PAPER

Quantum entanglement of bound particles under free center of mass dispersion

To cite this article: Fernanda Pinheiro and A F R de Toledo Piza 2012 *Phys. Scr.* **85** 065002

View the article online for updates and enhancements.

Quantum entanglement of bound particles under free center of mass dispersion

Fernanda Pinheiro[1,2,3] **and A F R de Toledo Piza**[1]

[1] Instituto de Física, Universidade de São Paulo, São Paulo, Brazil
[2] Department of Physics, Stockholm University, 106 91 Stockholm, Sweden
[3] NORDITA, 106 91 Stockholm, Sweden

E-mail: fep@fysik.su.se

Abstract
On the basis of the full analytical solution of the overall unitary dynamics, the time evolution of entanglement is studied in a simple bipartite model system evolving unitarily from a pure initial state. The system consists of two particles in one spatial dimension bound by harmonic forces and having its free center of mass initially localized in space in a minimum uncertainty wavepacket. The existence of such initial states in which the bound particles are not entangled is discussed. Galilean invariance of the system ensures that the dynamics of entanglement between the two particles is independent of the wavepacket mean momentum. In fact, as shown, it is driven by the dispersive center of mass free dynamics, and evolves in a time scale that depends on the interparticle interaction in an essential way.

PACS number: 03.65.Ud

(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum entanglement, given its relevance as a resource in the context of quantum information [1, 2], has been extensively studied [3], mostly in contexts involving non-interacting subsystems of composite quantum systems. This is in fact the context in which the consequences of the existence of non-classical correlations involving separate subsystems were first quantitatively formulated by Bell [4], in the wake of the seminal paper of Einstein *et al* [5]. Although originally cast in terms of systems involving dynamical variables with continuous spectra (position, momentum), the basic issues involving non-classical correlations were soon recast in terms of spin, or photon polarization degrees of freedom, which were in fact the ones used in connection with decisive experimental work [6]. On the conceptual side, the focus on non-interacting subsystems (after mutual interaction has taken place) has led to the important role played by 'local (subsystem) operations' in the characterization and quantification of quantum entanglement [3, 7].

A major issue concerning the dynamics of entanglement involves the effects of dissipative environment couplings, leading, in particular, to the development of the 'quantum channel phenomenology' for the dynamics of open quantum systems [8–10]. However, the effects of *internal* couplings on the dynamics of entanglement of subsystems involving continuous dynamical variables have received theoretical attention, notably in connection with scattering situations ([11, 12], and [13] and references therein) and also in connection with interacting particles bound in external potential traps. Relevant issues in the latter connection have been the role played by the interaction between subsystems in determining the non-classical correlations between them in stationary states [14, 15] and in affecting the dynamics of entanglement in non-stationary situations [16].

In this work, we try to pin down further the role of interparticle interactions in the dynamics of entanglement by considering a simple model system in which their effect is non-perturbatively strong and stationary. We thus consider a Galilean invariant system of two distinguishable, mutually interacting particles in one dimension and restrict the role

of the interparticle interaction to that of maintaining a stationary two-particle bound state. In a sense, this can be seen as an extreme case of the resonant scattering effects treated in [11]. We deal with states which are product states of center of mass and relative variables and choose the individual particle variables as the 'local' variables[4] . The initial state of the free center of mass is prepared as a Gaussian wavepacket, so that the system is in a non-stationary state which subsequently evolves dispersively under the *free* center of mass Hamiltonian. This evolution induces a monotonic increase in time of the entanglement of the two particles. The associated time scale is independent of the center of mass momentum and differs from that of the dispersive spreading of the wavepacket, as it *also* depends on parameters related to properties of the two-particle bound state and hence on the interaction between the particles.

The paper is organized as follows. In section 2, the characterization of the model system is given and its dynamics is fully worked out analytically. The presence of entanglement in the adopted initial states is also discussed. Section 3 constitutes the main body of the work. It is divided into three subsections, dealing respectively with the independence of the entanglement dynamics from the mean center of mass momentum, with the entanglement properties within the chosen manifold of initial states and with the time evolution of the entanglement. Section 4 contains the concluding remarks.

