

RT-MAT 95-02

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an elementary proof

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Fevereiro 1995

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## 1 Introduction

In this work we present, via linear system of differential equations, a natural and elementary proof for the Jordan canonical form. The idea of this work appeared during the preparation of a class on resolution of linear system of ordinary differential equations for a course that we taught in the second semester of 1994 in the Instituto de Matemática e Estatística da Universidade de São Paulo. We were considering a linear system of ordinary differential equations  $\dot{x} = Ax$ , where  $A$  is an  $n \times n$  complex matrix,  $x$  a column vector of coordinates  $x_1, x_2, \dots, x_n$  and  $\dot{x} = \frac{dx}{dt}$ ,  $t$  real. It is well known (see [5]) that there is a variable change that transforms  $\dot{x} = Ax$  in a system in a triangular inferior form. Now, if the system is in triangular inferior form, how do we put it in the *Jordan canonical form*? This was the natural question that came to our mind. Thus we started to play around with some examples. Miracle! Very soon we felt that we had in hands a natural and elementary proof for the *Jordan canonical form*. The proof is constructive and we think that the idea of the proof is new. In the references appear other interesting proofs for the Jordan canonical form.

We dedicate this work to the responsables for it: our students.

## 2 Definition of Jordan partial chain

Let  $A$  be an  $n \times n$  complex matrix, triangular inferior and with a unique eigenvalue

$\lambda$ . Let us consider the system  $\dot{x} = Ax$ ,  $A = (a_{ij})$ . Suppose that there exist  $p_1 < p_2 < \dots < p_s$ ,  $s \geq 1$ , such that

$$a_{ip_s} = 0 \text{ for } p_s < i \leq n$$

and for  $k = 2, 3, \dots, s$ , since  $s > 1$ ,

$$\begin{aligned} a_{p_k p_{k-1}} &= 1, \\ a_{p_k j} &= 0 \text{ for } 1 \leq j < p_k \text{ and } j \neq p_{k-1}, \end{aligned}$$

and

$$a_{i p_{k-1}} = 0 \text{ for } p_{k-1} < i \leq n \text{ and } i \neq p_k.$$

In these conditions, we say that the set of the  $s$  equations of the system  $\dot{x} = Ax$ :

$$\begin{aligned} \dot{x}_{p_1} &= \sum_{j=1}^q a_{p_1 j} x_j + \lambda x_{p_1} \\ \dot{x}_{p_2} &= \quad \quad \quad x_{p_1} + \lambda x_{p_2} \\ &\vdots \\ \dot{x}_{p_s} &= \quad \quad \quad \quad \quad \quad x_{p_{s-1}} + \lambda x_{p_s} \end{aligned}$$

where  $q = p_1 - 1$ , is a *Jordan partial chain of size  $s$  starting at  $\lambda x_{p_1}$* . If we have also  $a_{p_1 j} = 0$  for  $j = 1, 2, \dots, q$ , we say that the Jordan partial chain is a *Jordan chain starting at  $\lambda x_{p_1}$* .

**Example 2.1** Consider the system

$$\begin{aligned} \dot{x}_1 &= \lambda x_1 \\ \dot{x}_2 &= 2x_1 + \lambda x_2 \\ \dot{x}_3 &= x_1 + x_2 + \lambda x_3 \\ \dot{x}_4 &= \quad \quad \quad \lambda x_4 \\ \dot{x}_5 &= \quad \quad \quad x_3 + \lambda x_5 \\ \dot{x}_6 &= \quad \quad \quad x_4 + \lambda x_6 \\ \dot{x}_7 &= \quad \quad \quad x_5 + \lambda x_7 \end{aligned}$$

We have a Jordan partial chain of size 3 starting at  $\lambda x_3$  and a Jordan chain of size 2 starting at  $\lambda x_4$ .

### 3 Proof of the Jordan canonical form for a particular case

In this section we will prove the theorem on the Jordan canonical form for an  $n \times n$  complex matrix  $A$  that admits a unique eigenvalue  $\lambda$ . For this we need three lemmas, whose proofs are immediate.

