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LINDELÖF DOMINATION VERSUS ω-DOMINATION OF DISCRETE SUBSETS

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Abstract. A discrete subset is said to be Lindelöf dominated (respectively, ω-dominated) if it is contained in the closure of a Lindelöf (respectively, a countable) subspace. We continue the study of spaces begun in [1] in which every discrete subset is Lindelöf dominated (respectively, ω-dominated). We generalize results of [1] and [15] concerning perfect Hausdorff spaces and give a ZFC example of a perfect space in which all discrete subsets are Lindelöf dominated but not ω-dominated.

1. Introduction

Given a space X, a family $\mathcal{A}$ of subsets of X is said to be *dominated* by a family $\mathcal{B}$ of subsets of X if for any set $A \subseteq X$ with $A \in \mathcal{A}$, there is some $B \in \mathcal{B}$ such that $A \subseteq \mathrm{cl}(B)$. When $\mathcal{A}$ is the family of discrete subsets (respectively, closed and discrete subsets) of a topological space X and $\mathcal{B}$ is the family of countable subsets (respectively, Lindelöf subspaces), then we say that all discrete subsets (respectively, closed discrete subsets) are *ω-dominated* (respectively, *Lindelöf dominated*). In [10] a stronger property than Lindelöf domination of discrete subsets was introduced: A space is said to be *almost discretely Lindelöf* if every discrete subspace is contained in a Lindelöf subspace.

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The following propositions were proved in [1] and suggest why these concepts are worthy of study. For a definition of the terms *star Lindelöf* and *star countable* we refer the reader to [11].

PROPOSITION 1.1. *If X is a space in which Lindelöf* (*respectively, countable*) *subsets dominate closed discrete subsets. Then X is star Lindelöf* (*respectively, star countable*).

It is easy to see that ω-domination of closed discrete sets is strictly stronger than star countability. However, there is an important class of spaces in which they coincide.

PROPOSITION 1.2. *A first countable space X is star Lindelöf* (*countable*) *if and only if Lindelöf* (*countable*) *subsets of X dominate closed discrete subsets of X.*

Notation and terminology follow [6]; definitions of, and relations between, cardinal functions can be found in [9], but note that in the latter work, the hereditary Lindelöf number of a space X is denoted by $h(X)$ and the extent is denoted by $p(X)$, whereas here we use $hl(X)$ and $e(X)$, respectively. We say that a filter is *open* if it has a base of open sets. All spaces, including those which are designated regular or normal, are assumed to be Hausdorff.

2. The results

For the purposes of the first theorem we denote the first weakly inaccessible cardinal (in any model of ZFC in which it exists) by $\mathfrak{w}$.

THEOREM 2.1. *If $d(X) \leq s(X) < \mathfrak{w}$, then all discrete subsets of $X \times X$ are ω-dominated if and only if X is separable.*

PROOF. The sufficiency is clear, since $X \times X$ is separable if and only if X is separable. For the necessity, suppose first that $s(X) = \kappa < \mathfrak{w}$, where κ is a singular cardinal and let $\{d_\alpha : \alpha < \kappa\}$ be a dense subset of X. For each $\lambda < \kappa$, there is a discrete subset $S^\lambda = \{s^\lambda_\alpha : \alpha < \lambda\} \subseteq X$ of size λ. Then for each $\lambda < \kappa$, set $A_\lambda = \{(s^\lambda_\alpha, d_\alpha) : \alpha < \lambda\}$ is a discrete subset of $X \times X$ and so there is a countable set $C_\lambda \subseteq X \times X$ such that $A_\lambda \subseteq \mathrm{cl}(C_\lambda)$. Then if π denotes the projection onto the second coordinate space, $\pi[C_\lambda]$ is a countable subset of X whose closure contains $\{d_\alpha : \alpha < \lambda\}$. Let I be a cofinal subset of κ of cardinality $cf(\kappa) < \kappa$; it follows that $\bigcup\{\pi[C_\lambda] : \lambda \in I\}$ is a set whose cardinality is $\omega \cdot cf(\kappa) = cf(\kappa) < \kappa$ whose closure contains $\{d_\alpha : \alpha < \kappa\}$, and which therefore must be dense in X; thus $d(X) < \kappa$.