2. Two decompositions of a flying two-body bound state

The system we consider consists of a flying bound state of two (spinless) particles in one dimension, the binding being provided by a simple harmonic potential in the relative coordinate of the two particles. If the masses and positions of the particles are $\{m_1, x_1\}$ and $\{m_2, x_2\}$, respectively, the dynamics of the system is described by the Hamiltonian

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{\gamma}{2}(x_2 - x_1)^2$$

where p_1 and p_2 are the momenta canonically conjugated to x_1 and x_2. These variables will be considered as the 'local' subsystem variables. The dynamics associated with this Galilean invariant Hamiltonian is, however, much more conveniently worked out in terms of ('non-local') center of mass and relative variables

$$R = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \quad P = p_1 + p_2;$$

$$r = x_2 - x_1, \quad p = \frac{m_1 p_2 - m_2 p_1}{m_1 + m_2}, \quad (2.1)$$

in terms of which it appears in the guise

$$H = \frac{P^2}{2M} + \left(\frac{p^2}{2\mu} + \frac{\mu\omega^2}{2}r^2\right) \equiv H_{\mathrm{CM}} + H_{\mathrm{rel}}.$$

---

[4] The importance of the choice of the subsystem dynamical variables to be considered 'local' in evaluating entanglement measures has been stressed in [13].

Here the new mass parameters are, respectively, the total mass $M = m_1 + m_2$ and the reduced mass $\mu = m_1 m_2 / (m_1 + m_2)$. The oscillator frequency is $\omega = \sqrt{\gamma/\mu}$.

It is worth stressing that in both cases the system maintains a composite character, but in the latter rendering this reduces to two decoupled, 'non-local' (with respect to the particle variables) parts [13]. The well-known stationary solutions to the Schrödinger equation consist of the product of a center of mass momentum eigenfunction with a stationary state of the harmonic oscillator equation, the latter governing the dynamics of the relative motion. Taking the relevant state to be the oscillator ground state one has

$$\Psi_K(R, r) = \mathrm{e}^{\mathrm{i}KR} \frac{1}{\pi^{\frac{1}{4}}\sqrt{b}} \mathrm{e}^{-\frac{1}{2}\left(\frac{r}{b}\right)^2}, \quad b = \sqrt{\frac{\hbar}{\mu\omega}}.$$

Note that one faces here the well-known normalization problem in the case of the continuous spectrum of the center of mass Hamiltonian. We will not have to deal explicitly with this, however, since we consider a (non-stationary) normalized Gaussian wavepacket of width B and mean momentum $\hbar K$ as the wavefunction describing the initial state of the center of mass. The relative part remains in its chosen ground stationary state, so that our normalized initial state is

$$\Psi(R, r, t=0) = \frac{1}{\pi^{\frac{1}{4}}\sqrt{B}} \mathrm{e}^{-\frac{1}{2}\left(\frac{R}{B}\right)^2} \mathrm{e}^{\mathrm{i}K.R} \frac{1}{\pi^{\frac{1}{4}}\sqrt{b}} \mathrm{e}^{-\frac{1}{2}\left(\frac{r}{b}\right)^2}. \quad (2.2)$$

The exact time evolution of this state under the assumed Hamiltonian can be easily obtained analytically. Performing a double Fourier transform on the initial state (2.2) before acting on it with the evolution operator, one obtains

$$\Psi(\kappa, \xi, t) = \mathrm{e}^{-\mathrm{i}\frac{Ht}{\hbar}} \int \mathrm{e}^{-\mathrm{i}\kappa R} \mathrm{e}^{-\mathrm{i}\xi r} \Psi(R, r) \mathrm{d}R \mathrm{d}r$$

$$= 4\sqrt{\frac{Bb}{\pi}} \, \mathrm{e}^{-\frac{\mathrm{i}\hbar\kappa^2 t}{2M}} \mathrm{e}^{-2B^2(\kappa - K)^2} \, \mathrm{e}^{-\mathrm{i}\frac{\omega}{2}t} \mathrm{e}^{-2b^2\xi^2}. \quad (2.3)$$

We may now revert to 'local' subsystem variables by means of the transformation $\kappa = k_1 + k_2$, $\xi = (m_1 k_2 - m_2 k_1)/M$ (see equations (2.1)), where k_1 and k_2 are the wave numbers associated with the momentum eigenfunctions of particles 1 and 2, respectively, and the result is

$$\Psi(k_1, k_2, t) = 4\sqrt{\frac{Bb}{\pi}} \, \mathrm{e}^{\frac{-\mathrm{i}\hbar t}{2M}(k_1 + k_2)^2} \mathrm{e}^{-2B^2(k_1 + k_2 - K)^2}$$

$$\times \mathrm{e}^{-\mathrm{i}\frac{\omega}{2}t} \mathrm{e}^{-\frac{2b^2}{M^2}(m_1 k_2 - m_2 k_1)^2}. \quad (2.4)$$

