \mathcal{H}$;

(iv) *Given any pair* $u^+ \in \mathcal{E}^\infty(\overline{H}^+)$, $u^- \in \mathcal{E}^\infty(\overline{H}^-)$ *satisfying* $Mu^\pm = f|H^\pm$ *there exists* $F \in \mathcal{H}$ *such that* $u^+ - u^- = F$ *on* $t = 0$.

<u>*Proof:*</u> That (ii) implies (i) is trivial and that (iv) implies (iii) follows from proposition 3.1. Assume now that (iii) holds and define

$$u(x,t) = \begin{cases} u^+(x,t) & t \geq 0 \\ u^-(x,t) + F(x+it^2/2), & t < 0. \end{cases}$$

The verification that $u \in \mathcal{E}^\infty(\mathbb{R}^2)$ and that $Mu = f$ is immediate; thus (iii) implies (ii). It remains to prove that (i) implies (iv). Suppose that $Mu = f$ for some $u \in \mathcal{D}'(\mathbb{R}^2)$. Then u is smooth outside $t = 0$ and it is in fact a smmoth function of t valued in the space of distributions in x. We define

(3.2) $v(x,t) = u^+(x,t) - u^-(x,-t) - u(x,t) + u(x,-t)$, for $t > 0$.

Since $Mv = 0$ on H^+ we have $v(x,t) = F(x+it^2/2)$, where F is holomorphic; furthermore $\lim_{t\to 0} F(\cdot+it) = (u^+ - u^-)|_{t=0}$ in the weak sense. Then $F \in \mathcal{H}$ and (iv) follows. Q.E.D.

We can derive two immediate consequences of theorem 3.2. The first one is that $u \in \mathcal{D}'(\mathbb{R}^2)$, $Mu \in \mathcal{E}^\infty(\mathbb{R}^2)$ imply that $u-v \in \mathcal{E}^\infty(\mathbb{R}^2)$ for some $v \in \mathcal{D}'(\mathbb{R}^2)$ in the kernel of M; in other words the singularities of u are carried by the solutions of the homogeneous equation. The second one is a kind of " symmetric global hypoellipticity property": if $Mu \in \mathcal{D}'(\mathbb{R}^2)$ has even part in t of class $\mathcal{E}^\infty$, then the odd part in t of u is of class $\mathcal{E}^\infty$.

We have then obtained a characterization of the space $M\mathcal{E}^\infty(\mathbb{R}^2) \subsetneq \mathcal{E}^\infty(\mathbb{R}^2)$; for the space $M\,\mathcal{E}^\infty(\mathbb{R}^2) \cap \mathcal{E}^\infty_c(\mathbb{R}^2)$ a sharper result can be achieved:

<u>*Theorem 3.3*</u> *Let* $f \in \mathcal{E}^\infty_c(\mathbb{R}^2)$. The following properties are equivalent:

(3.3) *There exists* $u \in \mathcal{E}^\infty(\mathbb{R}^2)$ *such that* $Mu = f$;

(3.4) $\int (p \circ W) f = 0$ *for every polynomial* $p \in \mathbb{C}[X]$.

In particular $M\,\mathcal{E}^\infty(\mathbb{R}^2) \cap \mathcal{E}^\infty_c(\mathbb{R}^2)$ *is closed in* $\mathcal{E}^\infty_c(\mathbb{R}^2)$.

Here we are still denoting

$$W(x,t) = x + it^2/2.$$

<u>*Proof:*</u> We first show that (3.3) implies (3.4). Since $Mu=f$ implies $M[(p \circ W)u] = (p \circ W)\, f$ for every $p \in \mathbb{C}[X]$ it suffices to show that $\int f = 0$.

We assume that supp$f$ is contained in an open disc Ω

centered at the origin. We decompose $f = f_e + f_o$, $u = u_e + u_o$, where the subscripts "e" and "o" stand for even and odd parts in t respectively. Thus

$$Mu_o = f_e, \quad \operatorname{supp} f_e \subset \Omega$$

and hence

$$\int f = \int_{\Omega} f_e = 2\int_{\Omega \cap H^+} f_e = 2\int_{\Omega \cap H^+} Mu_o =$$

$$= \int_{\partial(\Omega \cap H^+)} u_o dW.$$

We claim that u_o vanishes on $\partial(\Omega \cap H^+)$, which implies what we want. In fact, u_o is identically zero on $\Omega \cap \{t=0\}$ simply because it is an odd function of t. Next Mu vanishes identically outside Ω which implies that u is a holomorphic function of W outside Ω. Since $u_o = 0$ for $t = 0$ this holomorphic function vanishes identically; in particular u vanishes on $\partial\Omega$.