Now consider the general case and suppose that $s(X) = \kappa < \mathfrak{w}$; if κ is singular, then by the argument of the previous paragraph, $d(X) = \gamma < \kappa$ and there is a discrete subset of size γ. If on the other hand κ is a regular cardinal, then there is a discrete subset of X of size κ, so in either case, there is a discrete subset of X of size $d(X)$. We now repeat the argument of the first paragaraph. Let $\{d_\alpha : \alpha < \kappa\}$ be a dense subset of X and $S = \{s_\alpha : \alpha < \kappa\} \subseteq X$ a discrete set of size κ. The set $A = \{(s_\alpha, d_\alpha) : \alpha < \kappa\}$ is a discrete subset of $X \times X$ and so there is a countable set $C \subseteq X \times X$ such that $A \subseteq \mathrm{cl}(C)$. Then if π denotes the projection onto the second coordinate space, $\pi[C]$ is a countable subset of X whose closure contains $\{d_\alpha : \alpha < \kappa\}$ and hence is dense in X. $\square$

The previous result should be compared with [1, Theorem 3.19], where it was shown that if X is a Lindelöf-p space and discrete subsets of $X \times X$ are ω-dominated, then X is separable. Theorem 2.1 seems to be considerably more general, but we note that under CH there are compact spaces (hence Lindelöf-p spaces) to which Theorem 2.1 does not apply (see of [1, Example 3.20]).

The proof of the next theorem is similar to that of the second paragraph of the proof of the previous result and we omit it.

THEOREM 2.2. *If $d(X) \leq s(X)$, $s(X)$ is attained and all discrete subsets of $X \times X$ are Lindelöf dominated then X has a dense Lindelöf subspace.*

PROPOSITION 2.3. *If X is hereditarily collectionwise Hausdorff, then the discrete subsets of X are ω-dominated if and only if $s(X) = \omega$.*

PROOF. The necessity is trivial. To prove the sufficiency, suppose that D is a discrete subset of X such that $|D| > \omega$ and consider the subspace $Y = X \setminus (\mathrm{cl}(D) \setminus D)$ which is open in X. Then D is a closed and discrete subset of Y and since Y is collectionwise Hausdorff, there is a cellular family in Y, hence also in X, of uncountable cardinality. The result now follows from [1, Proposition 3.1(c)], where it was shown that if discrete subsets are ω-dominated, then X has countable cellularity. $\square$

COROLLARY 2.4. *The discrete subsets of a monotonically normal space are ω-dominated if and only if it is hereditarily Lindelöf.*

PROOF. If X is monotonically normal then it is hereditarily collectionwise normal and $c(X) = hl(X)$ (for example, see [7]). $\square$

COROLLARY 2.5. *The discrete subsets of a generalized ordered space are ω-dominated if and only if it is hereditarily Lindelöf.*

COROLLARY 2.6. *If X is a hereditarily collectionwise Hausdorff space in which discrete subsets are ω-dominated and $\psi(X) = \omega$, then $|X| \leq \mathfrak{c}$.*

PROOF. This follows from the Hajnal–Juhász inequality $|X| \leq 2^{s(X)\psi(X)}$ (see [9, 2.15]). $\square$

It was shown in [1, Proposition 3.12] that if X is a regular space in which all discrete subsets are ω-dominated then $hl(X) \leq \mathfrak{c}$ and as a consequence,

$s(X) \le \mathfrak{c}$ and $\psi(X) \le \mathfrak{c}$. However, the Katětov extension of ω shows that the first two inequalities are false in the class of Hausdorff spaces. The next result should be compared to [8, Proposition 4.11].