**Lemma 3.1** Consider the system  $\dot{x} = Ax$ , where  $A = (a_{ij})$  is triangular inferior, with a unique eigenvalue  $\lambda$  and given by

$$\begin{aligned}
 \dot{x}_1 &= \lambda x_1 \\
 \dot{x}_2 &= a_{21}x_1 + \lambda x_2 \\
 \dot{x}_3 &= a_{31}x_1 + a_{32}x_2 + \lambda x_3 \\
 &\vdots \\
 \dot{x}_{q_1} &= \sum_{j=1}^r a_{q_1,j}x_j + \lambda x_{q_1} \\
 \dot{x}_{q_2} &= \sum_{j=1}^r a_{q_2,j}x_j + \lambda x_{q_2} \\
 &\vdots \\
 \dot{x}_{q_{k-1}} &= \sum_{j=1}^r a_{q_{k-1},j}x_j + \lambda x_{q_{k-1}} \\
 \dot{x}_{q_k} &= \sum_{j=1}^r a_{q_k,j}x_j + x_{q_1} + \lambda x_{q_k} \\
 &\vdots
 \end{aligned}$$

where  $r = q_1 - 1$ ,  $\lambda$  is a complex number,  $q_m = q_{m-1} + 1$  for  $m = 2, 3, \dots, k$  and  $a_{iq_1} = 0$  for  $q_k < i \leq n$ . Then such a system can be transformed in a system where the  $q_1$ th and  $q_k$ th equations are, respectively,

$$\begin{aligned}
 \dot{y}_{q_1} &= \sum_{j=1}^r b_{q_1,j}x_j + \lambda y_{q_1} \\
 \dot{x}_{q_k} &= y_{q_1} + \lambda x_{q_k}
 \end{aligned}$$

and the other equations remain unchanged. The elements  $b_{q_1,j}$ ,  $j = 1, 2, \dots, r$  and  $y_{q_1}$  are given by

$$\begin{aligned}
 b_{q_1,j} &= a_{q_1,j} + \sum_{i=j+1}^r a_{ij}a_{q_k,i}, \quad j = 1, 2, \dots, r-1, \quad b_{q_1,r} = a_{q_1,r} \\
 y_{q_1} &= \sum_{j=1}^r a_{q_k,j}x_j + x_{q_1}.
 \end{aligned}$$

**Proof.** It is enough to multiply the  $p$ th equation,  $p = 1, 2, \dots, r$  by  $a_{q_k p}$  to add with the  $q_1$ th and to make

$$y_{q_1} = \sum_{j=1}^r a_{q_k j} x_j + x_{q_1} \quad \square$$

The main idea of this work is contained in the next lemma.

**Lemma 3.2** *Let  $A = (a_{ij})$  be an  $n \times n$  complex matrix, in triangular inferior form that admits a unique eigenvalue  $\lambda$ . Suppose that the  $p_1$ th equation of the system  $\dot{x} = Ax$  has the form*

$$\dot{x}_{p_1} = \sum_{j=1}^k a_{p_1 j} x_j + \lambda x_{p_1}$$

and the  $q_1$ th has the form

$$\dot{x}_{q_1} = \sum_{j=1}^k a_{q_1 j} x_j + \lambda x_{q_1}$$

with  $k < \min\{p_1, q_1\}$  and  $a_{q_1 k} = 1$ . Suppose also there is a Jordan partial chain of size  $r$  starting at  $\lambda x_{p_1}$  and other of size  $s$  starting at  $\lambda x_{q_1}$ , with  $r \leq s$ . Then the given system can be transformed in a system where the  $p_1$ th equation is

$$\dot{u}_{p_1} = \sum_{j=1}^{k-1} b_{p_1 j} x_j + \lambda u_{p_1}$$

where  $b_{p_1 j} = a_{p_1 j} - a_{p_1 k} a_{q_1 j}$ ,  $j = 1, 2, \dots, k-1$ , the  $p_i$ th equation,  $i = 2, 3, \dots, r$ , is

$$\dot{u}_{p_i} = u_{p_{i-1}} + \lambda u_{p_i}$$

where  $x_{p_i} - a_{p_i k} x_{q_1} = u_{p_i}$ ,  $i = 1, 2, \dots, r$ , and the other equations remain unchanged.

**Proof.** Multiplying the  $q_1$ th equation by  $-a_{p_1 k}$  adding with the  $p_1$ th and making  $x_{p_1} - a_{p_1 k} x_{q_1} = u_{p_1}$  we have

$$\dot{u}_{p_1} = \sum_{j=1}^{k-1} b_{p_1 j} x_j + \lambda u_{p_1}$$

where  $b_{p_1 j} = a_{p_1 j} - a_{p_1 k} a_{q_1 j}$ ,  $j = 1, 2, \dots, k-1$ .