Conversely, let us assume that (3.4) holds. With the notation

$$F(x,t,y,s,\xi) = e^{i[W(x,t)-W(y,s)]\xi} f(y,s)/2\pi,$$

we define

$$
(3.5)\quad \begin{cases} u^+(x,t)=\left\{\int_0^t\int_0^\infty\int_{-\infty}^\infty - \int_t^\infty\int_{-\infty}^0\int_{-\infty}^\infty\right\}F(x,t,y,s,\zeta)dyd\zeta ds, & t \geq 0 \\ u^-(x,t)=\left\{\int_t^0\int_0^\infty\int_{-\infty}^\infty + \int_{-\infty}^t\int_{-\infty}^0\int_{-\infty}^\infty\right\}F(x,t,y,s,\zeta)dyd\zeta ds, & t \leq 0. \end{cases}
$$

It is a straightforward verification that $u^+\in \mathcal{E}^\infty(\overline{H}^+)$, $u^-\in \mathcal{E}^\infty(\overline{H}^-)$, $Mu^\pm = f|H^\pm$. Furthemore

$$
(3.6)\qquad u^+(x,0)-u^-(x,0)=
$$

$$
\frac{e^{ix}}{2\pi}\int_{-\infty}^0 d\zeta \left[\int\int_{\mathbb{R}^2} e^{-i\zeta W} f\right].
$$

If we approximate $e^{-i\zeta W}$ by a sequence of polynomials in W uniformly over suppf and make use of hypothesis (3.4)it follows that $u^+(.,0)-u^-(.,0)$ vanishes identically; thus $f\in M\,\mathcal{E}^\infty(\mathbb{R}^2)$ by theorem 3.2. Q.E.D.

We now describe the global range of the operators M_δ defined in (2.12). In this case we will denote by $\mathcal{J}_\delta$ the space of all holomorphic functions in the annulus $A(\delta)$ [see section 2]that are smooth up to S^1.

<u>Theorem 3.4</u> *The following statements are equivalent for a function* $f\in \mathcal{E}^\infty(\mathbb{R}^2)$:

(i) *There exists* $u\in \mathcal{D}'(\mathbb{R}^2)$ *such that* $M_\delta u= f$;

(ii) *There exists* $u\in \mathcal{E}^\infty(\mathbb{R}^2)$ *such that* $M_\delta u= f$;

(iii) *There exist* $u_1 \in \mathcal{E}^\infty(\bar{D})$, $u_2 \in \mathcal{E}^\infty(\mathbb{R}^2 \setminus D)$ *and* $F \in \mathcal{H}_\delta$ *such that* $M_\delta u_1 = f|D$, $M_\delta u_2 = f|\mathbb{R}^2 \setminus \bar{D}$ *and such that* $u_1 - u_2 = F$ *on* S^1.

(iv) *Given any pair* $u_1 \in \mathcal{E}^\infty(\bar{D})$, $u_2 \in \mathcal{E}^\infty(\mathbb{R}^2 \setminus D)$ *such that* $M_\delta u_1 = f|D$, $M_\delta u_2 = f|\mathbb{R}^2 \setminus \bar{D}$ *there exists* $F \in \mathcal{H}_\delta$ *such that* $u_1 - u_2 = F$ *on* S^1.

The proof is similar to that of theorem 3.2 and will be ommited.

Finally we study the range of M_δ in $\mathcal{E}_c^\infty(\mathbb{R}^2)$. In order to do so, we shall need right inverses for M_δ over D and $R^2 \setminus \bar{D}$ which can be constructed by means of the Fourier series in the variable θ ; we sketch their construction now.

Let $f \in \mathcal{E}^\infty(D)$ and let

$$\hat{f}_n(r) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\beta} f(re^{i\beta})\, d\beta, \quad n \in \mathbb{Z}. \tag{3.7}$$

Then

$$f(re^{i\theta}) = \sum_{n \in \mathbb{Z}} \hat{f}_n(r) e^{in\theta} \tag{3.8}$$

with convergence in $\mathcal{E}^\infty(\bar{D})$. Furthermore, if $u \in \mathcal{E}^\infty(D)$ then $M_\delta u = f$ is equivalent to

$$\partial_r [e^{-n\int_1^r \phi(s)ds} \hat{u}_n(r)] = \tag{3.9}$$

$$e^{-n\int_1^r \phi(s)ds}\ \hat{f}_{n+1}(r), \quad n \in \mathbb{Z},$$

by direct computation.

