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On two parameter envelopes of N-Spheres in a real space form

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by

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1. Introduction

The main purpose of this paper is to study two-parameter envelopes of n-spheres M^n $(n \geq 4)$ in a simply connected space form Q_c^{n+1} where $c = 0$, 1 or -1. The euclidean case, $c = 0$, was studied in [AD] and a natural problem, not considered there, is to characterize these immersions having constant mean curvature H (zero included), constant Gauss-Kronecker curvature K or, more generally, satisfying a linear Weingarten relation $A + BnH + CK = 0$, where A, B and C are real constants. *We must point out that the results, presented here, are related to special types of two-parameter envelopes, which are constructed from a given two-dimensional focal set and a given function;* see (1.4), (1.6) and (1.10), bellow. In our approach, they play a fundamental role in the characterization of the envelope and in the construction of the examples. To state our results, we need some terminology.

Let Q_c^N be a space form, as above, having metric $< , >_c$. In particular Q_{-1}^N will be the hyperbolic space $\mathbb{H}^N$, given by

$$\mathbb{H}^N = \{x \in \mathbb{L}^{N+1} ; < x, x >_{-1} = -1 \,,\ x_1 > 0\} ,$$

where $\mathbb{L}^{N+1}$ is the lorentzian space, i.e., the space $\mathbb{R}^{N+1}$ with the metric

$$< x, y >_{-1} = -x_1 y_1 + x_2 y_2 + \ldots + x_{N+1} y_{N+1} \ ,$$

for $x = (x_1, x_2, \ldots, x_{N+1})$ and $y = (y_1, y_2, \ldots, y_{N+1})$

1.1. Let $f : M^m \rightarrow Q_c^N$, $2 \leq m < N$, be an isometric immersion of a riemannian manifold M^m and $\alpha : T_pM \times T_pM \rightarrow (T_pM)^\perp$ the second fundamental form of f at $p \in M$; here T_pM and $(T_pM)^\perp$ are the tangent and the normal space of M at p. Let η be a unit field normal to the immersion and A_η the self adjoint map of tangent spaces corresponding to α along η, that is $< A_\eta(x), y >_c = < \alpha(x, y), \eta >_c$, for all $x, y \in T_pM$, where $< , >_c$

denotes, in this case, both the induced metric and the ambient space metric. We denote, as usual, by ∇ and $\overline{\nabla}$ the Levi-Civita connection of M^m and Q_c^n and by $\nabla^\perp$ the normal connection of f.

The following definitions can be found in [AD], for the case $c = 0$.

1.2. Let $x : M^n \to Q_c^{n+1}$ as in (1.1) and assume here and in the sequel that M^n *is connected and orientable with a given orientation.* In this case, we choose the unit normal vector N that gives the orientation of M and denote by $k_1, k_2, \ldots, k_n$ the real eigenvalues of A_N.

1.3. Definition. *We say that an isometric immersion $x : M^n \to Q_c^{n+1}$, as above, with $n \geq 4$, is a p-parameter envelopes of n-spheres, $p \leq n-2$, (briefly p-PES), if at each point of M, $k_1 = k_2 = \ldots = k_{n-p} = \lambda$ and $k_j \neq \lambda$ if $n\text{-}p\text{+}1 \leq j \leq n$, where $\lambda \neq 0$ if $c \geq 0$ and $|\lambda| > 1$ if $c < 0$.*

We now describe how to construct a p-PES in Q_c^{n+1}, for all c.

1.4. Let $L = L^p$ be a p-dimensional riemannian manifold and $g : L \to Q_c^{n+1}$ an isometric immersion. Let r be a function in $C^\infty(L)$ satisfying: r is a positive function if $c \leq 0$; $r \neq (2k+1)\frac{\pi}{2}$ and $r \neq k\pi$, $k \in \mathbb{Z}$, if $c = 1$; and if r is not constant, $0 < |\nabla r|^2 < 1$, for all c. Now, take functions α and β in $C^\infty(L)$ given by

$$(1.5) \qquad (\alpha, \beta) = (1, r), \quad (\cos r, \sin r) \text{ or } (\cosh r, \sinh r)$$

according to $c = 0$, 1 or -1.