If  $r = 1$  the lemma is proved. Otherwise,

$$\dot{x}_{p_2} = u_{p_1} + a_{p_1 k} x_{q_1} + \lambda x_{p_2} .$$

Remembering that  $s \geq r$ , multiplying the  $q_2$ th equation by  $-a_{p_1, k}$ , adding with the  $p_2$ th and making  $x_{p_2} - a_{p_1, k} x_{q_2} = u_{p_2}$  we get

$$\dot{u}_{p_2} = u_{p_1} + \lambda u_{p_2} .$$

If  $r = 2$  the lemma is proved. Otherwise it is enough to repeat the process.  $\square$

**Example 3.1** Consider the system

$$\begin{aligned} \dot{x}_1 &= \lambda x_1 \\ \dot{x}_2 &= x_1 + \lambda x_2 \\ \dot{x}_3 &= 2x_1 + x_2 + \lambda x_3 \\ \dot{x}_4 &= x_1 + 2x_3 + \lambda x_4 \\ \dot{x}_5 &= 2x_1 + x_2 + x_3 + \lambda x_5 \\ \dot{x}_6 &= x_5 + \lambda x_6 \\ \dot{x}_7 &= x_4 + \lambda x_7 . \end{aligned}$$

The Jordan partial chain starting at  $\lambda x_4$  has size 2 and the one starting at  $\lambda x_5$  has also size 2. So, applying lemma 3.2 we get

$$\begin{aligned} \dot{x}_1 &= \lambda x_1 \\ \dot{x}_2 &= x_1 + \lambda x_2 \\ \dot{x}_3 &= 2x_1 + x_2 + \lambda x_3 \\ \dot{u}_4 &= -3x_1 - 2x_2 + \lambda u_4 \\ \dot{x}_5 &= 2x_1 + x_2 + x_3 + \lambda x_5 \\ \dot{x}_6 &= x_5 + \lambda x_6 \\ \dot{u}_7 &= u_4 + \lambda u_7 . \end{aligned}$$

where  $x_4 - 2x_5 = u_4$  and  $x_7 - 2x_6 = u_7$ . Now applying lemma 3.1 we obtain

$$\begin{aligned} \dot{x}_1 &= \lambda x_1 \\ \dot{x}_2 &= x_1 + \lambda x_2 \\ \dot{u}_3 &= 3x_1 + x_2 + \lambda u_3 \\ \dot{u}_4 &= -3x_1 - 2x_2 + \lambda u_4 \\ \dot{x}_5 &= u_3 + \lambda x_5 \\ \dot{x}_6 &= x_5 + \lambda x_6 \\ \dot{u}_7 &= u_4 + \lambda u_7 . \end{aligned}$$

where  $2x_1 + x_2 + x_3 = u_3$ .

The next lemma is suggested by the following example.

**Example 3.2** Consider the system

$$\begin{aligned} \dot{x}_1 &= \lambda x_1 \\ \dot{x}_2 &= \lambda x_2 \\ \dot{x}_3 &= \lambda x_3 \\ \dot{x}_4 &= x_2 + \lambda x_4 \\ \dot{x}_5 &= x_4 + \lambda x_5 \\ \dot{x}_6 &= x_3 + \lambda x_6 \end{aligned}$$

We have a Jordan chain of size 1 starting at  $\lambda x_1$ , a Jordan chain of size 3 starting at  $\lambda x_2$  and a Jordan chain of size 2 starting at  $\lambda x_3$ . The system can be rewritten in the following form

$$\begin{aligned} \dot{x}_2 &= \lambda x_2 \\ \dot{x}_4 &= x_2 + \lambda x_4 \\ \dot{x}_5 &= x_4 + \lambda x_5 \\ \dot{x}_3 &= \lambda x_3 \\ \dot{x}_6 &= x_3 + \lambda x_6 \\ \dot{x}_1 &= \lambda x_1 \end{aligned}$$

which is the *Jordan canonical form* of the given system.

In the following, we indicate by  $J_{\lambda,k}$  a square matrix of order  $n_k$  given by

$$J_{\lambda,k} = \lambda I + E$$

where  $I$  is the identity matrix of order  $n_k$  and  $E = (e_{ij})$  is a square matrix of order  $n_k$  given by  $e_{ij} = 1$  if  $j = i - 1$  and  $e_{ij} = 0$  if  $j \neq i - 1$ . So

$$J_{\lambda,k} = \begin{pmatrix} \lambda & 0 & 0 & \dots & 0 & 0 \\ 1 & \lambda & 0 & \dots & 0 & 0 \\ 0 & 1 & \lambda & \dots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \dots & 1 & \lambda \end{pmatrix}$$

The matrix  $J_{\lambda,k}$  is called a *Jordan block of order  $n_k$* .

