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LINEAR CALIBRATION IN MULTIPLICATIVE MEASUREMENT ERROR MODELS

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Summary

In this paper, we consider linear calibration in the functional multiplicative measurement error model. Classical and inverse type estimators are proposed. First order approximations are obtained for the bias and mean square erros of the estimators considered, from which interesting comparisons result. A simulation study seems to corroborate the expressions obtained and also to indicate in conjunction with the theoretical results that the classical estimator is the one that should be prefered.

1. Introduction

Linear calibration studies under ordinary regression setups have been the subject of several important studies reported in the literature. A summary of the most important results is presented in Brown (1993). Shukla (1972) obtained first order approximations for the classical and inverse estimators under the ordinary regression set up and concluded that the classical estimator seems to present an overall better large sample behavior. The multiplicative measurement error model (Hwang, 1986) can be represented by the equations

$$y_i = \alpha + \beta x_i + e_i, \tag{1.1}$$

$$X_i = x_i u_i, \tag{1.2}$$

$i = 1, \ldots, n$. The errors e_i and u_i are independent with $E[e_i] = 0$, $E[u_i] = 1$, $E[e_i^2] = \mu_e^{(2)}$ and $E[u_i^2] = \mu_u^{(2)}$, which is assumed to be known, $i = 1, \ldots, n$. The multiplicative functional model follows when the x_i are considered to be fixed unknown quantities. Otherwise, if x_i, $i = 1, \ldots, n$, are random quantities then the structural model follows. Hwang (1986) considers consistent estimation of α and β under the structural multiplicative model and also some central limit theorems related to the estimators considered. In this paper, we consider linear calibration under the multiplicative functional model. The calibration experiment consists of two stages. In the first, the calibration stage, based on a sample of size n, estimators of α and β are obtained according to the multiplicative model described by equations (1.1) and (1.2). In a second stage a sample of size k is obtained, corresponding to an unknown value x_0, that is,

$$y_{0j} = \alpha + \beta x_0 + e_{0j}, \tag{1.3}$$

$j = 1, \ldots, k$, where e_{0j} are independent and also independent of u_i, $i = 1, \ldots, n$, $E[e_{0j}] = 0$ and $E[e_{0j}^2] = \mu_e^{(2)}$ is finite, $j = 1, \ldots, k$. The main interest is on estimating the unknown x_0.

In this paper, under the model specified by equations (1.1)-(1.3), classical and inverse type estimators are considered. The estimators considered are based on the ordinary least squares estimators, which are not consistent, and also on consistent estimators of β. First order approximations for the bias and mean squared error of the estimators considered are obtained. The expressions obtained allow the comparisons of the estimators, with interesting conclusions. The results indicate that the classical estimator based on the consistent estimator of β seems to present an overall best asymptotic behavior. The parameter α is considered known, and taken to be equal zero, without loss of generality. The general case with α unknown can be treated similarly, requiring just additional tecnical details. The outline of the paper is as follows. Section 2 discusses the functional multiplicative model. Inverse and classical estimators based on consistent and least squares estimators are considered. Conditions for strong consistency of the estimators considered are studied. First order approximation for the bias and mean squared error of the estimators of α and β are obtained, which allows derivation of first order approximation for the estimators considered for x_0. Section 3 presents a simulation study where it is studied the behavior of the bias and mean squared error derived in Section 2. Section 3 presents a simulation study aimed at verifying the accurateness of the expressions derived in Section 2 for the bias and mean squared error of the estimators considered.

2. Estimators and properties

In this section, classical and inverse estimators of x_0 are considered and some of its properties like asymptotic bias and mean squared error (MSE) are studied under the functional multiplicative model defined in Equations (1.1)-(1.3). The classical and inverse estimators of x_0 considered are represented as

$$\hat{x} = \frac{\bar{y}_0}{\hat{\beta}} \quad \text{and} \quad \tilde{x} = \hat{\phi}\bar{y}_0,$$

where $\hat{\beta}$ and $\hat{\phi}$ are estimators of β and ϕ, with $\phi = 1/\beta$ arising in the inverse regression seting and $\bar{y}_0 = \sum_{j=1}^{k} y_{0j}/n$. Motivations for considering inverse type estimators are given in Krutchkoff (1969). The least squares estimators of β and ϕ are defined by

$$\tilde{\beta} = \frac{\sum_{i=1}^{n} X_i y_i}{\sum_{i=1}^{n} X_i^2} \quad \text{and} \quad \tilde{\phi} = \frac{\sum_{i=1}^{n} X_i y_i}{\sum_{i=1}^{n} y_i^2}.$$

In the following we assume that the sequence $\{x_i\}_{i\geq 1}$ satisfy the following conditions:

$$(2.1)\qquad \lim_{n\to\infty} \bar{x} = \mu_x < \infty, \quad \lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i^2}{n} = \mu_x^{(2)} < \infty \quad \text{and} \quad \lim_{n\to\infty} \sum_{i=1}^{n} \frac{x_i^4}{n} = \mu_x^{(4)} < \infty,$$

with $\bar{x} = \sum_{i=1}^{n} x_i/n$. Moreover, it is considered that the population moments $\mu_u^{(4)} = E[u_i^4]$ and $\mu_e^{(4)} = E[e_i^{(4)}]$ are finite. The next lemma presents a consistent estimator of β based on the least squares estimator $\tilde{\beta}$.