Unlike the description in terms of center of mass and relative variables, equation (2.3), this wavefunction in general does not factor in terms of the particle variables, indicating entanglement of the two particles. Wavefunction factorization in the 'local' variables is, however, not altogether excluded [17], as it does occur at $t = 0$ for the particular choice B_0 of the center of mass wavepacket width parameter given by

$$B^2 \to B_0^2 = \frac{m_1 m_2}{M^2}b^2 = \frac{\mu}{M}b^2. \quad (2.5)$$

While it can be seen here as an artifact of the Gaussian character of the assumed wavefunctions, it provides, on the

other hand, a counter-example disallowing direct association of dynamical correlations (here embodied in the two-particle bound state) to entanglement. The factorized amplitudes of the wavefunction cast in terms of the 'local' variables are in this case

$$\psi_1(k_1) = N_1 e^{-2b^2 \frac{m_2}{M} k_1^2 + 4b^2 \frac{\mu}{M} k_1 K} \quad \text{and}$$

$$\psi_2(k_2) = N_2 e^{-2b^2 \frac{m_1}{M} k_2^2 + 4b^2 \frac{\mu}{M} k_2 K},$$

where N_1 and N_2 are appropriate normalization constants. It is apparent from these expressions that, while the *quantum kinematical* property of amplitude factorization ensures lack of entanglement, each one of the two factor amplitudes involves parameters referring to both particles and has a bearing on their dynamical correlation.

3. Dynamics of entanglement

We next examine the time dependence of the entanglement of the two bound particles under the free dispersive dynamics of their center of mass. For this purpose we first obtain the reduced density matrix for one of the particles using the joint wavefunction (2.4), i.e.

$$\rho^{(1)}(k_1, k_1'; t) \equiv \int dk_2 \, \Psi(k_1, k_2, t) \Psi^*(k_1', k_2, t)$$

$$= \frac{8BbM}{\pi \sqrt{M^2 B^2 + m_1^2 b^2}} e^{-\frac{4K^2 B^2 b^2 m_1^2}{M^2 B^2 + m_1^2 b^2}} e^{-\frac{\hbar^2 t^2}{16(M^2 B^2 + m_1^2 b^2)}(k_1 - k_1')^2}$$

$$\times e^{4KB^2(k_1 - k_1')} e^{\frac{(\mu b^2 - MB^2)^2}{M^2 B^2 + m_1^2 b^2}(k_1 + k_1')^2}$$

$$\times e^{2KB^2 M \left(\frac{\mu b^2 - MB^2}{M^2 B^2 + m_1^2 b^2} \right)(k_1 + k_1')}$$

$$\times e^{-\frac{2}{M^2}(M^2 B^2 + m_2^2 b^2)(k_1^2 + k_1'^2)} e^{-\frac{i\hbar t}{2} \left(\frac{m_1 b^2}{M^2 B^2 + m_1^2 b^2} \right)(k_1^2 - k_1'^2)}.$$
$$(3.1)$$

The complementary reduced density $\rho^{(2)}(k_2, k_2'; t)$ can of course be read off from equation (3.2) simply by replacing all indices 1 by 2 and vice versa. In order to gauge entanglement on the basis of the reduced densities, we evaluate the linear entropy Δ defined as[5]

$$\Delta(t) \equiv 1 - \text{Tr}[\rho^{(1)}(t)]^2$$

$$= 1 - \frac{1}{\sqrt{\frac{\hbar^2 t^2}{16 M^2 B^2 b^2} + \frac{(M^2 B^2 + m_1^2 b^2)(M^2 B^2 + m_2^2 b^2)}{M^4 B^2 b^2}}}$$

$$\equiv 1 - \frac{b}{B} \frac{\tau_B}{\tau} \frac{1}{\sqrt{1 + \frac{t^2}{\tau^2}}}. \qquad (3.2)$$