Let Λ^1 be the unit normal bundle over the immersion g, i.e., the set of the points (y, ξ_y), where $y \in L$ and ξ_y is a unit vector normal to g, at the point y. By (1.1), we know that for each point $y \in L$, ξ_y describes a unit n-p sphere centered at the origin in the (n-p+1)-space, orthogonal to g. Define $\psi : \Lambda^1 \to Q_c^{n+1}$, by

$$(1.6) \qquad \psi(y, \xi_y) = \alpha(y)g(y) - \beta(y)\nabla r(y) - \beta(y)\sqrt{1 - |\nabla r(y)|^2}\, \xi_y$$

1.7. Theorem. *The hypersurface given by (1.6) is (in the open subset of the regular points) a p-parameter envelope of spheres. Conversely, every p-PES is locally of the form (1.6) where $r = \dfrac{1}{\lambda}$, $\cot^{-1}\lambda$ or $\coth^{-1}\lambda$ according to $c = 0$, 1 or -1.*

1.8. Let $x : M^n \to Q_c^{n+1}$ be a p-PES and D_λ the smooth distribution given by taking at each $p \in M$, the (n-p)-dimensional eigenspace of A_N, corresponding to the eigenvalue λ. Following [AD], we say that x is a *special p-parameter envelope of spheres* (briefly p-SPES) if the distribution $D_\lambda^\perp$ is integrable.

Now, for each $q \in L$, let $B(q) \subset T_qL$ be the *relative nullity subspace* of the immersion g (given by Theorem 1.7), defined by

$$B(q) = \{X \in T_qL : \alpha(X,Y) = 0, \ \forall \ Y \in T_qL\},$$

where α stands for the second fundamental form of g.

1.9. Theorem. *Let $x : M^n \to Q_c^{n+1}$ be a p-PES. Then x is a p-SPES if and only if $g : L \to Q_c^{n+1}$ has flat normal bundle and $\nabla r(q) \in B(q)$ for all $q \in L$, where r is given by* (1.7).

We will not present the proof of this theorem here because it can be found in [AD], for $c = 0$. We remark that a 1-PES ($p = 1$) is a general conformally flat hypersurface, defined and studied in [CDM]. Then, *from now on, we restrict ourselves to n-dimensional* 2*-PES,* $n \geq 4$.

1.10. Assume that the *index of relative nullity of $g : L^2 \to Q_c^{n+1}$, defined by $\mu(q) = dim\ B(q)$, is constant.* The possibilities are $\mu \equiv 0$, $\mu \equiv 1$ or $\mu \equiv 2$ and we can extend for the other ambients, the considerations found in [AD], for $\mathbb{R}^{n+1}$. By Theorem 1.9, if $\nabla r(q) \neq 0$ then $\mu(q) \neq 0$, which implies that the gaussian curvature of the immersion g at the point q, $K_g(q)$, is equal to c. On the other hand, if $\mu \equiv 0$, then $\nabla r \equiv 0$ and so r must be constant. So, we can conclude that, locally, we have only three types of 2-SPES:

Type I. A normal bundle of spheres with radius r constant over a surface $g : L \to Q_c^{n+1}$, having $\mu \equiv 0$ ($K_g \neq c$ at every point).

Type II. A 2-SPES where $g(L)$ is a ruled surface with $K_g = c$ in Q_c^{n+1}, without umbilic points ($\mu \equiv 1$).

Type III. A 2-SPES where $g(L)$ is a part of a 2-tottaly geodesic submanifold in $Q_c^{n+1}(\mu \equiv 2)$.