Lemma 2.1. *Under the calibration model (1.1)-(1.3) with the assumption (2.1), and with finite fourth moments $\mu_u^{(4)}$ and $\mu_e^{(4)}$, it follows that $\tilde{\beta}_c = \mu_u^{(2)}\tilde{\beta}$, is a strongly consistent estimator of β.*

Proof. Notice that $X_1^2, X_2^2, \ldots$, are independent random variables with $E[X_i^2] = \mu_u^{(2)} x_i^2$ and $Var[X_i^2] = x_i^4(\mu_u^{(4)} - (\mu_u^{(2)})^2)$, $i = 1, 2, \ldots$, which is finite according to the above assumptions and implies that $\sum_{n=1}^{n} Var[X_n]/n^2$ is finite. By using the Kolmogorov's strong law of large numbers it follows that $\sum_{i=1}^{n} \frac{X_i^2}{n} - \mu_u^{(2)} \sum_{i=1}^{n} \frac{x_i^2}{n} \to 0$, almost surely. Thus, $\sum_{i=1}^{n} \frac{X_i^2}{n} \to \mu_u^{(2)}\mu_2^{(2)}$, almost surely. Similarly, it can be shown that $\sum_{i=1}^{n} \frac{X_i y_i}{n} \longrightarrow \beta\mu_x^{(2)}$. The result follows because $\tilde{\beta}_c$ is a continuous function of $\sum_{i=1}^{n} X_i^2/n$ and $\sum_{i=1} X_i y_i/n$.

Two consistent estimators of ϕ are considered next.

Lemma 2.2. *Under model (1.1), (1.2) and assumption (2.1), with $\mu_u^{(2)}$ and $\mu_e^{(2)}$ known, it follows that*

$$\hat{\phi} = \frac{\sum_{i=1}^{n} X_i y_i/n}{\sum_{i=1}^{n} y_i^2 - \mu_e^{(2)}} \quad \text{and} \quad \hat{\phi}^* = \frac{\sum_{i=1}^{n} X_i^2}{\mu_u^{(2)} \sum_{i=1}^{n} X_i y_i},$$

are strongly consistent estimators of ϕ.

The proof is similar to that of Lemma 2.1. Moreover, notice that $\hat{\phi}^* = 1/\tilde{\beta}_c$, so that

$$(2.2)\qquad \hat{x} = \frac{\bar{y}_0}{\tilde{\beta}_c} = \hat{\phi}^*\bar{y}_0 = \tilde{x},$$

meaning that in this case the classical and inverse estimators coincide. By considering the estimators of β and ϕ defined above the following estimators of x_0 are considered:

$$\hat{x}_c = \frac{\bar{y}_0}{\tilde{\beta}_c}, \qquad \hat{x}_L = \frac{\bar{y}_0}{\tilde{\beta}},$$

$$\tilde{x}_c = \hat{\phi}\bar{y}_0 \quad \text{and} \quad \tilde{x}_L = \tilde{\phi}\bar{y}_0.$$

According to (2.2), there is no need to consider $\tilde{x} = \hat{\phi}^*\bar{y}_0$ because it coincides with $\hat{x}_c$. Notice that $\hat{x}_c$ and $\tilde{x}_c$ are the classical and inverse estimators of x_0 combined with the consistent estimators of β and ϕ, respectively. On the other hand, $\hat{x}_L$ and $\tilde{x}_L$ are the classical and inverse estimators of x combined with the least squares estimators of β

and ϕ, respectively. In the following lemma, first order approximations are obtained for the expected value and variance of the least squares and consistent estimators of β. Davies and Hulton (1975) report results of studies on the asymptotic bias and MSE of the least squares estimators $\tilde{\beta}$ under additive models. However the study presented in that paper deals only with the leading terms of the approximations, without specifying the orders involved. We point out that the approximations obtained are independent of the assumtions (2.1).

Lemma 2.3. *Under the model (1.1) and (1.2), with $\mu_e^{(i)}$ and $\mu_u^{(i)}$ finite, $1 \le i \le 4$, and $\mu_u^{(2)}$ known, it follows that*

(i) $$E(\tilde{\beta}) = \frac{\beta}{\mu_u^{(2)}} + \frac{\beta\left(\sum_{i=1}^n x_i^4/n\right)}{n(\mu_u^{(2)})^2\left(\sum_{i=1}^n x_i^2/n\right)^2}\left(\frac{\mu_u^{(4)}}{\mu_u^{(2)}} - \mu_u^{(3)}\right) + O(n^{-2})$$

and

$$\mathrm{Var}(\tilde{\beta}) = \frac{\mu_e^{(2)}}{n\mu_u^{(2)}\left(\sum_{i=1}^n x_i/n\right)} + \frac{\beta^2\left(\sum_{i=1}^n x_i^4/n\right)}{n\mu_u^{(2)}\left(\sum_{i=1}^n x_i^2/n\right)^2}\left[1 + \frac{\mu_u^{(4)}}{(\mu_u^{(2)})^3} - \frac{2\mu_u^{(3)}}{(\mu_u^{(2)})^2}\right] + O(n^{-2}),$$

(ii) $$E(\hat{\beta}) = \beta + \frac{\beta\left(\sum_{i=1}^n x_i^4/n\right)}{n\mu_u^{(2)}\left(\sum_{i=1}^n x_i^2/n\right)^2}\left(\frac{\mu_u^{(4)}}{\mu_u^{(2)}} - \mu_u^{(3)}\right) + O(n^{-2})$$

and

$$\mathrm{Var}(\hat{\beta}) = \frac{\mu_u^{(2)}\mu_e^{(2)}}{n\left(\sum_{i=1}^n x_i^2/n\right)} + \frac{\beta^2\mu_u^{(2)}\left(\sum_{i=1}^n x_i^4/n\right)}{n\left(\sum_{i=1}^n x_i^2/n\right)^2}\left[1 + \frac{\mu_u^{(4)}}{(\mu_u^{(2)})^3} - \frac{2\mu_u^{(3)}}{(\mu_u^{(2)})^2}\right] + O(n^{-2}).$$