As shown, the result is a simple analytical expression for the time dependence of the linear entropy. In the last expression, $\tau_B = MB^2/\hbar$ is the characteristic time associated with the dispersive spreading of the center of mass wavepacket, and τ has been introduced as the characteristic

---

[5] The quantity $1 - \Delta(t)$, referred to as *purity*, or its inverse $K = 1/(1 - \Delta(t))$, referred to as *Schmidt number*, are sometimes adopted instead of $\Delta(t)$.
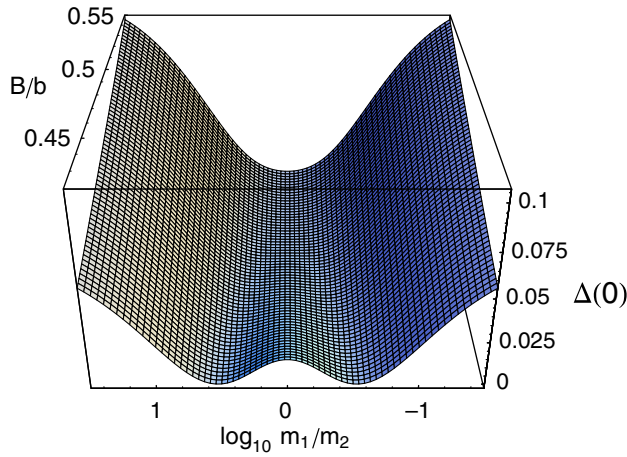
---

time associated with the time dependence of the linear entropy. It is given by

$$\tau \equiv \tau_B \sqrt{\left(1 + \frac{m_1^2 b^2}{M^2 B^2}\right)\left(1 + \frac{m_2^2 b^2}{M^2 B^2}\right)}, \qquad (3.3)$$

an expression which is symmetric under the interchange of indices $1 \leftrightarrow 2$, consistent with the fact that the same result for the linear entropy is obtained if one uses $\rho^{(2)}(k_2, k_2', t)$, instead of $\rho^{(1)}(k_1, k_1', t)$, in its evaluation.

3.1. Independence of center of mass momentum

A first, general feature to be noted in connection with the expression obtained for the linear entropy $\Delta(t)$ is that it turns out to be independent of the mean center of mass momentum K. This indicates Galilean invariance of the dynamics of entanglement in the present case, and is similar to what happens in connection with the dispersive spreading of wavepackets in position space. Actually this feature can be examined in very simple terms by considering the action of the unitary center of mass momentum translation operator

$$G(\zeta) = e^{i\zeta X} = e^{i\zeta \frac{m_1 x_1 + m_2 x_2}{M}} = G_1(\zeta) \otimes G_2(\zeta).$$

The unitary factorization of $G(\zeta)$ results from the fact that the operators x_1 and x_2 commute. The action of this operator on the Schmidt decomposition of the wavefunction for the composite system then gives

$$G(\zeta)\Psi(k_1, k_2, t) = \sum_n \sqrt{\lambda_n} \left(G_1(\zeta)\chi_n^{(1)}(k_1, t) \right)$$
$$\times \left(G_2(\zeta)\chi_n^{(2)}(k_2, t) \right),$$

showing that the amplitudes $\sqrt{\lambda_n}$, are preserved, while the orthonormality of the associated subsystem wavefunctions is preserved by unitarity. The eingenvalues of the reduced densities are therefore preserved and so is, in particular, the value of the linear entropy.

3.2. Initial state entanglement

The linear entropy (3.2), evaluated at $t = 0$, is given in terms of the mass parameters m_1, m_2 and of the Gaussian width parameters b, B as

$$\Delta_0 \equiv \Delta(t = 0) = 1 - \left[\frac{B^2}{b^2} \left(1 - \frac{b^2}{B^2} \frac{m_1/m_2}{(1 + m_i/m_2)^2} \right)^2 + 1 \right]^{-\frac{1}{2}}, \qquad (3.4)$$

which, as shown, depends just on the two dimensionless quantities B/b and m_1/m_2. Their value, supplemented by one additional scale-setting quantity, which can be conveniently taken to be the total mass M, completely determines particular initial states of the assumed form (2.2). Note that the linear entropy is independent of the scale setting, so that its behavior over different domains in the considered manifold of initial states may be mapped by analyzing Δ_0 as a function of B/b and m_1/m_2. A view of the behavior of this function in a relevant domain can be seen as a surface plot in figure 1.