Here, a *normal bundle of spheres with radius r constant* is a hypersurface given by taking the function $r \equiv$constant in (1.6). A *ruled surface* $g : L^2 \to Q_c^{n+1}$ with no umbilics is given by

(1.11) $$g(s,t) = exp_{\gamma(s)}t\ v(s)$$

where $\gamma(s)$ is a curve in Q_c^{n+1} and v is a normal vector field with $|\dot\gamma| = |v| = 1$ (here $\cdot = d/ds$). In this case, by Gauss equation, we have that $K_g = c$ if and only if there exists a function $\rho = \rho(s)$ such that $\dot v(s) = \rho(s)\dot\gamma(s)$.

Now we can state the results obtained for each one of the above types of 2-SPES. For type I, we have

1.12. Theorem. *Let $\psi : \Lambda^1 \to Q_c^{n+1}$ be a 2-SPES of type I, having neither H nor K constant. Then they satisfy the Weingarten linear relation*

$$(1.13)\quad -nc^2\alpha^{n-2}\beta^4 + n\alpha^{n+2} - 2\alpha^n + (c^2\alpha^{n-3}\beta^5 - \alpha^{n+1}\beta)nH + 2\beta^n K = 0$$

for all (y, ξ_y) in Λ^1 if and only if $g : L^2 \to Q_c^{n+1}$ is a minimal surface.

As an example, take a *catenoid*, that is, a minimal rotation hypersurface $g : L^2 \to Q_c^3 \subset Q_c^{n+1}$ (see [CD]) and construct over it a normal bundle of spheres, with constant r, being carefull with the regularity condition, see 2.10.

For 2-SPES of type II, we have

1.14. Theorem. *Let $\psi : \Lambda^1 \to Q_c^{n+1}$, $c \neq 0$ be a 2-SPES of type II. If the function r is a constant satisfying (1.4) then the the Weingarten relation*

$$c\alpha^{-1}\beta - c(n-2)\alpha\beta^{-1} + cnH + \alpha^{-n+3}\beta^{n-3}K = 0$$

holds for all $(y, \xi_y) \in \Lambda^1$.

Now, *let $\psi : \Lambda^1 \to Q_c^{n+1}$ be a 2-SPES of type III.* In this case the respective $g : L^2 \to Q_c^{n+1}$ is totally geodesic and we can assume that $g = g(u_1, u_2)$, where (u_1, u_2) are orthogonal parameters for L, i.e., $\left\langle \frac{\partial g}{\partial u_1}, \frac{\partial g}{\partial u_2} \right\rangle_c = 0$ (see 3.24). Let $b_i := \left\| \frac{\partial g}{\partial u_i} \right\|^{-1} \cdot \frac{\partial g}{\partial u_i}$, $i = 1, 2$, and consider the operators P and T on TL given by (see also (2.2) and (2.5))

$$(1.15)\quad \begin{aligned} P(b) &= \alpha b - \alpha b(r)\nabla r - \beta\,\mathrm{Hess}_r b\,, \\ T(b) &= -(\alpha\,\mathrm{Hess}_r b + c\beta b + b(\alpha)\nabla r)\,, \end{aligned}$$

where $b \in T_yL$ and $\mathrm{Hess}_r b := \nabla_b(\nabla r)$ is the hessian of r relative to b. Our main results for this type of envelope are

1.16. Theorem. *Let $\psi : \Lambda^1 \to Q_c^{n+1}$ be a 2-SPES of type III ($n \geq 4$). Then H is a constant k if and only if*

(1.17)

$$((n-2)\alpha - nk\beta)\det P + \beta(T_{11}P_{22} - T_{21}P_{12} - T_{12}P_{21} + T_{22}P_{11}) = 0$$

where, for $1 \le i, j \le 2$, $P_{ij} = < P(b_i), b_j >$ *and* $T_{ij} = < T(b_i), b_j >$.

The expression (1.17) can be considerably simplified with an additional hypothesis on r, if $c = 0$.