Proof. Notice that $\hat{\beta}_Q$ is a continuous function of $(\sum_{i=1}^n X_i^2, \sum_{i=1}^n X_i y_i)$. Taking the expectation and the variance of a Taylor series expansion about the point $\mathbf{a} = (E(\sum_{i=1}^n X_i^2), E[\sum_{i=1}^n X_i y_i))$, it follows that

$$E(\hat{\beta}) = \frac{\beta}{\mu_u^{(2)}} + \frac{\beta}{n^2(\mu_u^{(2)})^3\left(\sum_{i=1}^n x_i^2/n\right)^2}\mathrm{Var}(\ell_2) - \frac{1}{n^2(\mu_u^{(2)})^2\left(\sum_{i=1}^n x_i^2/n\right)^2}\mathrm{Cov}(\ell_1, \ell_2)$$

$$+O(n^{-2}) \tag{2.3}$$

and

$$\mathrm{Var}(\hat{\beta}) = \frac{1}{n^2(\mu_u^{(2)})^2\left(\sum_{i=1}^n x_i^2/n\right)^2}\mathrm{Var}(\ell_1) + \frac{\beta^2}{n^2(\mu_u^{(2)})^4\left(\sum_{i=1}^n x_i^2/n\right)^2}\mathrm{Var}(\ell_2)$$

$$-\frac{2\beta\mathrm{Cov}(\ell_1, \ell_2)}{n^2(\mu_u^{(2)})^3\left(\sum_{i=1}^n x_i^2/n\right)^2} + O(n^{-2}), \tag{2.4}$$

where $\ell_1 = \sum_{i=1}^n X_i y_i$ and $\ell_2 = \sum_{i=1}^n X_i^2$. It can be shown that

$$E(\ell_1) = n\beta\left(\sum_{i=1}^n x_i^2/n\right), \tag{2.5}$$

(2.6) $$E(\ell_2) = n\mu_u^{(2)} \left(\sum_{i=1}^{n} x_i^2/n\right),$$

(2.7) $$\mathrm{Var}(\ell_1) = n\beta^2\mu_u^{(2)} \left(\sum_{i=1}^{n} x_i^4/n\right) - n\beta^2 \left(\sum_{i=1}^{n} x_i^4/n\right) + n\mu_e^{(2)}\mu_u^{(2)} \left(\sum_{i=1}^{n} x_i^2/n\right),$$

(2.8) $$\mathrm{Var}(\ell_2) = n(\mu_u^{(4)} - (\mu_u^{(2)})^2) \left(\sum_{i=1}^{n} x_i^4/n\right)$$

and

(2.9) $$\mathrm{Cov}(\ell_1, \ell_2) = n\beta(\mu_u^{(3)} - \mu_u^{(2)}) \left(\sum_{i=1}^{n} x_i^4/n\right).$$

Replacing (2.5) - (2.9) in expressions (2.3) and (2.4), the results follow after ordinary algebraic manipulations.

Results about estimators of ϕ are presented next. The proof is similar to the one presented in Lemma 2.3.

Lemma 2.4. *Under the model (1.1) and(1.2) with the assumption that $\mu_e^{(i)}$ and$\mu_u^{(i)}$, $1 \leq i \leq 4$, exist and $\mu_e^{(2)}$ known, it follows that*

(i) $$E(\widehat{\phi}_L) = \frac{\beta\left(\sum_{i=1}^{n} x_i^2/n\right)}{d} + \frac{\beta\left(\sum_{i=1}^{n} x_i^2/n\right)}{nd^3}[\mu_e^{(4)} - 3(\mu_e^{(2)})^2] + \frac{2\beta^3\mu_e^{(2)}\left(\sum_{i=1}^{n} x_i^2/n\right)}{nd^3}$$

$$+\frac{\mu_e^{(3)}\bar{x}}{nd^3}\left[3\beta^2\left(\sum_{i=1}^{n} x_i^2/n\right) - \mu_e^{(2)}\right] + O(n^{-2}),$$

and

$$\mathrm{Var}(\widehat{\phi}_L) = \frac{\beta^2(\mu_u^{(2)} - 1)\left(\sum_{i=1}^{n} x_i^4/n\right) + \mu_e^{(2)}\mu_u^{(2)}\left(\sum_{i=1}^{n} x_i^2/n\right)}{nd^2}$$

$$+\frac{\beta^2(\mu_e^{(4)} - 5(\mu_e^{(2)})^2)\left(\sum_{i=1}^{n} x_i^2/n\right)^2}{nd^3} + \frac{2\beta\mu_e^{(3)}\bar{x}\left(\sum_{i=1}^{n} x_i^2/n\right)\left[\left(\sum_{i=1}^{n} x_i^2/n\right) - \mu_e^{(2)}\right]}{nd^4}$$

$$+O(n^{-2}),$$

(ii) $$E(\widehat{\phi}_C) = \frac{1}{\beta} + \frac{2\mu_e^{(2)}}{n\beta^3\left(\sum_{i=1}^{n} x_i^2/n\right)} + \frac{3\mu_e^{(3)}\bar{x}}{n\beta^4\left(\sum_{i=1}^{n} x_i^2/n\right)^2} + \frac{(\mu_e^{(4)} - (\mu_e^{(2)})^2)}{n\beta^5\left(\sum_{i=1}^{n} x_i^2/n\right)^2} + O(n^{-2})$$

and

$$\mathrm{Var}(\widehat{\phi}_C) = \frac{\left(\sum_{i=1}^{n} x_i^4/n\right)}{n\beta^2\left(\sum_{i=1}^{n} x_i^2/n\right)^2}(\mu_u^{(2)} - 1) + \frac{\mu_e^{(2)}\mu_u^{(2)}}{n\beta^4\left(\sum_{i=1}^{n} x_i^2/n\right)} + \frac{2\mu_e^{(3)}\bar{x}}{n\beta^5\left(\sum_{i=1}^{n} x_i^2/n\right)^2}$$