**Lemma 3.3** Let  $A = (a_{ij})$  be an  $n \times n$  complex matrix, in triangular inferior form, with a unique eigenvalue  $\lambda$  and let  $r$  be the number of linearly independent eigenvectors associated with  $\lambda$ , that is,  $r = \dim\{v \in \mathbb{C}^n \mid Av = \lambda v\}$ . Suppose, if  $r < n$ , there are

$j_1, j_2, \dots, j_{n-r}$ , with  $j_\alpha \neq j_\beta$  for  $\alpha \neq \beta$ ,  $j_k \in \{1, 2, \dots, n\}$  and  $j_k < r+k$ ,  $k = 1, 2, \dots, n-r$ , such that

$$a_{(r+k)j_k} = 1, \quad k = 1, 2, \dots, n-r,$$

and

$$a_{ij} = 0 \text{ for } j < i \text{ and } (i, j) \notin \{(r+k, j_k) | k = 1, 2, \dots, n-r\}.$$

In these conditions, there exists an  $n \times n$  nonsingular matrix  $P$  such that  $P^{-1}AP = J_\lambda$  where  $J_\lambda = \text{diag}(J_{\lambda,1}, J_{\lambda,2}, \dots, J_{\lambda,r})$ ,  $\sum_{k=1}^r n_k = n$  and  $n_1 \geq n_2 \geq \dots \geq n_r$ , where  $n_k$ ,  $k = 1, 2, \dots, r$ , is the order of the matrix  $J_{\lambda,k}$ .

**Proof.** If  $r = n$  the lemma is immediate. So, let us suppose  $1 \leq r < n$ . It is immediate that we have a Jordan chain starting at  $\lambda x_i$ ,  $i = 1, 2, \dots, r$ , and if  $r < m \leq n$ , there is  $i \in \{1, 2, \dots, r\}$  such that the  $m$ th equation belongs to the Jordan chain starting at  $\lambda x_i$ . Now, let  $i_k \in \{1, 2, \dots, r\}$  and let  $n_k$  be the size of the Jordan chain starting at  $\lambda x_{i_k}$ . It is clear that we can choose  $i_k$ ,  $k = 1, 2, \dots, r$ , such that  $n_1 \geq n_2 \geq \dots \geq n_r$ . Now, interchanging conveniently the positions of the equations (and changing the names of the variables) our system can be rewritten in the following form:

$$\begin{aligned} \dot{y}_1 &= J_{\lambda,1} y_1 \\ \dot{y}_2 &= J_{\lambda,2} y_2 \\ &\vdots \\ \dot{y}_r &= J_{\lambda,r} y_r \end{aligned}$$

where  $y_k$ ,  $k = 1, 2, \dots, r$ , is a column vector of coordinates  $y_{k,1}, y_{k,2}, \dots, y_{k,n_k}$ . To close, let  $y$  be the column vector of coordinates  $y_{1,1}, y_{1,2}, \dots, y_{1,n_1}, y_{2,1}, \dots, y_{2,n_2}, \dots, y_{r,n_r}$ . It is clear that the matrix  $P$  such that  $x = Py$  is nonsingular. From  $\dot{x} = Ax$ , it follows that  $P\dot{y} = APy$  and therefore  $\dot{y} = P^{-1}APy$ . So  $J_\lambda = P^{-1}AP$ .  $\square$

In the following we enunciate and prove the theorem on the Jordan canonical form for a matrix  $A$  with a unique eigenvalue  $\lambda$ .

**Theorem 3.1** (Jordan canonical form) *Let  $A$  be an  $n \times n$  complex matrix with a unique eigenvalue  $\lambda$ . Then there exists an  $n \times n$  nonsingular matrix  $M$  such that*

$$M^{-1}AM = J_\lambda$$

where  $J_\lambda = \text{diag}(J_{\lambda,1}, J_{\lambda,2}, \dots, J_{\lambda,r})$ ,  $r = \dim\{v \in \mathbb{C}^n | Av = \lambda v\}$ ,  $\sum_{k=1}^r n_k = n$  and  $n_1 \geq n_2 \geq \dots \geq n_r$ .