1.18. Corollary. *Let* $\psi : \Lambda^1 \to \mathbb{R}^{n+1}$ *be a 2-SPES of type III* ($n \ge 4$) *and assume that* $r = r(u_1)$. *Then* H *is a constant* k *if and only if*

$$[(1 - kr)n - 2] \cdot \left(1 - \left(\frac{\partial r}{\partial u_1}\right)^2 - r\frac{\partial^2 r}{\partial {u_1}^2}\right) - r\frac{\partial^2 r}{\partial {u_1}^2} = 0 .$$

We can find examples that verifies this corollary in the following result.

1.19. Theorem. *Let* $\psi : \Lambda^1 \to \mathbb{R}^{n+1}$ *be a 2-SPES of type III* ($n \ge 4$) *with* $r = r(u_1)$. *Then it is a cylinder over a rotation hypersurface* $M^{n-1} \subset \mathbb{R}^n$. *Furthermore, the mean curvature* H *of the 2-SPES is a constant* k *if and only if* M^{n-1} *has constant mean curvature* H' *equal to* $\frac{nk}{n-1}$. *In particular, the 2-SPES is minimal if and only if it is a cylinder over part of a catenoid* M^{n-1} *contained in* $\mathbb{R}^n \subset \mathbb{R}^{n+1}$.

For the caracterization of a rotation hypersurface in $\mathbb{R}^n$, having constant mean curvature, see [CD].

We can also show the non-existence of 2-SPES of types I and II under certain conditions on H and K. For technical reasons, *for these results we assume that the leaves of* D_λ *are complete.*

1.20. Theorem. *i) There is not a 2-SPES of type I in* Q_c^{n+1}, *with constant* H *neither with constant* K. *In particular, there is not a minimal one*

ii) There is not a 2-SPES of type II in $\mathbb{R}^{n+1}$, *with* $A + BnH + CK = 0$ ($B \ne 0$). *In particular, there is not an example having constant* H. *Also, we cannot obtain a 2-SPES of type II in* Q_c^{n+1}, $c \ne 0$, *with constant* H *neither with constant* $K \ne 0$.

However, there do exist examples of 2-SPES of type II with $K \equiv 0$ (see Proposition 3.21).

The results of this paper are essencially chapters I and II of the author's doctoral dissertation [Ch], presented at IME-USP. There, in chapter III, we have also studied the 2-PES M^n in H^{n+1} which can be isometrically immersed in $\mathbb{R}^{n+2}$ ($n \ge 5$) and we obtained similar results to [AD]. But, at the same time, Dajczer and Tojero, see [DT] obtained a general theorem on composition of immersions that contains the results of chapter III mencioned.

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2. Regular points of ψ; proof of Theorem 1.7.

2.1. Let $g : L \to Q_c^{n+1}$ be an isometric immersion and ∇ the Levi-Civita *connection of* $L = L^p$. Let $A_\xi : T_yL \to T_yL$ be as in (1.1) and $P : T_yL \to T_yL$ the linear operator given by

$$P(b) = \alpha b - \alpha b(r)\nabla r - \beta\,\mathrm{Hess}_r b + \beta\sqrt{1 - |\nabla r|^2}\;A_\xi(b)\,. \tag{2.2}$$

Taking a local orthonormal frame field $\{b_1, b_2, ..., b_p\}$ *of* TL, we can see that the space $\psi_{*(y,\xi_y)}(T\Lambda^1)$ decomposes *into* a direct sum *of subspaces* $\mathcal{H}$ *(horizontal) and* $\mathcal{V}$, *where* $\mathcal{H}$ *is the* subspace spanned by

$$e_j = b_j + (-1)^{c+1}\frac{b_j(r)}{\sqrt{1 - |\nabla r|^2}}\;\xi\,,\ j = 1, 2, ...p. \tag{2.3}$$

and $\mathcal{V}$ is the n-p dimensional subspace orthogonal to ξ and to the immersion g.