$$+\frac{(\mu_e^{(4)} - (\mu_e^{(2)})^2)}{n\beta^6\left(\sum_{i=1}^{n} x_i^2/n\right)^2} + O(n^{-2}),$$

where

$$d = \beta^2 \left(\sum_{i=1}^{n} x_i^2/n \right) + \mu_e^{(2)}. \tag{2.10}$$

Proof. A proof is provided for (i) only. The other results can be proved similarly. Notice that $\widehat{\phi}_L$ is a function of $(\sum_{i=1}^n y_i^2, \sum_{i=1}^n X_i y_i)$. Taking expectation and variance of a Taylor series expansion of $\tilde{\phi}$ about $\mathbf{a} = (E(\sum_{i=1}^n y_i^2), E(\sum_{i=1}^n X_i y_i))$, it follows that

$$E(\widehat{\phi}_L) = \frac{\beta(\sum_{i=1}^n x_i^2/n)}{d} + \frac{\beta(\sum_{i=1}^n x_i^2/n)}{n^2 d^3}\mathrm{Var}(\ell_3) - \frac{\mathrm{Cov}(\ell_1, \ell_3)}{n^2 d^2} + O(n^{-2})$$

and

$$\mathrm{Var}(\widehat{\phi}_L) = \frac{1}{n^2 d^2}\mathrm{Var}(\ell_1) + \frac{\beta^2 (\sum_{i=1}^n x_i^2/n)^2}{n^2 d^4}\mathrm{Var}(\ell_3) - \frac{2\beta(\sum_{i=1}^n x_i^2/n)\,\mathrm{Cov}(\ell_1, \ell_3)}{n^2 d^3} + O(n^{-2}),$$

where ℓ_1 is as in Lemma 2.3 and $\ell_3 = \sum_{i=1} y_i^2$.

It is easy to show that

$$E(\ell_3) = nd, \tag{2.11}$$

$$\mathrm{Var}(\ell_3) = 4n\beta^2\mu_e^{(2)} \left(\sum_{i=1}^{n} x_i^2/n \right) + 4\beta\mu_e^{(3)}\bar{x} + n\mu_e^{(4)} - n(\mu_e^{(2)})^2, \tag{2.12}$$

$$\mathrm{Cov}(\ell_1, \ell_3) = 2n\beta\mu_e^{(2)} \left(\sum_{i=1}^{n} x_i^2/n \right) + n\mu_e^{(3)}\bar{x}. \tag{2.13}$$

The result follows by replacing (2.11)-(2.13) in the above expressions for the expectation and variance of $\hat{\phi}$.

Theorem 2.1. *Under model (1.1)-(1.3) with the assumptions that $\mu_e^{(i)}$ and $\mu_u^{(i)}$, $1 \le i \le 4$, are finite with $\mu_u^{(2)}$ known, it follows that*

(i) $$E(\hat{x}_L) = \mu_u^{(2)} x_0 + \frac{x_0 \mu_e^{(2)} (\mu_u^{(2)})^2}{n\beta^2 (\sum_{i=1}^n x_i^2/n)} + \frac{x_0 (\sum_{i=1}^n x_i^4/n)}{n(\sum_{i=1}^n x_i^2/n)^2}((\mu_u^{(2)})^2 - \mu_u^{(3)}) + O(n^{-2})$$

and

$$\mathrm{Var}(\hat{x}_L) = \frac{\mu_e^{(2)} (\mu_u^{(2)})^2}{k\beta^2} + \frac{3(\mu_u^{(2)})^3 (\mu_e^{(2)})^2}{nk\beta^4 (\sum_{i=1}^n x_i^2/n)} + \frac{3(\mu_u^{(2)})^3 \mu_e^{(2)} (\sum_{i=1}^n x_i^4/n)}{nk\beta^2 (\sum_{i=1}^n x_i^2/n)^2}$$

$$+ \frac{\mu_u^{(2)} \mu_e^{(2)} (\sum_{i=1}^n x_i^4/n)}{nk\beta^2 (\sum_{i=1}^n x_i^2/n)^2} \left(\frac{\mu_u^{(4)}}{\mu_u^{(2)}} - 4\mu_u^{(3)} \right) + \frac{x_0^2 (\mu_u^{(2)})^3 \mu_e^{(2)}}{n\beta^2 (\sum_{i=1}^n x_i^2/n)}$$

$$+ \frac{x_0^2 (\mu_u^{(2)})^3 (\sum_{i=1}^n x_i^4/n)}{n(\sum_{i=1}^n x_i^2/n)^2} \left[1 + \frac{\mu_u^{(4)}}{(\mu_u^{(2)})^3} - \frac{2\mu_u^{(3)}}{(\mu_u^{(2)})^2} \right] + O(n^{-2});$$

(ii) $E(\hat{x}_c) = x_0 + \dfrac{x_0\mu_u^{(2)}\mu_e^{(2)}}{n\beta^2\left(\sum_{i=1}^n x_i^2/n\right)} + \dfrac{x_0\left(\sum_{i=1}^n x_i^4/n\right)}{n\mu_u^{(2)}\left(\sum_{i=1}^n x_i^2/n\right)^2}((\mu_u^{(2)})^2 - \mu_u^{(3)}) + O(n^{-2})$

and

$$\mathrm{Var}(\hat{x}_c) = \frac{\mu_e^{(2)}}{k\beta^2} + \frac{3\mu_u^{(2)}(\mu_e^{(2)})^2}{nk\beta^4\left(\sum_{i=1}^n x_i^2/n\right)} + \frac{x_0^2\mu_u^{(2)}\mu_e^{(2)}}{n\beta^2\left(\sum_{i=1}^n x_i^2/n\right)}$$

$$+\frac{\mu_e^{(2)}\left(\sum_{i=1}^n x_i^4/n\right)}{nk\beta^2\mu_u^{(2)}\left(\sum_{i=1}^n x_i^2/n\right)^2}\left(\frac{\mu_u^{(4)}}{\mu_u^{(2)}} - 4\mu_u^{(3)} + 3(\mu_u^{(2)})^2\right)$$

$$+\frac{x_0^2\mu_u^{(2)}\left(\sum_{i=1}^n x_i^4/n\right)}{n\left(\sum_{i=1}^n x_i^2/n\right)^2}\left[1 + \frac{\mu_u^{(4)}}{(\mu_u^{(2)})^3} - \frac{2\mu_u^{(3)}}{(\mu_u^{(2)})^2}\right] + O(n^{-2}). \tag{2.14}$$