Figure 1. View of a relevant section of the initial linear entropy landscape as a function of $\log_{10} m_1/m_2$ and B/b. The section shown here contains the point $m_1/m_2 = 1$, $B/b = 0.5$ where the valley of $\Delta(t = 0)$ at $m_1/m_2 = 1$ for $B/b > 0.5$ branches into two valleys where $\Delta(0) = 0$, separated by a ridge still at $m_1/m_2 = 1$, for $B/b < 0.5$.

We now discuss some features of this initial entanglement landscape, which can be scrutinized by analyzing equation (3.4) in some detail. First of all, one easily finds that, for any given value of the mass ratio m_1/m_2, the relevant extremum of Δ_0 as a function of B/b occurs for $B/b \rightarrow \sqrt{m_1 m_2}/(1 + m_1/m_2)$. Substitution back into the expression for Δ_0 shows that one has at these points $\Delta_0 = 0$. We thus re-obtain, in the guise of the vanishing of the initial value of the linear entropy, the parametric conditions for the factorization of the initial wavefunction (see equation (2.5)) when cast in terms of particle variables.

All the possible values of B/b associated with this set of unentangled initial states lie in the interval $0 \leqslant B/b \leqslant 0.5$, as follows from the range available for m_1/m_2, namely $0 \leqslant m_1/m_2 < \infty$. They are therefore confined to a front vertical slice of figure 1. In order to explore the behavior of the initial value of the linear entropy for $B/b > 0.5$ one may fix the value of this variable and study the dependence of the initial linear entropy on m_1/m_2. What one finds in this way is that there is an extremum at $m_1/m_2 = 1$ for all possible values of B/b, in addition to two other extrema with m_1/m_2 values given by

$$\left(\frac{m_1}{m_2}\right)_{\pm} = \frac{b^2}{2B^2}(1 \pm \sqrt{1 - 4B^2/b^2}) - 1,$$

which are real, non-negative numbers only for $B/b \leqslant 0.5$. It turns out that the extremum at $m_1/m_2 = 1$ is in fact a (non-vanishing) minimum of the initial linear entropy for $B/b > 0.5$, which however becomes a maximum for $B/b < 0.5$. In this range, the additional extrema are the minima for which $\Delta(0) = 0$, which have already been identified earlier. The fact that there are *two* such minima is due to the symmetry of the linear entropy under interchange of the masses m_1 and m_2. As a check on this, one can easily verify that $(m_1/m_2)_{-}(m_1/m_2)_{+} = 1$ for all values of B/b. At $B/b = 0.5$, $\Delta(0)$ has a very flat minimum (a zero of the fourth order) as a function of m_1/m_2 at $m_1/m_2 = 1$. These features are displayed in figure 1, which covers the relevant domain of the initial linear entropy landscape, namely $0.031 \leqslant m_1/m_2 \leqslant 31$

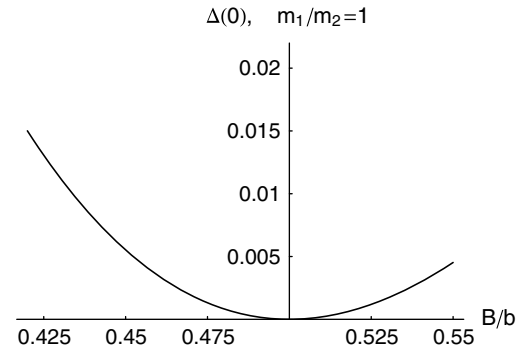

Figure 2. A cut of the surface shown in figure 1 at the plane $\alpha = m_1/m_2 = 1$, showing values of the maxima (for $B/b < 0.5$) and minima (for $B/b > 0.5$) of the initial linear entropy. The plotted range of B/b is the same as in figure 1.

and $0.42 \leqslant B/b \leqslant 0.55$. A cut of this surface graph, showing the values of $\Delta(0)$ at $m_1/m_2 = 1$ as a function of B/b, is shown in figure 2. In this graph the ranges $B/b < 0.5$ and $B/b > 0.5$ correspond, respectively, to a ridge and to a valley in the $\Delta(0)$ surface.