Thus, in order to solve the equation $M_\delta u = f$ over $\bar{D}$ we must solve equations (3.9) with the additional requirement that series $\sum_{n\in\mathbb{Z}} \hat{u}_n(r)e^{in\theta}$ converges in $\mathcal{C}^\infty(\bar{D})$. This in fact can be achieved with the choice

$$(3.10) \qquad \hat{u}_n(r) = \int_{\ell(n)}^{r} e^{n\int_\rho^r \phi(s)ds}\ \hat{f}_{n+1}(\rho)d\rho$$

where $\ell(n)=0$ if $n<0$, $\ell(n)=1$ if $n\geq 0$.

Proceeding exactly in the same way on $\mathbb{R}^2\setminus D$ we will reach the following conclusion:

<u>*Proposition 3.5*</u> *The operators* $K_1: \mathcal{C}^\infty(\bar{D}) \to \mathcal{C}^\infty(\bar{D})$, $K_2: \mathcal{C}_c^\infty(\mathbb{R}^2\setminus D) \to \mathcal{C}^\infty(\mathbb{R}^2\setminus D)$ *defined by:*

$$(3.11) \quad K_1[f](re^{i\theta}) = \sum_{n\in\mathbb{Z}} \left[\int_{\ell(n)}^{r} e^{n\int_\rho^r \phi(s)ds}\ \hat{f}_{n+1}(\rho)d\rho\right] e^{in\theta}$$

$$(3.12)\quad \overline{K}_2[f](re^{i\theta}) = \sum_{n\in\mathbb{Z}} \left[\int_{\overline{\ell}(n)}^{r} e^{n\int_\rho^r \phi(s)ds} f_{n+1}(\rho)d\rho\right] e^{in\theta}$$

where $\ell(n)$ *has the meaning above and* $\overline{\ell}(n)=1$ *if* $n \geq 0$, $\overline{\ell}(n)=\infty$ *if* $n<0$, *are right inverses for* M_δ *over* $\overline{D}$ *and* $\mathbb{R}^2\setminus D$ *respectively.*

Corollary 3.6 *Let* $f\in\mathcal{E}_c^\infty(\mathbb{R}^2)$ *and define the sequence*

$$(3.13)\qquad a_p(f)=\int_0^\infty e^{-p\int_\rho^1 \phi(s)ds} f_{-p+1}(\rho)d\rho,\quad p>1.$$

Then there exists $u\in\mathcal{E}^\infty(\mathbb{R}^2)$ *such that* $M_\delta u=f$ *if and only if*

$$\overline{\lim_{p\to\infty}}\, |a_p(f)|^{1/p} \leq \delta.$$

Proof: If we combine theorem 3.4 with proposition 3.5 we conclude that $f\in M_\delta\mathcal{E}^\infty(\mathbb{R}^2)$ if and only if $K_1[f]-K_2[f]$ equals, over S^1, the boundary value of an element in $\mathcal{H}_\delta$. But, from (3.11) and (3.12) we have

$$(K_1[f]-K_2[f])\,(e^{i\theta})=\sum_{p=1}^\infty a_p(f)\, e^{-ip\theta},$$

hence the corollary.

To conclude we will show that $M_\delta \mathcal{E}^\infty(\mathbb{R}^2) \cap \mathcal{E}_c^\infty(\mathbb{R}^2)$ is not a closed subspace of $\mathcal{E}_c^\infty(\mathbb{R}^2)$, as it was in the case of the Mizohata operator. To start the study we take $q \in \mathcal{E}^\infty(A(\delta))$ nowhere zero and consider the following "boundary value" problem:

$$(3.15) \qquad \begin{cases} q \dfrac{\partial u}{\partial \bar{\zeta}} = f \quad \text{on } A(\delta) \\ u|_{S^1} = 0 \\ u \in \mathcal{E}^\infty(A(\delta)) \text{ and smooth up to } S^1. \end{cases}$$

Here $f \in \mathcal{E}_c^\infty(A(\delta))$. We will denote by $\mathbb{E}(q)$ the subspace of $\mathcal{E}_c^\infty(A(\delta))$ consisting of all functions f for which (3.15) has a solution.

Lemma 3.7 $\mathbb{E}(q)$ is not closed in $\mathcal{E}_c^\infty(A(\delta))$.

Proof: We first take

$$h(\zeta) = \sum_{\ell=1}^{\infty} \frac{b_\ell}{\zeta^\ell}$$

which is holomorphic for $|\zeta| > (1+\delta)/2$ without holomorphic extension to any larger domain.