Proof. The prove is given to (i) only. Notice that

$$E(\hat{x}_L) = E\left(\frac{\bar{Y}_0}{\tilde{\beta}}\right) = E(\bar{Y}_0)E\left(\frac{1}{\tilde{\beta}}\right) = \beta x_0 E\left(\frac{1}{\tilde{\beta}}\right).$$

A Taylor series expansion of $1/\tilde{\beta}$ allows to write

$$E(\hat{x}_L) = \beta x_0\left[\frac{\mu_u^{(2)}}{\beta} - \frac{\left(\sum_{i=1}^n x_i^4/n\right)}{n\beta\left(\sum_{i=1}^n x_i^2/n\right)^2}\left(\frac{\mu_u^{(4)}}{\mu_u^{(2)}} - \mu_u^{(3)}\right) + \frac{(\mu_u^{(2)})^3\mathrm{Var}(\tilde{\beta})}{\beta^3}\right] + O(n^{-2}).$$

Replacing $\mathrm{Var}(\tilde{\beta})$ given in Lemma 2.3 in the above expression, we arrive at the expectation of $\hat{x}_L$.

Moreover, algebraic manipulations yields

$$\begin{aligned}
\mathrm{Var}(\hat{x}_L) &= \mathrm{Var}\left(\frac{\bar{Y}_0}{\tilde{\beta}}\right) = E\left(\frac{\bar{Y}_0^2}{\tilde{\beta}^2}\right) - E^2(\bar{Y}_0)E^2\left(\frac{1}{\tilde{\beta}}\right) \\
&= \left(\beta^2x_0^2 + \frac{\sigma_\varepsilon^2}{k}\right)E\left(\frac{1}{\tilde{\beta}^2}\right) - \beta^2x_0^2E^2\left(\frac{1}{\tilde{\beta}}\right) \\
&= \frac{\sigma_\varepsilon^2}{k}E\left(\frac{1}{\tilde{\beta}^2}\right) + \beta^2x_0^2\mathrm{Var}\left(\frac{1}{\tilde{\beta}}\right).
\end{aligned}$$

Using the results in Lemma 2.3, we can write

$$\begin{aligned}
E\left(\frac{1}{\tilde{\beta}^2}\right) &= \frac{1}{E^2(\tilde{\beta})} + \frac{3\mathrm{Var}(\tilde{\beta})}{E^4(\tilde{\beta})} + O(n^{-2}) \\
&= \frac{(\mu_u^{(2)})^2}{\beta^2} - \frac{2\mu_u^{(2)}\left(\sum_{i=1}^n x_i^4/n\right)}{n\beta^2\left(\sum_{i=1}^n x_i^2/n\right)^2}\left(\frac{\mu_u^{(4)}}{\mu_u^{(2)}} - \mu_u^{(3)}\right) + \frac{3(\mu_u^{(2)})^4}{\beta^4}\mathrm{Var}(\tilde{\beta}) + O(n^{-2})
\end{aligned}$$

and

$$\mathrm{Var}\left(\frac{1}{\tilde{\beta}}\right) = \frac{\mathrm{Var}(\tilde{\beta})}{E^4(\tilde{\beta})} + O(n^{-2}) = \frac{(\mu_u^{(2)})^4\mathrm{Var}(\tilde{\beta})}{\beta^4} + O(n^{-2}).$$

Thus,

$$\mathrm{Var}(\hat{x}_L) = \frac{\mu_e^{(2)}(\mu_u^{(2)})^2}{k\beta^2} - \frac{2\mu_e^{(2)}\mu_u^{(2)}\left(\sum_{i=1}^n x_i^4/n\right)}{nk\beta^2\left(\sum_{i=1}^n x_i^2/n\right)^2}\left(\frac{\mu_u^{(4)}}{\mu_u^{(2)}} - \mu_u^{(3)}\right)$$

$$+\frac{3\mu_e^{(2)}(\mu_u^{(2)})^4}{k\beta^4}\mathrm{Var}(\tilde{\beta}) + \frac{x_0^2(\mu_u^{(2)})^4\mathrm{Var}(\tilde{\beta})}{\beta^2} + O(n^{-2}).$$

The result follows now from Lemma 2.3.