THEOREM 2.7. *If X is a Hausdorff space in which all discrete subsets are ω-dominated and $s(X) \le \mathfrak{c}$, then $\psi(X) \le \mathfrak{c}$.*

PROOF. Fix $p \in X$; for each $q \in X \setminus \{p\}$ we may find an open neighbourhood V_q of q such that $\mathrm{cl}(V_q) \subseteq V \setminus \{p\}$. The family $\{V_q : q \in X \setminus \{p\}\}$ is an open cover of $X \setminus \{p\}$ and it then follows from a result of Šapirovskiĭ in [14], (as detailed in [6, Problem 3.12.9]), that there are sets $A, B \subseteq X \setminus \{p\}$ each of cardinality at most $\mathfrak{c}$, such that A is discrete and $X \setminus \{p\} \subseteq \mathrm{cl}(A) \cup \bigcup \{V_q : q \in B\}$. Now suppose that $Y \subseteq X$ is a separable subspace of X which contains the discrete set A, and let $D \subseteq Y$ be a countable dense subset of Y. Let

$$\mathcal{S} = \big\{X \setminus \mathrm{cl}(C) : C \subseteq D \text{ and } p \notin \mathrm{cl}(C)\big\} \cup \big\{X \setminus \mathrm{cl}(V_q);\ q \in B\big\}.$$

Since D is countable and $|B| \le \mathfrak{c}$, it is clear that $|\mathcal{S}| \le \mathfrak{c}$. We shall show that $\bigcap \mathcal{S} = \{p\}$. To this end, suppose that $x \in \bigcap \mathcal{S}$ and $x \ne p$; it follows from the definition of $\mathcal{S}$ that $x \in \mathrm{cl}(A)$, since if not, $x \in \mathrm{cl}(V_q)$ for some $q \in B$ and $X \setminus \mathrm{cl}(V_q) \in \mathcal{S}$. But then, if U is a neighbourhood of X such that $p \notin \mathrm{cl}(U)$, the set $U \cap D$ is a non-empty subset of D and $p \notin \mathrm{cl}(U \cap D)$, so $X \setminus \mathrm{cl}(U \cap D) \in \mathcal{S}$ which contradicts the fact that $x \in \mathrm{cl}(U \cap D)$. $\square$

A space is said to be *perfect* if every closed set is a G_δ. If X is a regular perfect Lindelöf space then it is both perfectly normal and hereditarily Lindelöf (see [6, 3.8A]). The next result should be compared to [4, Corollary 7] and [1, Corollary 3.13].

PROPOSITION 2.8. *If a space X is perfect and almost discretely Lindelöf, then $s(X) = \omega$ and $|X| \le \mathfrak{c}$.*

PROOF. Since each discrete set is contained in a Lindelöf subspace, it follows that $e(X) = \omega$ and since X is perfect and $s(X) \le e(X)\psi(X)$ (see [9, 2.30]) that $s(X) = \omega$. That $|X| \le \mathfrak{c}$ now follows from the previously cited Hajnal–Juhász inequality $|X| \le 2^{s(X)\psi(X)}$. $\square$

COROLLARY 2.9. *In an almost discretely Lindelöf perfect space, all discrete sets are ω-dominated.*

The next lemma has a proof similar to that of [9, 2.30]

LEMMA 2.10. *If a space is perfect then all its closed and discrete subsets are ω-dominated (respectively, Lindelöf dominated) if and only if all its discrete subsets are ω-dominated (respectively, Lindelöf dominated).*