Now, define the maps $Q : T_yL \to \mathcal{H}$, $T : T_yL \to \mathcal{H}$ and $U : T_yL \to \mathcal{H}$, by

$$Q(b) = P(b) - \frac{< P(b), \nabla r >_c}{\sqrt{1 - |\nabla r|^2}}\;\xi\,, \tag{2.4}$$

$$T(b) = -\left[\alpha\left(\mathrm{Hess}_r\, b - \sqrt{1 - |\nabla r|^2}\;A_\xi(b)\right) + c\beta b + b(\alpha)\nabla r\right]\,, \tag{2.5}$$

$$U(b) = T(b) - \frac{< T(b), \nabla r >_c}{\sqrt{1 - |\nabla r|^2}}\;\xi\,. \tag{2.6}$$

Now, we can state

2.7. Proposition.

(i) $\psi : \Lambda^1 \to Q_c^{n+1}$ *has maximal rank* n *at the point* (y, ξ_y) *if and only if* P *is not singular (that is,* Q *is not singular)*

(ii) In the regular points (y, ξ_y) *given by (i), the unit normal field* N *of* ψ *is given by*

$$N = c\beta g + \alpha(\nabla r + \sqrt{1 - |\nabla r|^2}\;\xi) \tag{2.8}$$

and the second fundamental form A_N of ψ has the following restrictions to the subspaces $\mathcal{V}$ and $\mathcal{H}$: $A_N|_{\mathcal{V}} = \lambda I$ and $A_N|_{\mathcal{H}} = UQ^{-1}$.

All eigenvalues of the map UQ^{-1} are different from λ, in any point.

2.9. Proof. To prove that all eigenvalues of UQ^{-1} are different from λ, the argument is exactly the same of the proof of Theorem 1.5 [AD, p.624]. To prove (i) and (ii) we follow closely the proof of Proposition 1.8 in [DG]. Since $T\Lambda^1 = \mathcal{H} \oplus \mathcal{V}$, to compute $\psi_{*(y,\xi_y)}$, we will first compute $\psi_*|_{\mathcal{V}}$. For this, take $v \in \mathcal{V}$ and $\gamma : (-\epsilon, \epsilon) \to \Lambda^1$ the curve given by $\gamma(t) = (y, \xi(t))$ with $\gamma(0) = (y, \xi_y)$ and $\dot{\xi}(0) = v$. Then

$$\psi_{*(y,\xi_y)}(v) = \frac{d}{dt}(\psi \circ \gamma)(t)|_{t=0} = -\beta(y)\sqrt{(1 - |\nabla r|^2(y)}\ \ v \tag{2.10}$$

Now to compute $\psi_*|_{\mathcal{H}}$, we observe that any tangent vector in $\mathcal{H}$ can be written as $\xi_*(b)$ where $b \in T_y L$ and ξ is a local section of Λ^1, through (y, ξ_y). By identifying $g_{*y}(b)$ with b, we can see that

$$\begin{aligned} \psi_*(\xi_*(b)) &= \overline{\nabla}_b(\psi \circ \xi) = \\ &= \left(\alpha Id - \alpha < \nabla r, \cdot >_c \nabla r - \beta \,\mathrm{Hess}_r + \beta\sqrt{1 - |\nabla r|^2} A_\xi\right)(b) \\ &\quad -b\left(\beta\sqrt{1 - |\nabla r|^2}\right)\xi - \beta\sqrt{1 - |\nabla r|^2}\ \nabla_b^{\perp}\xi - \beta\alpha(b, \nabla r). \end{aligned} \tag{2.11}$$

Let $\alpha_1(b, \nabla r) := \alpha(b, \nabla r) - < \alpha(b, \nabla r), \xi > \ \xi$ and let $R : T_y L \to \mathcal{V}$ be the operator given by $R(b) = \alpha_1(b, \nabla r) + \sqrt{1 - |\nabla r|^2}\ \nabla_b^{\perp} \xi$. It is easy to see that $\psi_*(\xi_*(b)) = Q(b) - \beta R(b)$ and then $Im\ \psi_* = Im\ Q \oplus \mathcal{V}$. Since $\dim \mathcal{V} = n - p$, (i) follows.