3.3. Time evolution of entanglement

The time dependence of the linear entropy measure of particle–particle entanglement consists in a simple, monotonic approach to the upper bound $\Delta = 1$ as seen in equation (3.2). It involves an algebraic expression which is of the same form as that which governs the dispersive spreading of the center of mass wavepacket albeit involving a time scale $\tau \neq \tau_B$. Since τ_B is the relevant time scale in the unitary dynamical evolution of the initial state of the composite system, it is both natural and convenient to use this time scale to analyze the dependence of the characteristic time τ of the dynamics of entanglement on the different initial states of the form (2.2).

The quantity τ/τ_B is found from equation (3.3) to be closely related to the initial state value of the linear entropy (3.4). In particular, it is also independent of the scale setting parameter M, and can be written explicitly as

$$\frac{\tau}{\tau_B} = \frac{b/B}{1 - \Delta(0)} = \left[\left(1 - \frac{b^2}{B^2}\frac{m_1/m_2}{(1 + m_1/m_2)^2}\right)^2 + \frac{b^2}{B^2}\right]^{\frac{1}{2}}.$$

(3.5)

This implies that the main general features of the $\Delta(0)$ landscape are essentially carried over to the τ/τ_B landscape, as can in fact be seen in figure 3. One finds here the same overall structure of valleys and ridges seen in the corresponding initial linear entropy plot, in figure 1. As a consequence of inversion and of an additional b/B factor, however, one sees that τ/τ_B is bounded by unity *below*, while $\Delta(0)$ is similarly bounded *above*. For sufficiently large values of b/B the characteristic time for the evolution of entanglement becomes substantially longer than τ_B.

Note, however, that more tightly localized center of mass wavepackets (smaller values of B) spread faster (have smaller values of τ_B), so that this does not necessarily imply very long 'absolute' times for entanglement evolution. In fact, for given
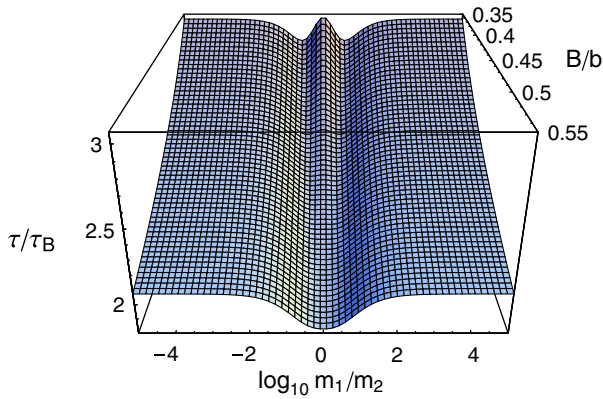
Figure 3. Closer view of the valley bifurcation at $m_1/m_2 = 1$, $B/b = 0.5$ in the τ/τ_B landscape. Compare with figure 1.

M and m_1/m_2, the limit of very large b/B in equation (3.5) gives

$$\lim_{b/B \to \infty} \tau = \frac{\mu b^2}{\hbar},$$

which is just the inverse of the oscillator frequency ω.

4. Concluding remarks

We have examined in some detail the quantum entanglement of two particles in one dimension whose center of mass evolves as a free Gaussian wavepacket with mean momentum $\hbar K$, when they are bound in the ground state of a two-body harmonic interaction potential. While the occurrence of entanglement is of no surprise in view of the strong mutual dynamical correlations (e.g. in position space) of the two particles [18], one can easily see in this case that the entanglement may nevertheless be made to vanish in the initial state by an appropriate choice of the width parameter of the center of mass wavepacket. The fact that dynamically correlated, e.g. bound, subsystems can, at least in this case, be represented in unentangled form (which means being represented by factorized amplitudes) indicates that the two concepts are in fact distinct.

Given the stationary character of the bound state of the two particles, the dynamics of their entanglement is driven by the unitary, dispersive spreading of the center of mass wavepacket. In particular, it comes out as independent of the mean value of the center of mass momentum, a fact which can be understood in terms of the preservation of the coefficients in the Schmidt decomposition of the two-particle wavefunction under center of mass boosts. The presence and the properties of a two-particle bound state still plays an essential role in the dynamics of the entanglement of the 'local' particle variables, however. This may be inferred directly from the fact that entanglement becomes time independent if the binding harmonic potential is turned off, since in this case time evolution amounts to unitary local operations only [17]. Furthermore, the characteristic time τ associated with the entanglement dynamics depends on the bound state parameters. It has been shown, in particular, that the quadratic decrease, as $B \to 0$, of the characteristic time τ_B for wavepacket spreading is not paralleled by

the characteristic time for the evolution of entanglement, which approaches a finite limit involving properties of the two-particle bound state.