PROOF. Suppose that X has the property in the hypothesis and let D be a discrete subset of X. For each $d \in D$, let U_d be an open set such that $U_d \cap D = \{d\}$. The open set $W = \bigcup\{U_d : d \in D\}$ is an F_σ set and hence there is a countable family of closed sets D_n so that $W = \bigcup\{D_n : n \in \omega\}$. Each of the sets $D \cap D_n$ is then a closed discrete subset of X and hence there is a countable set A_n (respectively, a Lindelöf subspace A_n) such that $D_n \subseteq \mathrm{cl}(A_n)$. It follows that $A = \bigcup\{A_n : n \in \omega\}$ is a countable subset (respectively, a Lindelöf subspace) of X and $D \subseteq \mathrm{cl}(A)$. □

It was shown in [15, Proposition 2.6] that if X is perfect and all closed discrete subsets are ω-dominated, then X has countable cellularity; the following result generalizes this.

THEOREM 2.11. *A perfect space in which all closed discrete sets* (*or equivalently, all discrete sets*) *are Lindelöf dominated has countable cellularity.*

PROOF. Suppose that the space X satisfies the hypothesis of the theorem but $c(X) \geq \omega_1$. There is some cellular family $\{U_\alpha : \alpha < \omega_1\}$ of non-empty open sets and for each $\alpha < \omega_1$ we may pick $x_\alpha \in U_\alpha$. Since $\bigcup\{U_\alpha : \alpha \in \omega_1\}$ is an F_σ-set, there is a family $\{F_n : n \in \omega\}$ of closed sets such that $\bigcup\{U_\alpha : \alpha \in \omega_1\} = \bigcup\{F_n : n \in \omega\}$ and hence there is $m \in \omega$ such that $J = \{\alpha \in \omega_1 : x_\alpha \in F_m\}$ is uncountable; clearly $A = \{x_\alpha : \alpha \in J\}$ is a closed and discrete subset of X. There is some Lindelöf subspace $L \subseteq X$ such that $A \subseteq \mathrm{cl}(L)$ and so $L \cap U_\alpha \neq \emptyset$ and we may pick $l_\alpha \in L \cap U_\alpha$ for each $\alpha \in J$. The uncountable set $\{l_\alpha : \alpha \in J\} \subseteq L$ is discrete and the family of open sets $\{U_\alpha : \alpha \in J\}$ witnesses this. Again since $\bigcup\{U_\alpha : \alpha \in J\}$ is an F_σ-set, as previously, we may find an uncountable subset $K \subseteq J$ such that $\{l_\alpha : \alpha \in K\}$ is closed and discrete, which contradicts the fact that L is Lindelöf. □

Clearly, if all discrete subsets of a space are ω-dominated, then they are also Lindelöf dominated, and the inverse implication was proved in the class of semi-stratifiable spaces in [15]. Recall that if X is a Tychonoff space, a point $p \in \beta X \setminus X$ is a *remote point* of X if it is not in the closure of any nowhere dense subset of X. It was shown in [5] that any non-pseudocompact space with countable cellularity and π-weight at most ω_1 has a remote point. We now give a ZFC example of a perfect regular space in which all discrete subsets are Lindelöf dominated but not all are ω-dominated.

EXAMPLE 2.12. There is a perfect regular space in which all discrete subsets are Lindelöf dominated but in which there is a closed and discrete subset which is not ω-dominated.

PROOF. Let $\mathcal{L}$ be the space constructed in [12], that is, $\mathcal{L}$ is a hereditarily Lindelöf non-separable Tychonoff space and we may assume that no non-empty open subset of $\mathcal{L}$ is separable; furthermore, $\pi w(\mathcal{L}) \leq w(\mathcal{L}) =$

$d(\mathcal{L}) = \omega_1$. Let $\mathcal{M}$ be an almost disjoint family on ω of cardinality ω_1 and denote by X the set $(\mathcal{L} \times \omega) \cup \mathcal{M}$. Since $\mathcal{L}$ is not pseudocompact and has countable cellularity, it follows from [5, Theorem 2.4] that $\mathcal{L}$ has a remote point p and let $\mathcal{F}$ be the open filter which is the the trace of the neighbourhood system of p in $\beta\mathcal{L}$ on $\mathcal{L}$. For future reference, we note that the regularity of $\beta\mathcal{L}$ implies that $\mathcal{F}$ has a base of regular closed sets.