For part (ii), an easy computation shows that N is unitary and normal to the generators of both $\mathcal{H}$ and $\mathcal{V}$. Now, to compute $A_N|_{\mathcal{V}}$, take $\eta = \psi_*(v) \in \mathcal{V}$ as in (2.10) and $\Gamma(t) = \psi(\gamma(t))$ a curve such that $\eta = \dot{\Gamma}(0)$. Since $N(t) = N(\Gamma(t))$ can be view as a curve in $\mathbb{R}^{n+2}$ ($\mathbb{R}^{n+1}$, if $c = 0$), we obtain

$$A_N(\eta) = -\overline{\nabla}_{\dot{\Gamma}(0)} N = -N'(0) = -\alpha\sqrt{1 - |\nabla r|^2}\ v = \frac{\alpha}{\beta}\ \eta = \lambda\, \eta\,. \tag{2.12}$$

To compute $A_N|_{\mathcal{H}}$, take $u = \psi_*(\xi_* b)$ as in (2.11). Since $R(b) \in \mathcal{V}$, on one hand we have

$$A_N(u) = A_N(Q(b) - \beta R(b)) = A_N(Q(b)) - \beta\lambda R(b)\,. \tag{2.13}$$

On the other hand, looking N as a map of M^n into a vector space, we obtain

$$
\begin{aligned}
A_N(u) &= -N_*(u) = -(N\circ\psi\circ\xi)_*(u)\\
&= -\overline{\nabla}_b\left(c\beta g + \alpha\nabla r + \alpha\sqrt{1-|\nabla r|^2}\ \xi\right)\\
(2.14)\qquad &= -c\beta b - b(\alpha)\nabla r - \alpha\,\mathrm{Hess}_r\, b - \alpha(\alpha(b,\nabla r))\\
&\quad -b\left(\alpha\sqrt{1-|\nabla r|^2}\right)\xi + \alpha\sqrt{1-|\nabla r|^2}\ A_\xi(b)\\
&\quad -\alpha\sqrt{1-|\nabla r|^2}\ \nabla_b^{\perp}\xi\ .
\end{aligned}
$$

Now (2.13) and (2.14) gives
(2.15)

$$
\begin{aligned}
A_N(Q(b)) &= -c\beta b - b(\alpha)\nabla r - \alpha\,\mathrm{Hess}_r b - b\left(\alpha\sqrt{1-|\nabla r|^2}\right)\xi\\
&\quad +\alpha\sqrt{1-|\nabla r|^2}\ A_\xi(b) - \alpha\ < A_\xi(b),\nabla r >\ \xi\ ,
\end{aligned}
$$

which implies by (2.6) that $A_N(Q(b)) = U(b)$ and $A_N|_{\mathcal{H}} = UQ^{-1}$. This complete our proof.

Remark. We observe that the operators P, T and U defined so far, appear naturally in the above computations.

By 2.7, we can see that the matrix of the operator A_N, relative to a basis on which the $n-p$ first vectors are in $\mathcal{V}$ and the last p-vectors are in $\mathcal{H}$, takes the form

$$
(2.16)\qquad \begin{bmatrix} [\lambda I]_{n-p\times n-p} & 0\\ 0 & [UQ^{-1}]_{p\times p}\end{bmatrix}.
$$