**Proof.** Without loss of generality we can suppose that  $A$  is in triangular inferior form. If  $r = n$  the theorem is immediate. So we can suppose  $1 \leq r < n$ . From lemma 3.3, it is sufficient to prove there exists an  $n \times n$  nonsingular matrix  $Q$  such that  $Q^{-1}AQ = B$  where  $B = (b_{ij})$  is triangular inferior and given by: there are  $j_1, j_2, \dots, j_{n-r}$ , with  $j_\alpha \neq j_\beta$  for  $\alpha \neq \beta$ ,  $j_k \in \{1, 2, \dots, n\}$  and  $j_k < r + k$ ,  $k = 1, 2, \dots, n - r$ , such that

$$b_{(r+k)j_k} = 1, \quad k = 1, 2, \dots, n - r$$

and

$$b_{ij} = 0 \text{ for } j < i \text{ and } (i, j) \notin \{(r+k, j_k) | k = 1, 2, \dots, n-r\}.$$

Let

$$p = \max\{j \mid \text{there is } i, i > j, \text{ with } a_{ij} \neq 0\}.$$

(such a  $p$  exists because  $r < n$ ). Without loss of generality we can suppose  $a_{np} = 1$ . The Jordan partial chain starting at  $\lambda x_{p+k}$ ,  $k = 1, 2, \dots, n - p$ , has size 1. From lemmas 3.1 and 3.2, the system  $\dot{x} = Ax$  can be transformed in a system  $\dot{y} = Sy$ ,  $S = (s_{ij})$ , where

$$\begin{aligned} s_{np} &= 1, \\ s_{nj} &= 0 \text{ for } j \neq p \text{ and } j < n \end{aligned}$$

and

$$s_{ij} = 0 \text{ for } p \leq j < i < n.$$

It is clear that the  $n \times n$  matrix  $M_1$  such that  $x = M_1 y$  is nonsingular. By induction, suppose now there is an  $n \times n$  nonsingular matrix  $M_2$  such that  $M_2^{-1}AM_2 = D$ ,  $D = (d_{ij})$ , where  $D$  is given by: there are  $j_1^*, j_2^*, \dots, j_{n-q}^*$ , with  $q \geq r$ ,  $j_\alpha^* \neq j_\beta^*$  for  $\alpha \neq \beta$ ,  $j_1^* = \min\{j_1^*, j_2^*, \dots, j_{n-q}^*\}$ ,  $j_k^* \in \{1, 2, \dots, n\}$  and  $j_k^* < q + k$  for  $k = 1, 2, \dots, n - q$  such that

$$d_{(q+k)j_k^*} = 1 \text{ for } k = 1, 2, \dots, n - q,$$

$$d_{ij} = 0 \text{ for } i \geq q + 1, j < i \text{ and } (i, j) \notin \{(q + k, j_k^*) | k = 1, 2, \dots, n - q\}$$

and

$$d_{ij} = 0 \text{ for } j_1^* \leq j < i < q + 1.$$

Observe that for  $q = n - 1$  such a matrix  $M_2$  there exists:  $M_2 = M_1$ . Consider the system  $\dot{\mathbf{u}} = D\mathbf{u}$ . If  $q = r$  the theorem is proved. So, let us suppose  $q > r$ . Let

$$h = \max\{j < j^* \mid \text{there is } i, j < i \leq q, \text{ with } d_{ij} \neq 0\}.$$

(Such an  $h$  exists because  $q > r$ .) There is  $q_1 \in \{h + i \mid i = 1, 2, \dots, q - h\}$  such that  $d_{q_1, h} \neq 0$  and the size of the Jordan partial chain starting at  $\lambda u_{q_1}$  is  $\geq$  to the size of the Jordan partial chain starting at  $\lambda u_{h+i}$ ,  $i = 1, 2, \dots, q - h$ , with  $d_{(h+i)h} \neq 0$ . Without loss of generality we can suppose  $d_{q_1, h} = 1$ . Now, interchanging the positions of the  $q$ th and  $q_1$ th equations and applying lemmas 3.1 and 3.2 our system  $\dot{\mathbf{u}} = D\mathbf{u}$ , where  $\mathbf{x} = M_2\mathbf{u}$ , is transformed in  $\dot{\mathbf{z}} = H\mathbf{z}$ ,  $H = (h_{ij})$ , where  $H$  is given by: there are  $\bar{j}_1, \bar{j}_2, \dots, \bar{j}_{n-m}$ , with  $m = q - 1 \geq r$ ,  $\bar{j}_\alpha \neq \bar{j}_\beta$  for  $\alpha \neq \beta$ ,  $\bar{j}_1 = h$ ,  $\bar{j}_1 = \min\{\bar{j}_1, \bar{j}_2, \dots, \bar{j}_{n-m}\}$ ,  $\bar{j}_k \in \{1, 2, \dots, n\}$  and  $\bar{j}_k < m + k$  for  $k = 1, 2, \dots, n - m$  such that