Then for each nowhere dense subset $A \subseteq \mathcal{L}$, and hence for each separable subset $S \subseteq \mathcal{L}$, there is $F \in \mathcal{F}$ such that $S \cap F = A \cap F = \emptyset$. We define a topology on X as follows:

(i) $\mathcal{L} \times \omega$ has the product topology,

(ii) A basic neighbourhood of a point $M \in \mathcal{M}$ is any set of the form

$$\{M\} \cup \bigcup \big\{F(n) \times \{n\} : F(n) \in \mathcal{F} \text{ and } n \in M \setminus T\big\},$$

where $T \subseteq \omega$ is finite.

Since the dense subspace $\mathcal{L} \times \omega$ is hereditarily Lindelöf, it is clear that all discrete subsets of X are Lindelöf dominated. To show that the discrete subset $\mathcal{M}$ is not ω-dominated, it suffices to note that if $C \subseteq X$ is separable, then for each $n \in \omega$, $C \cap (\mathcal{L} \times \{n\})$ is nowhere dense in $\mathcal{L} \times \{n\}$ and hence for each $n \in \omega$, there are sets $G(n) \in \mathcal{F}$ such that $(G(n) \times \{n\}) \cap C = \emptyset$ for each $n \in \omega$; thus no point of $\mathcal{M}$ lies in the closure of a countable subset of $\mathcal{L} \times \omega$. It is easy to see that every point of X has a local base of closed neighbourhoods and hence X is regular. Thus it only remains to show that X is perfect. To this end, let C be a closed subset of X and let $C_{\mathcal{L}} = C \cap (\mathcal{L} \times \omega)$ and $C_O = C \cap \mathcal{M}$; $C_{\mathcal{L}}$ is a closed subspace of $\mathcal{L} \times \omega$ and C_O is closed in X. Since $\mathcal{L} \times \omega$ is perfectly normal, $C_{\mathcal{L}}$ is a G_δ-set in $\mathcal{L} \times \omega$ and hence in X. Furthermore, since $C_O = \bigcap\{\bigcup\{\mathcal{L} \times \{n\} : n \geq k\} \cup C_O : k \in \omega\}$ it follows that C_O is a G_δ-set in X; as a consequence, C is a G_δ in X. $\square$

Recall that a *Q-set* is an uncountable subset of the real line in which every subset is a G_δ. The space described in Example 2.12 is not necessarily normal; however the following modification provides a perfectly normal example assuming there exists a Q-set (whose existence is a consequence of $MA + \neg CH$).

Example 2.13. Assuming that there exists a Q-set, there is a perfectly normal space in which all discrete subsets are Lindelöf dominated but in which there is a closed and discrete subset which is not ω-dominated.

Proof. Suppose that $\mathcal{M}$ is the almost disjoint family on ω which corresponds to an ω_1-Cantor tree K – which, assuming the existence of a Q-set is a normal space (see [13, IV(5)]). The notation and the topology of $X = (\mathcal{L} \times \omega) \cup \mathcal{M}$ are as described in Example 2.12. It remains only to show that X is normal. To this end, suppose that C, D are disjoint closed subsets of X. It is easy to see that there are essentially only three distinct cases to consider:

(1) $C, D \subseteq \mathcal{L} \times \omega$, or

(2) $C, D \subseteq \mathcal{M}$, or

(3) $C \subseteq \mathcal{L} \times \omega$ and $D \subseteq \mathcal{M}$.

Case (1). That C and D can be separated follows from the fact that $\mathcal{L} \times \omega$ is perfectly normal and open in X.