Considering the results in the above theorem, the mean squared error of the estimators can be obtained by adding the variance of the estimator to the square of the bias. Because the bias of the estimator $\hat{x}_c$ is of the order n^{-1}, the mean squared error of the estimator coincides with its variance. With respect to the estimador $\hat{x}_L$, it follows that

$$MSE(\hat{x}_L) = x_0^2(\mu_u^{(2)} - 1)^2 + \frac{3(\mu_u^{(2)})^3(\mu_e^{(2)})^2}{nk\beta^4\left(\sum_{i=1}^n x_i^2/n\right)} + \frac{\mu_e^{(2)}(\mu_u^{(2)})^2}{k\beta^2}$$

$$+\frac{\mu_u^{(2)}\mu_e^{(2)}\left(\sum_{i=1}^n x_i^4/n\right)}{nk\beta^2\left(\sum_{i=1}^n x_i^2/n\right)^2}\left(\frac{\mu_u^{(4)}}{\mu_u^{(2)}} - 4\mu_u^{(3)} + 3(\mu_u^{(2)})^2\right) + \frac{x_0^2(\mu_u^{(2)})^2\mu_e^{(2)}(3\mu_u^{(2)} - 2)}{n\beta^2\left(\sum_{i=1}^n x_i^2/n\right)}$$

(2.15) $$+\frac{x_0^2\mu_u^{(2)}\left(\sum_{i=1}^n x_i^4/n\right)}{n\left(\sum_{i=1}^n x_i^2/n\right)^2}\left[3(\mu_u^{(2)})^2 - 4\mu_u^{(3)} - 2\mu_u^{(2)} + \frac{\mu_u^{(4)}}{\mu_u^{(2)}} + \frac{2\mu_u^{(3)}}{\mu_u^{(2)}}\right] + O(n^{-2}).$$

The proof of the next theorem is similar to that of Theorem 2.1 and so it is not presented.

Theorem 2.2. *Under model (1.1)-(1.3) with the assumption that $\mu_e^{(i)}$ and $\mu_u^{(i)}$, $1 \le i \le 4$, are finite and with $\mu_e^{(2)}$ known, it follows that*

$$E(\tilde{x}_L) = \frac{x_0\beta^2\left(\sum_{i=1}^n x_i^2/n\right)}{d} + \frac{x_0\beta^2\left(\sum_{i=1}^n x_i^2/n\right)}{nd^3}(\mu_e^{(4)} - 3(\mu_e^{(2)})^2)$$

$$+\frac{2x_0\beta^4\mu_e^{(2)}\left(\sum_{i=1}^n x_i^2/n\right)^2}{nd^3} + \frac{x_0\beta\mu_e^{(3)}\bar{x}}{nd^3}[3\beta^2\left(\sum_{i=1}^n x_i^2/n\right) - \mu_e^{(2)}] + O(n^{-2})$$

and

$$\mathrm{Var}(\tilde{x}_L) = \frac{\beta^2\mu_e^{(2)}\left(\sum_{i=1}^n x_i^2/n\right)^2}{kd^2} + \frac{\mu_e^{(2)}\left[\beta^2(\mu_u^{(2)} - 1)\left(\sum_{i=1}^n x_i^4/n\right) + \mu_e^{(2)}\mu_u^{(2)}\left(\sum_{i=1}^n x_i^2/n\right)\right]}{nkd^2}$$

$$+\frac{\beta^2\mu_e^{(2)}\left(\sum_{i=1}^n x_i^2/n\right)^2}{nkd^4}(3\mu_e^{(4)} - 11(\mu_e^{(2)})^2) + \frac{4\beta^4(\mu_e^{(2)})^2\left(\sum_{i=1}^n x_i^2/n\right)^3}{nkd^4}$$

$$+\frac{4\beta\mu_e^{(2)}\mu_e^{(3)}\bar{x}\left(\sum_{i=1}^n x_i^2/n\right)}{nkd^4}\left[2\beta^2\left(\sum_{i=1}^n x_i^2/n\right) - \mu_e^{(2)}\right]$$

$$+\frac{\beta^2 x_0^2}{nd^2}\left[\beta^2(\mu_u^{(2)}-1)\left(\sum_{i=1}^n x_i^4/n\right)+\mu_e^{(2)}\mu_u^{(2)}\left(\sum_{i=1}^n x_i^2/n\right)\right]$$

$$+\frac{\beta^4 x_0^2(\mu_e^{(4)}-5(\mu_e^{(2)})^2)(\sum_{i=1}^n x_i^2/n)^2}{nd^4}+\frac{2\beta^3 x_0^2\mu_e^{(3)}\bar{x}(\sum_{i=1}^n x_i^2/n)}{nd^4}\left[\beta^2\left(\sum_{i=1}^n x_i^2/n\right)-\mu_e^{(2)}\right]+O(n^{-2})$$

Moreover,

$$E(\tilde{x}_c)=x_0+\frac{2x_0\mu_e^{(2)}}{n\beta^2(\sum_{i=1}^n x_i^2/n)}+\frac{3x_0\bar{x}\mu_e^{(3)}}{n\beta^3(\sum_{i=1}^n x_i^2/n)^2}+\frac{x_0(\mu_e^{(4)}-(\mu_e^{(2)})^2)}{n\beta^4(\sum_{i=1}^n x_i^2/n)^2}+O(n^{-2})$$

and

$$\mathrm{Var}(\tilde{x}_c)=\frac{\mu_e^{(2)}}{k\beta^2}+\frac{x_0^2(\sum_{i=1}^n x_i^4/n)}{n(\sum_{i=1}^n x_i^2/n)^2}(\mu_u^{(2)}-1)+\frac{x_0^2\mu_e^{(2)}\mu_u^{(2)}}{n\beta^2(\sum_{i=1}^n x_i^2/n)}+\frac{2\mu_e^{(3)}\bar{x}x_0^2}{n\beta^3(\sum_{i=1}^n x_i^2/n)^2}$$

$$+\frac{x_0^2(\mu_e^{(3)}-(\mu_e^{(2)})^2)}{n\beta^4(\sum_{i=1}^n x_i^2/n)^2}+\frac{\mu_e^{(2)}(\sum_{i=1}^n x_i^4/n)}{nk\beta^2(\sum_{i=1}^n x_i^2/n)^2}(\mu_u^{(2)}-1)+\frac{\mu_u^{(2)}(\mu_e^{(2)})^2}{nk\beta^4(\sum_{i=1}^n x_i^2/n)}$$

$$+\frac{8\mu_e^{(2)}\mu_e^{(3)}\bar{x}}{nk\beta^5(\sum_{i=1}^n x_i^2/n)^2}+\frac{4(\mu_e^{(2)})^2}{nk\beta^4(\sum_{i=1}^n x_i^2/n)}+\frac{3\mu_e^{(2)}(\mu_e^{(3)}-(\mu_e^{(2)})^2)}{nk\beta^6(\sum_{i=1}^n x_i^2/n)^2}+O(n^{-2}), \tag{2.16}$$

where d is as given in (2.10).