$$h_{(m+k)\bar{j}_k} = 1 \text{ for } k = 1, 2, \dots, n - m,$$

$$h_{ij} = 0 \text{ for } i \geq m + 1 = q, j < i \text{ and } (i, j) \notin \{(m + k, \bar{j}_k) \mid k = 1, 2, \dots, n - m\}$$

and

$$h_{ij} = 0 \text{ for } h \leq j < i < q.$$

It is clear that  $\mathbf{u} = M_3\mathbf{z}$  where the  $n \times n$  matrix  $M_3$  is nonsingular. If  $m > r$ , it is enough to repeat the process.  $\square$

## 4 Jordan canonical form: the general case

In order to prove the general case we need three lemmas, whose proofs are immediate and will be omitted.

**Lemma 4.1** Let  $A, B$  and  $M$  be  $n \times n$  complex matrices, with  $M$  nonsingular, such that  $M^{-1}AM = B$ . Let  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{v}_1$  and  $\mathbf{v}_2$  be column vectors of  $\mathbb{C}^n$  such that  $M\mathbf{e}_1 = \mathbf{v}_1$  and  $M\mathbf{e}_2 = \mathbf{v}_2$ . Suppose there is a complex number  $\alpha$  such that  $(B - \lambda I)\mathbf{e}_1 = \alpha\mathbf{e}_2$ . Then, we have also  $(A - \lambda I)\mathbf{v}_1 = \alpha\mathbf{v}_2$ .

**Lemma 4.2** Let  $J_\lambda$  be an  $n \times n$  matrix given by  $J_\lambda = \text{diag}(J_{\lambda,1}, J_{\lambda,2}, \dots, J_{\lambda,r})$ . Let  $\mathbf{e}_i$ ,  $i = 1, 2, \dots, n$  be the  $i$ th canonical column vector of  $\mathbb{C}^n$ , that is,  $\mathbf{e}_i$  is the column vector of coordinates  $e_{ij}$ ,  $j = 1, 2, \dots, n$ , with  $e_{ij} = 0$  for  $i \neq j$  and  $e_{ij} = 1$  for  $i = j$ . Then

$$(J_\lambda - \lambda I)\mathbf{e}_i = \mathbf{e}_{i+1} \text{ for } i \notin \left\{ \sum_{q=1}^k n_q \mid k = 1, 2, \dots, r \right\}$$

and

$$(J_\lambda - \lambda I)\mathbf{e}_i = \mathbf{0} \text{ for } i \in \left\{ \sum_{q=1}^k n_q \mid k = 1, 2, \dots, r \right\}$$

where  $n_q$  is the order of the matrix  $J_{\lambda,q}$ .

**Lemma 4.3** Let  $A$  be an  $n \times n$  complex matrix with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_s$ ,  $\lambda_\alpha \neq \lambda_\beta$  for  $\alpha \neq \beta$ . Let  $m_k$ ,  $k = 1, 2, \dots, s$ , denote the algebraic multiplicity of  $\lambda_k$  and for  $k = 1, 2, \dots, s$  let  $r_k$  be the dimension of  $\{\mathbf{v} \in \mathbb{C}^n \mid A\mathbf{v} = \lambda_k\mathbf{v}\}$ . Then, for each  $k$ ,  $k = 1, 2, \dots, s$ , there is an  $n \times n$  nonsingular matrix  $M_k$  such that  $M_k^{-1}AM_k = H_k$  with

$$H_k = \begin{pmatrix} H_{nm} & O_{pq} \\ & J_{\lambda_k} \end{pmatrix}$$

where  $m = n - m_k$ ,  $p = n - m_k$ ,  $q = m_k$ ,  $H_{nm}$  is a  $n \times m$  complex matrix,  $O_{pq}$  a  $p \times q$  matrix where all the elements are equal to zero and  $J_{\lambda_k} = \text{diag}(J_{\lambda_k,1}, J_{\lambda_k,2}, \dots, J_{\lambda_k,r_k})$ .