Case (2). Consider C and D as disjoint subsets of the ω'th level of the tree K. By normality of K there are disjoint subsets I and J of ω such that $I \cup C$ and $J \cup D$ are open in K. Then $U = C \cup (\mathcal{L} \times I)$ and $V = D \cup (\mathcal{L} \times J)$ are disjoint open neighbourhoods of C and D respectively.

Case (3). It suffices to consider the case $D = \mathcal{M}$. For each $M \in \mathcal{M}$ there is a neighbourhood U of M which misses C, and hence for some finite set $T_M \subseteq M$ and for each $n \in M \setminus T_M$ there is $F_{M,n} \in \mathcal{F}$ such that $(F_{M,n} \times \{n\}) \cap C = \emptyset$. Since $\mathcal{F}$ has a base of regular closed sets, for each $M \in \mathcal{M}$ and each $n \in M \setminus T_M$; there is a regular closed set $G_{M.n} \in \mathcal{F}$ such that $G_{M,n} \subseteq F_{M,n}$. Thus for each $n \in \bigcup \{ M \setminus T_M : M \in \mathcal{M} \}$, we have found a regular closed set, which we denote by G_n such that $G_n \in \mathcal{F}$ and $(G_n \times \{n\}) \cap C = \emptyset$. Let

$$V = \mathcal{M} \cup \bigcup \big\{ G_n \times \{n\} : n \in \cup\{M \setminus T_M : M \in \mathcal{M}\} \big\}.$$

Then V is a closed neighbourhood of $\mathcal{M}$ which misses C and we are done. $\square$

QUESTION 2.14. *Is there a ZFC example of a perfectly normal space in which all discrete subsets are Lindelöf dominated, but not all discrete subsets are ω-dominated?*

With regard to this question, we note that in [2] and [3], Z. Balogh has constructed in ZFC regular spaces in which every subset is a G_δ, the space of [3] being perfectly normal and that of [2] having cardinality $\mathfrak{c}$, but which are not σ-discrete. However, we do not know if these spaces can be used to construct a space with the properties of Example 2.13.

Clearly, discrete subsets of a space X are dominated by separable Lindelöf subspaces if and only if discrete subsets are ω-dominated. A stronger condition is the following: A discrete subset of a topological space X will be said to be L_ω-embedded if it is contained in a separable Lindelöf space. Obviously, if discrete subsets of a space X are L_ω-embedded, then they are ω-dominated and the space X is almost discretely Lindelöf.

PROPOSITION 2.15. *A space is almost discretely Lindelöf and has all of its discrete subspaces ω-dominated if and only if all discrete subspaces are L_ω-embedded.*

PROOF. The sufficiency is trivial. To prove necessity, suppose that D is a discrete subset of an almost discretely Lindelöf space X which has the property that all its discrete subsets are ω-dominated. Then there is some Lindelöf subspace L which contains D and some countable set C such that $D \subseteq \mathrm{cl}_X(C)$. It follows that $\mathrm{cl}_X(C) \supseteq \mathrm{cl}_X(D) \supseteq \mathrm{cl}_L(D)$. But $\mathrm{cl}_L(D)$ is Lindelöf and so $C \cup \mathrm{cl}_L(D)$ is both separable and Lindelöf and contains D. $\square$

PROPOSITION 2.16. *If all discrete subsets of X are L_ω-embedded and $f\colon X \to Y$ is a continuous surjection, then all discrete subsets of Y are L_ω-embedded.*

PROOF. Suppose that D is a discrete subset of a space Y; clearly there is some discrete subset $C \subseteq X$ such that $f[C] = D$ and so there is a separable Lindelöf subspace $A \subseteq X$ such that $A \supseteq C$. Then $f[A]$ is a separable Lindelöf subspace of Y which contains D. $\square$

[1, Proposition 3.11(a)] states that ω-domination of discrete sets is a property which is inherited by open subspaces. However, neither an open nor a closed subset of a space in which all discrete sets are L_ω-embedded, need have this property; the one point compactification of an Isbell–Mrowka Ψ-space is one such example. This same example shows that part (d) of the above-mentioned theorem does not hold either. Nor is the property of having discrete sets L_ω-embedded productive; the Sorgenfrey line is the requisite example here.