From Theorem 2.2, the mean squared erros of both estimators can be obtained. Notice that $bias(\tilde{x}_c)=O(n^{-1})$, so that the first order approximation to the mean squared error of the estimator $\tilde{x}_c$ coincides with its variance. On the other hand, with respect to the estimator $\tilde{x}_L$, it follows that

$$MSE(\tilde{x}_L)=\frac{\beta^2\mu_e^{(2)}(\sum_{i=1}^n x_i^2/n)^2}{kd^2}+\frac{\mu_e^{(2)}\left[\beta^2(\mu_u^{(2)}-1)(\sum_{i=1}^n x_i^4/n)+\mu_e^{(2)}\mu_u^{(2)}(\sum_{i=1}^n x_i^2/n)\right]}{nkd^2}$$

$$+\frac{\beta^2\mu_e^{(2)}(\sum_{i=1}^n x_i^2/n)^2}{nkd^4}(3\mu_e^{(4)}-11(\mu_e^{(2)})^2)+\frac{4\beta^4(\mu_e^{(2)})^2(\sum_{i=1}^n x_i^2/n)^3}{nkd^4}$$

$$+\frac{4\beta\mu_e^{(2)}\mu_e^{(3)}\bar{x}(\sum_{i=1}^n x_i^2/n)}{nkd^4}\left(2\beta^2\left(\sum_{i=1}^n x_i^2/n\right)-\mu_e^{(2)}\right)+\frac{x_0^2(\mu_e^{(2)})^2}{d^2}$$

$$+\frac{\beta^2 x_0^2}{nd^2}\left[\beta^2(\mu_u^{(2)}-1)\left(\sum_{i=1}^n x_i^4/n\right)+\mu_e^{(2)}\mu_u^{(2)}\left(\sum_{i=1}^n x_i^2/n\right)\right]$$

$$-\frac{2x_0^2\mu_e^{(2)}\beta^2\left(\sum_{i=1}^n x_i^2/n\right)}{nd^4}(\mu_e^{(4)} - 3(\mu_e^{(2)})^2) + \frac{\beta^4 x_0^2\left(\sum_{i=1}^n x_i^2/n\right)^2}{nd^4}(\mu_e^{(4)} - 9(\mu_e^{(2)})^2)$$

$$-\frac{2\beta x_0^2\mu_e^{(2)}\mu_e^{(3)}\bar{x}}{nd^4}\left[3\beta^2\left(\sum_{i=1}^n x_i^2/n\right) - \mu_e^{(2)}\right]$$

$$+\frac{2\beta^3 x_0^2\mu_e^{(3)}\bar{x}\left(\sum_{i=1}^n x_i^2/n\right)}{nd^4}\left[\beta^2\left(\sum_{i=1}^n x_i^2/n\right) - \mu_e^{(2)}\right] + O(n^{-2}). \tag{2.17}$$

Thus, from Theorems 2.1 and 2.2, it follows that

$$\lim_{n\to\infty} E(\hat{x}_L) = \mu_u^{(2)} x_0,$$
$$\lim_{n\to\infty} E(\hat{x}_c) = x_0,$$
$$\lim_{n\to\infty} E(\tilde{x}_L) = \frac{x_0\beta^2\left(\sum_{i=1}^n x_i^2/n\right)}{d}$$
$$\text{and} \quad \lim_{n\to\infty} E(\tilde{x}_c) = x_0,$$

which imply that the classical estimators $\hat{x}_L$ and $\hat{x}_c$ are asymptotically unbiased. Morever, this result is independent of k. Comparisons between the mean squared errors of the estimators considered are based on expressions (2.14)–(2.17). Note that

$$\lim_{n\to\infty} MSE(\hat{x}_L) = x_0^2(\mu_u^{(2)} - 1)^2 + \frac{\mu_e^{(2)}(\mu_u^{(2)})^2}{k\beta^2},$$
$$\lim_{n\to\infty} MSE(\hat{x}_c) = \frac{\mu_e^{(2)}}{k\beta^2},$$
$$\lim_{n\to\infty} MSE(\tilde{x}_L) = \frac{x_0^2(\mu_e^{(2)})^2}{d^2} + \frac{\beta^2\mu_e^{(2)}\left(\sum_{i=1}^n x_i^2/n\right)^2}{kd^2},$$
$$\text{and} \quad \lim_{n\to\infty} MSE(\tilde{x}_c) = \frac{\mu_e^{(2)}}{k\beta^2}.$$