Now, let us consider the matrix  $H_k$  of the above lemma 4.3. Let  $\mathbf{f}_{k,i}$  denote the canonical column vector  $\mathbf{e}_{n-m_k+i}$ ,  $k = 1, 2, \dots, s$  and  $i = 1, 2, \dots, m_k$ . From lemmas 4.2 and 4.3, it follows immediately that

$$(H_k - \lambda_k I)\mathbf{f}_{k,i} = \mathbf{f}_{k,i+1} \text{ for } i \notin \left\{ \sum_{q=1}^h n_{k,q} \mid h = 1, 2, \dots, r_k \right\}$$

and

$$(H_k - \lambda_k J) \mathbf{f}_{k,i} = 0 \text{ for } i \in \left\{ \sum_{q=1}^h n_{k,q} \mid h = 1, 2, \dots, r_k \right\}$$

where  $n_{k,q}$  is the order of the matrix  $J_{\lambda_k, q}$ . Now, let us make

$$M_k \mathbf{f}_{k,i} = \mathbf{v}_{k,i},$$

$k = 1, 2, \dots, s$  and  $i = 1, 2, \dots, m_k$ . From lemma 4.1 it follows that

$$A \mathbf{v}_{k,i} = \lambda_k \mathbf{v}_{k,i} + \mathbf{v}_{k,i+1} \text{ for } i \notin \left\{ \sum_{q=1}^h n_{k,q} \mid h = 1, 2, \dots, r_k \right\} \quad (1)$$

and

$$A \mathbf{v}_{k,i} = \lambda_k \mathbf{v}_{k,i} \text{ for } i \in \left\{ \sum_{q=1}^h n_{k,q} \mid h = 1, 2, \dots, r_k \right\}. \quad (2)$$

By induction, one prove easily that the  $n$  column vectors  $\mathbf{v}_{k,i}$ ,  $k = 1, 2, \dots, s$  and  $i = 1, 2, \dots, m_k$ , are linearly independent.

Now let us consider the system  $\dot{\mathbf{x}} = A\mathbf{x}$  and let us make the variable change given by

$$\mathbf{x} = \sum_{k=1}^s \sum_{i=1}^{m_k} y_{k,i} \mathbf{v}_{k,i}.$$

We have

$$\dot{\mathbf{x}} = \sum_{k=1}^s \sum_{i=1}^{m_k} \dot{y}_{k,i} \mathbf{v}_{k,i} \quad \text{and} \quad A\mathbf{x} = \sum_{k=1}^s \sum_{i=1}^{m_k} y_{k,i} A \mathbf{v}_{k,i}.$$

From the relations (1) and (2) above, it follows immediately that

$$\dot{y}_k = J_{\lambda_k} y_k, \quad k = 1, 2, \dots, s,$$

where  $\mathbf{y}_k$  is the column vector of coordinates  $y_{k,1}, y_{k,2}, \dots, y_{k,m_k}$ . So  $\dot{\mathbf{y}} = J\mathbf{y}$  where  $J = \text{diag}(J_{\lambda_1}, J_{\lambda_2}, \dots, J_{\lambda_s})$  and  $\mathbf{y}$  is the column vector of coordinates  $y_{1,1}, y_{1,2}, \dots, y_{1,m_1}, \dots, y_{s,1}, y_{s,2}, \dots, y_{s,m_s}$ . Observe that the variable change

$$\mathbf{x} = \sum_{k=1}^s \sum_{i=1}^{m_k} y_{k,i} \mathbf{v}_{k,i}$$

can be rewritten in the following form

$$\mathbf{x} = M\mathbf{y}$$

where  $M$  is the  $n \times n$  nonsingular matrix whose column vectors are  $v_{1,1}, v_{1,2}, \dots, v_{1,m_1}, v_{2,1}, v_{2,2}, \dots, v_{2,m_2}, \dots, v_{s,1}, v_{s,2}, \dots, v_{s,m_s}$ . So  $J = M^{-1}AM$ . Thus we have proved the following theorem.