As mentioned previously, it follows from [1, Proposition 3.12] that if X is regular and its discrete subsets are ω-dominated, then $s(X) \le hl(X) \le \mathfrak{c}$. The following question then arises: If discrete subsets of X are L_ω-embedded, is it true that $hl(X) \le \mathfrak{c}$ and hence $s(X) \le \mathfrak{c}$?

To give a partial answer to this question, we need the following lemma which may be of interest in its own right.

LEMMA 2.17. *If X is a separable Lindelöf T_2-space, then $nw(X) \le \mathfrak{c}$.*

PROOF. Suppose that D is a countable dense subset of X; we shall show that the set $\mathcal{N}$ of all countable intersections of elements of the set $\{\mathrm{cl}(U \cap D) : U \in \tau\}$ is a network for X, thus proving the result, since $|\mathcal{P}(D)| \le \mathfrak{c}$. Fix $x \in X$; since X is Hausdorff,

$$\{x\} = \bigcap \Big\{ \mathrm{cl}(U) : a \in U \in \tau \Big\} = \bigcap \Big\{ \mathrm{cl}(U \cap D) : x \in U \in \tau \Big\},$$

and so $\psi_c(x, X) \le \mathfrak{c}$. Suppose that $\{x\} = \bigcap\{\mathrm{cl}(U_\alpha) : \alpha < \kappa\}$, where $x \in U_\alpha \in \tau$ and $\kappa \le \mathfrak{c}$. If V is an open neighbourhood of X; then $\bigcap\{\mathrm{cl}(U_\alpha) \setminus V : \alpha \in \kappa\} = \emptyset$ and since X is Lindelöf, there is a countable subset $I \subseteq \kappa$ such that $\bigcap\{\mathrm{cl}(U_\alpha) \setminus V : \alpha \in I\} = \emptyset$. Hence $x \in \bigcap\{\mathrm{cl}(U_\alpha) : \alpha \in I\} \subseteq V$ and clearly, $\bigcap\{\mathrm{cl}(U_\alpha) : \alpha \in I\} \in \mathcal{N}$. $\square$

THEOREM 2.18. *If X is a Hausdorff space in which discrete subsets are L_ω-embedded, then $s(X) \le \mathfrak{c}$.*

Proof. Let E be a discrete subset of X and Z a separable Lindelöf subspace of X which contains E. From the previous lemma we have that $nw(Z) \leq \mathfrak{c}$ and hence $s(Z) \leq \mathfrak{c}$. Thus $|E| \leq \mathfrak{c}$ and the result follows. $\square$

The following corollary is now an immediate consequence of the previous theorem and Theorem 2.7.

Corollary 2.19. *If X is a Hausdorff space in which discrete subsets are L_ω-embedded, then $\psi(X) \leq \mathfrak{c}$.*

We end with some open problems.

Question 2.20. *If discrete subsets of a Hausdorff space X are L_ω-embedded, is it true that $d(X) \leq \mathfrak{c}$?*

Question 2.21. *If discrete subsets of a Hausdorff space X are L_ω-embedded, is it true that $hl(X) \leq \mathfrak{c}$?*

Question 2.22. *If discrete subsets of a Hausdorff (respectively, regular) space X are L_ω-embedded, is it true that $nw(X) \leq \mathfrak{c}$?*

Question 2.23. *If discrete subsets of a Hausdorff (respectively, regular) space X are L_ω-embedded, is it true that $w(X) \leq \mathfrak{c}$?*

Of course, there are separable compact Hausdorff spaces of size $2^{\mathfrak{c}}$, (for example $\beta\omega$), but any such space must have weight at most $\mathfrak{c}$.

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