Thus, as $n \to \infty$, it follows that

$$MSE(\hat{x}_c) = MSE(\tilde{x}_c),$$

that is, in the limit, the mean squared errors of the classical and inverse estimators, combined with the estimators $\tilde{\beta}_c$ and $\tilde{\phi}$, have samilar behavior. Further, none of the estimators are consistent for finite k. However, as $k \to \infty$, $\hat{x}_c$ and $\tilde{x}_c$ are consistent. Moreover, calling both $\hat{x}_c$ and $\tilde{x}_c$ of $\hat{x}$, to simplify notation, it follows that

$$MSE(\hat{x}_L) - MSE(\hat{x}) = x_0^2(\mu_u^{(2)} - 1)^2 + \frac{\mu_e^{(2)}}{k\beta^2}((\mu_u^{(2)})^2 - 1),$$

so that, if $\mu_u^{(2)} > 1$, then $MSE(\tilde{x}) < MSE(\hat{x}_L)$. Additional algebraic manipulations yield

$$
\begin{aligned}
MSE(\tilde{x}_L) - MSE(\tilde{x}) &= \frac{x_0^2(\mu_e^{(2)})^2}{d^2} + \frac{\mu_e^{(2)}}{k}\left(\frac{\beta^2\left(\sum_{i=1}^n x_i^2/n\right)^2}{d^2} - \frac{1}{\beta^2}\right) \\
&= \frac{(\mu_e^{(2)})^2}{d^2}\left(x_0^2 - \frac{\left(d+\beta^2\left(\sum_{i=1}^n x_i^2/n\right)\right)}{k\beta^2}\right),
\end{aligned}
$$

so that, if $k > \frac{d+\beta^2\left(\sum_{i=1}^n x_i^2/n\right)}{\beta^2 x_0^2}$, $MSE(\tilde{x}) < MSE(\tilde{x}_L)$. Further, as $n \to \infty$ and $k \to \infty$, it follows that

$$
\begin{aligned}
& MSE(\hat{x}_L) \to x_0^2(\mu_u^{(2)} - 1)^2, \\
& MSE(\hat{x}_c) = MSE(\tilde{x}_c) = MSE(\tilde{x}) \to 0 \\
\text{e} \quad & MSE(\tilde{x}_L) \to \frac{x_0^2(\mu_e^{(2)})^2}{d^2}.
\end{aligned}
$$

Thus, the above results imply that the estimators $\hat{x}_c$ and $\tilde{x}_c$ are consistent. If $\mu_u^{(2)} = 1$, then the estimator $\hat{x}_L$ is also consistent. Comparisons between $\hat{x}_L$ and $\tilde{x}_L$ will depende on the model parameters, and can be studied numerically. Some results in this direction are reported in Tables 3.1 and 3.2.

3. A simulation study

In this section a simulation study is presented which illustrate the behavior of the bias and mean squred error of the estimators considered. 1,000 samples are generated according to the multiplicative model (1.1)-(1.3) considering $\beta = 0.5$, $x_0 = 15.0$, $k = 2$, and $n = 10, 20, 30, 50, 100$. Moreover, $\epsilon_1, \ldots, \epsilon_n$ are independent with $\epsilon_i \sim N(0, 0.1)$ and $u_i \sim U(1,2)$, $i = 1, \ldots, n$. The specification of the model parameters is completed by considering $x_i \sim N(10, 30)$, $i = 1, \ldots, n$, which are are also indepedent. For each selected sample, the estimators $\hat{x}_c$, $\hat{x}_L$, $\tilde{x}_L$ and $\tilde{x}_c$ were computed and their simulated bias and mean squared error evaluated by $\sum(\hat{x}_G - x_0)/1000$ and $\sum(\hat{x}_G - x_0)^2/1000$, where $\hat{x}_G$ is any one of the above estimators and $\sum$ is extended to the to the 1,000 generated samples. The results are presented in Tables 3.1 and 3.2 bellow, where the Theoretical values are computed from the corresponding expressions for the bias and mean squared errors given in Theorems 2.1 and 2.2 and the Simulated values from the simulated study.

Table 3.1: bias of the four estimators

	$\hat{x}_L$		$\hat{x}$		$\tilde{x}_L$		$\tilde{x}$	
n	Simulated	Theoretical	Simulated	Thorethical	Simulated	Theoretical	Simulated	Theoretical
10	1.344	1.359	-0.142	-0.128	-0.175	-0.119	-0.027	0.030
20	1.400	1.427	-0.091	-0.066	-0.167	-0.125	-0.029	0.014
30	1.405	1.448	-0.087	-0.047	-0.188	-0.129	-0.049	0.009
50	1.516	1.468	0.015	-0.028	-0.106	-0.139	0.040	0.006
100	1.491	1.484	-0.007	-0.014	-0.157	-0.147	-0.007	0.003

Table 3.2: mean squared error of the four estimators

n	$\hat{x}_L$ Simulated	$\hat{x}_L$ Theoretical	$\hat{x}$ Simulated	$\hat{x}$ Theoretical	$\bar{x}_L$ Simulated	$\bar{x}_L$ Theoretical	$\bar{x}$ Simulated	$\bar{x}$ Theoretical
10	7.031	7.211	4.338	4.450	4.873	4.847	4.952	4.933
20	5.726	5.957	3.122	3.245	3.385	3.443	3.427	3.493
30	5.696	5.564	3.084	2.867	3.320	3.000	3.353	3.040
50	5.056	5.219	2.279	2.532	2.374	2.606	2.412	2.638
100	4.954	4.950	2.255	2.271	2.245	2.300	2.267	2.325

With the above specifications, and from the results reported in Theorems 2.1 and 2.2, it follows that the bias and mean squared error of the estimators considered, as $n \to \infty$ are such that

$$\begin{array}{lcl} Bias(\hat{x}_L) \to 1.5 & \text{and} & MSE(\hat{x}_L) \to 2.42, \\ Bias(\hat{x}_c) \to 0 & \text{and} & MSE(\hat{x}_c) \to 2.0, \\ Bias(\bar{x}_L) \to -0.137 & \text{and} & MSE(\bar{x}_L) \to 1.982, \\ Bias(\bar{x}_c) \to 0 & \text{and} & MSE(\bar{x}_c) \to 2.0. \end{array}$$

Note from the tables that the bias and MSE of all estimators tend toward the limiting values presented above. An exception seems to be the estimator $\hat{x}_L$, for which the mean squared error appear to take longer to converge to the limiting values. The tables also show that, in general, the approximations are satisfactory, even for small sample sizes.

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