One must also note that the monotonic *increase* in time of both the spreading of the initial width B of the center of mass wavepacket and of the linear entropy Δ used to measure the degree of particle–particle entanglement hinge on the particular choice made of the initial state. A still particular, but different choice also leading to different behavior may be devised by taking advantage of the time-reversible character of the overall unitary dynamics. In fact, one may use as an alternative initial state $\tilde{\Psi}(k_1, k_2; 0)$; the time reversed counterpart of the state resulting from our adopted initial state (2.2) after it evolves for a time T, i.e.

$$\tilde{\Psi}(k_1, k_2; 0) \equiv \Psi^*(-k_1, -k_2; T).$$

Due to time-reversal invariance of the overall dynamics, the time evolution of this state is given as $\tilde{\Psi}(k_1, k_2; t) = \Psi^*(-k_1, -k_2; T - t)$, so that the time evolution from $\Psi(k_1, k_2; 0)$ to $\Psi(k_1, k_2, T)$ is traced backwards in time. As a result, the center of mass wavepacket will shrink and the linear entropy will decrease as t increases from 0 to T. In particular, if $\Psi(k_1, k_2; 0)$ is a particle-factorizable initial state, the linear entropy decreases smoothly from a non-zero value to zero in the finite time T, as in the cases of the so-called entanglement sudden death [19], but immediately and also smoothly rebounds to a monotonic increase for subsequent times.

Finally, it is worth stressing that our simple model system accounts for instances of quantum states consisting of parts which are strongly correlated dynamically but which are nevertheless *not entangled*. This is, in a way, complementary to the situation treated by Schrödinger [20], namely one in which entanglement persists after interaction between the two parts has ceased. Since entanglement correlations establish at the level of amplitudes, both cases seem to point to a picture in which entanglement is ultimately a quantum kinematical feature even if circumstantially affected by quantum dynamical processes.

Acknowledgments

References

[1] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)
[2] Preskill J 1998 Lecture Notes for quantum computation www.theory.caltech.edu/people/preskill/ph229/#lecture
[3] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 *Rev. Mod. Phys.* **81** 865
[4] Bell J 1987 *Speakable and Unspeakable in Quantum Mechanics* (Cambridge: Cambridge University Press)
[5] Einstein A, Podolski B and Rosen N 1935 *Phys. Rev.* **47** 777
[6] Aspect A, Grangier P and Roger G 1981 *Phys. Rev. Lett.* **47** 460
[7] Mintert F, Carvalho A R R, Kus M and Buchleitner A 2005 *Phys. Rep.* **415** 207

[8] Werner R F 2001 Quantum information theory—an invitation *Quantum Information* (*Springer Tracts in Modern Physics* vol 173) (Berlin: Springer)

[9] Konrad T, de Melo F, Tiersch M, Kasztelan C, Aragão A and Buchenleitner A 2008 *Nature Phys.* **4** 99

[10] Jiménez Farías O, Lombard Latune C, Walborn S P, Davidovich L and Souto Ribeiro P H 2009 *Science* **324** 1414

[11] Law C K 2004 *Phys. Rev.* A **70** 062311

[12] Buscemi F, Bordone P and Bertoni A 2006 *Phys. Rev.* A **73** 052312

[13] Harshman N L and Hutton G 2008 *Phys. Rev.* A **77** 042310

[14] Wang J, Law C K and Chu M-C 2005 *Phys. Rev.* A **72** 022346

[15] Sun B, Zhou D L and You L 2006 *Phys. Rev.* A **73** 012336

[16] Mack H and Freyberger M 2002 *Phys. Rev.* A **66** 042113
Bußhardt M and Freyberger M 2007 *Phys. Rev.* A **75** 052101

[17] Chan K W and Eberly J H 2004 arXiv:quant-ph/0404093v2

[18] Tommasini P, Timmermans E and de Toledo Piza A F R 1998 *Am. J. Phys.* **66** 881

[19] Yu T and Eberly J H 2004 *Phys. Rev. Lett.* **93** 140404
Yu T and Eberly J H 2009 *Science* **323** 598

[20] Schrödinger E 1935 *Proc. Camb. Phil. Soc.* **31** 555