**Theorem 4.1** (Jordan canonical form: general case) *Let  $A$  be an  $n \times n$  complex matrix with eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_s, \lambda_\alpha \neq \lambda_\beta$  for  $\alpha \neq \beta$ . Let  $m_k$  and  $r_k, k = 1, 2, \dots, s$ , be where  $m_k$  is the algebraic multiplicity of  $\lambda_k$  and  $r_k = \dim\{v \in \mathbb{C}^n | Av = \lambda_k v\}$ . Then there is an  $n \times n$  nonsingular matrix  $M$  such that  $M^{-1}AM = J$  where  $J = \text{diag}(J_{\lambda_1}, J_{\lambda_2}, \dots, J_{\lambda_s})$  and, for  $k = 1, 2, \dots, s, J_{\lambda_k}$  is a square matrix of order  $m_k$  given by  $J_{\lambda_k} = \text{diag}(J_{\lambda_k,1}, J_{\lambda_k,2}, \dots, J_{\lambda_k,r_k})$ . We have also, for  $k = 1, 2, \dots, s, \sum_{q=1}^{r_k} n_{k,q} = m_k$  and  $n_{k,1} \geq n_{k,2} \geq \dots \geq n_{k,r_k}$  where  $n_{k,q}$  is the order of the Jordan block  $J_{\lambda_k,q}$ .*

## References

- [1] Botha, J.D. A short treatment of the Jordan canonical form theory. *Expositiones Mathematicae*, vol. II, 4 (1993), 365-367.
- [2] Brualdi, R.A. The Jordan canonical form: an old proof. *Amer. Math. Monthly*, 94 (1987), 257-267.
- [3] Fletcher, R. and Sorensen, D.C. An algorithmic derivation of the Jordan canonical form. *Amer. Math. Monthly*, 90 (1983), 12-16.
- [4] Galperin, A. and Waksman, Z. An elementary approach to Jordan theory. *Amer. Math. Monthly*, 87 (1981), 728-732.
- [5] Plaat, O. *Ordinary differential equations*. Holden-Day, Inc., San Francisco, California (1974).

### Example

Consider the following  $9 \times 9$  real matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

With a slight rearrangement of the blocks (to meet the decreasing-size convention established above), we have the Jordan normal form

$$\begin{pmatrix} \boxed{1} & \boxed{1} & \boxed{0} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{0} & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{0} & \boxed{0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{0} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{2} \end{pmatrix}$$

The blocks correspond to the respective eigenvalues 1, 0, and 2. The characteristic polynomial of  $A$  is evidently

$$p(t) = (t-1)^5 t^3 (t-2)$$

but  $A$  also satisfies

$$q(t) = (t-1)^3 t^2 (t-2)$$

It is an easy exercise to show that  $q(t)$  is in fact the polynomial of smallest degree having leading coefficient 1 such that  $q(A) = 0$ . It is thus called the *minimal polynomial* of  $A$ .

## Errata

The matrix appearing in Theorem 10-17 on page 207 should have the following, more general form:

$$\begin{pmatrix} \begin{array}{c|c} \begin{matrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & \\ & & & & 0 \end{matrix} & \\ \hline & 0 \end{array} & \\ \dots & \\ \begin{array}{c|c} \begin{matrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & \\ & & & & 0 \end{matrix} & \\ \hline & 0 \end{array} \end{pmatrix}$$

Not all of these square subblocks are necessarily of the same size. (See overleaf for an example.) Note that this includes the possibility of the zero matrix, when all blocks are of size  $1 \times 1$ . Similarly, on page 208, the matrices  $A_j$  appearing in the block decomposition of  $A$  should also have the more general form:

$$A_j = \begin{pmatrix} \begin{array}{c|c} \begin{matrix} \lambda_j & 1 & & & \\ & \lambda_j & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & \\ & & & & \lambda_j \end{matrix} & \\ \hline & 0 \end{array} & \\ \dots & \\ \begin{array}{c|c} \begin{matrix} \lambda_j & 1 & & & \\ & \lambda_j & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & \\ & & & & \lambda_j \end{matrix} & \\ \hline & 0 \end{array} \end{pmatrix}$$

Note that this includes the possibility of a diagonal matrix, again when all blocks are of size  $1 \times 1$ . The subblocks appearing in  $A_j$  are called the *basic Jordan blocks* belonging to the eigenvalue  $\lambda_j$ . If these basic blocks occur in order of decreasing size, we then speak of the *Jordan normal form* of  $A$ , which is thus well defined up to the ordering of the  $A_j$  themselves.

Exercise 26 on page 212 must be reinterpreted accordingly: one should describe *all* possible Jordan normal forms that can arise from an endomorphism with the characteristic polynomial given.

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