We also take $\psi \in \mathcal{E}^\infty(\mathbb{R}^1)$, $\psi \equiv 1$ for $r < a^2$, $\psi \equiv 0$ for $r > b^2$, where

$$(1+\delta)/2 < a < b < 1$$

and define

$$(3.16)\quad \begin{cases} h_n(\zeta) = \sum_{p=1}^{n} b_p/\zeta^p \\ u_n(\zeta) = \psi(|\zeta|^2)\, h_n(\zeta) \\ f_n(\zeta) = \zeta\psi'(|\zeta|^2)\, q(\zeta)h_n(\zeta) \\ f(\zeta) = \zeta\psi'(|\zeta|^2)\, q(\zeta)h(\zeta). \end{cases}$$

Then f_n, $f \in \mathcal{C}^\infty_c(A(\delta))$, $f_n \to f$ in $\mathcal{C}^\infty_c(A(\delta))$ and each f_n belongs to $E(q)$, for $q\partial u_n/\partial\bar\zeta = f_n$, $u_n \in \mathcal{C}^\infty(A(\delta))$ and u_n vanishes identically in a neighborhood of S^1 in $A(\delta)$. We claim that $f \notin E(q)$. In fact suppose there is $h \in \mathcal{C}^\infty(A(\delta))$ solving (3.15).

Then

$$v(\zeta) = u(\zeta) - \psi(|\zeta|^2)\, h(\zeta),\ \zeta \in A\left(\frac{1+\delta}{2}\right)$$

is a holomorphic function vanishing identically on S^1. Thus $v \equiv 0$ and since $\frac{\partial u}{\partial\zeta} = 0$ for $\delta < |\zeta| < a$ it follows that the function defined by

$$\tilde h(\zeta) = \begin{cases} u(\zeta) & \delta < |\zeta| \leq \frac{1+\delta}{2} \\ h(\zeta) & |\zeta| > \frac{1+\delta}{2}. \end{cases}$$

is a holomorphic extension of h, a contradiction.

The lemma is proved.

As a corollary, we shall show that $M_\delta \mathcal{E}^\infty(\mathbb{R}^2) \cap \mathcal{E}^\infty_c(\mathbb{R}^2)$ is not closed in $\mathcal{E}^\infty_c(\mathbb{R}^2)$. Let us call $\bar{Z}_\delta$ the restriction of Z_δ to $\mathbb{R}^2 \setminus \bar{D}$ [see (2.13); $\bar{Z}_\delta$ is in fact the biholomorphism Z_2 in the notation of section 2] and let then γ denote the isomorphism (of topological vector spaces) $\mathcal{E}^\infty_c(A(\delta)) \to \mathcal{E}^\infty_c(\mathbb{R}^2 \setminus \bar{D})$ defined by

$$\gamma[f] = f \circ \bar{Z}_\delta .$$

Theorem 3.8 *Let* $q_\delta \in \mathcal{E}^\infty(A(\delta))$ *be the nowhere zero function defined by*

$$(\bar{Z}_\delta)_* M_\delta = q_\delta \frac{\partial}{\partial \bar{\zeta}} .$$

Then

$$\gamma\, \mathbb{E}(q_\delta) = M_\delta \mathcal{E}^\infty(\mathbb{R}^2) \cap \mathcal{E}^\infty_c(\mathbb{R}^2, \bar{D}) \tag{3.17}$$

and thus the range of M_δ *in* $\mathcal{E}^\infty_c(\mathbb{R}^2)$ *is not closed.*

Proof If $g \in \mathcal{E}^\infty_c(\mathbb{R}^2 \setminus \bar{D})$ can be written as $g = \gamma[f]$ with $f \in \mathbb{E}(q_\delta)$ then the function $v = \gamma[u]$, where u is the solution of (3.15) for such f, satisfies $M_\delta v = g$ on $\mathbb{R}^2 \setminus D$ and vanishes on S^1. Extending v as zero inside D we get a global solution of $M_\delta v = g$.

Conversely, if $M_\delta v = g$ with $v \in \mathcal{E}^\infty(\mathbb{R}^2)$ and $g \in \mathcal{E}^\infty_c(\mathbb{R}^2 \setminus \bar{D})$ then, by lemma 2.5, we can find a continuous functions F on $\bar{D}$, holomorphic on D such that $u = F \circ Z_\delta$ on $r < 1 + \varepsilon$ for some $\varepsilon > 0$.

Thus $w = v - F \circ Z_\delta$ on $\mathbb{R}^2$ satisfies $M_\delta w = g$ and vanishes identically for $r < 1 + \varepsilon$. Let $u = \gamma^{-1}[w]$. Then $u \in \mathcal{E}^\infty(A(\delta))$, u vanishes identically in a neighborhood of S^1 and $q_\delta \frac{\partial u}{\partial \bar{\zeta}} = \gamma^{-1}[g]$. Thus $\gamma^{-1}[g] \in \mathbb{E}(q_\delta)$ and the proof is complete.

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