It follows that

$$
(2.17)\qquad H = \frac{1}{n}\mathrm{tr}[A_N] = \frac{1}{n}\left[(n-p)\lambda + \mathrm{tr}[UQ^{-1}]\right]\ ,
$$

$$
(2.18)\qquad K = \det[A_N] = (\lambda)^{n-p}\det[UQ^{-1}]\ .
$$

For each (y,ξ_y) in Λ^1, let $B = \{b_1(y),\dots,b_p(y)\}$ be any orthonormal basis of T_yL and let $E = \{e_1(y),\dots,e_p(y)\}$ be the correspondent basis of $\mathcal{H}_{(y,\xi_y)}$, where the e_j are given by (2.3). We know that $[UQ^{-1}]_{E,E} = [U]_{B,E}\cdot[Q^{-1}]_{E,B}$, where, for instance, $[UQ^{-1}]_{E,E}$ denotes the matrix of UQ^{-1} relative to the basis E.

Since $[U]_{B,E} = [T]_{B,B}$, $[Q]_{B,E} = [P]_{B,B}$ and $[Q^{-1}]_{E,B} = [Q]^{-1}_{B,E}$, we get $[UQ^{-1}]_{E,E} = [T]_{B,B}\cdot[P]^{-1}_{B,B}$.

When $p = 2$, we obtain

$$(2.19)\qquad [UQ^{-1}]_{E,E} = \frac{1}{\det P}\begin{bmatrix} T_{11}P_{22} - T_{21}P_{12} & -T_{11}P_{21} + T_{21}P_{11} \\ T_{12}P_{22} - T_{22}P_{12} & -T_{12}P_{21} + T_{22}P_{11} \end{bmatrix},$$

where $T_{ij} = < T(b_i), b_j >_c$ and $P_{ij} = < P(b_i), b_j >_c$. *Then*

$$(2.20)\qquad \det P = P_{11}P_{22} - P_{12}P_{21}\ ,$$

$$(2.21)\qquad \mathrm{tr}[UQ^{-1}]_{E,E} = (T_{11}P_{22} - T_{21}P_{12} - T_{12}P_{21} + T_{22}P_{11})\cdot(\det P)^{-1}$$

and

$$(2.22)\qquad \det[UQ^{-1}]_{E,E} = (T_{11}T_{22} - T_{12}T_{21})\cdot(\det P)^{-1}\ .$$

2.23. Proof of Theorem 1.7.

We will only present here the case $c = -1$, the others being similar. In any case, our proof follows closely the proof of Theorem 1.5 in [AD].

Let $g : L \to \mathbb{H}^{n+1}$, $r \in C^\infty(L)$ as in (1.4) and let $\psi : \Lambda^1 \to \mathbb{H}^{n+1}$ be as in (1.6). By Proposition 2.7, ψ is a p-PES, in the regular points (y, ξ_y). We can see that $\lambda > 1$ holds, because in this case we take $\lambda = \dfrac{\alpha(r)}{\beta(r)}$ where $\alpha(r) = \cosh r$ and $\beta(r) = \sinh r$.

For the converse, let $x : M^n \to \mathbb{H}^{n+1}$ be a p-PES and $D_\lambda : M \to TM$ the distribution given by

$$D_\lambda(q) = \{X \in T_qM \mid A_N(X) = \lambda X\}$$

for each $q \in M$. It is known (see [R]) that D_λ is differentiable and involutive, λ is constant along each leaf Σ_λ of D_λ and Σ_λ is a n-p-umbilic submanifold of both M^n and $\mathbb{H}^{n+1}$. Then (see [Sp]), Σ_λ^{n-p} is either a totally geodesic submanifold, a horo-sphere, or an equidistant hypersurface in some totally geodesic submanifold $\mathbb{H}^{n-p+1}$ of $\mathbb{H}^{n+1}$. In this way, Σ_λ^{n-p} is also umbilic in Σ_λ^{n-p+1}. Since $\lambda > 1$, it is not difficult to see (see [Sp]), that Σ_λ^{n-p} is actually a geodesic sphere. For this, we call briefly each Σ_λ^{n-p} a sphere.

Now we choose local coordinates $(u_1, ..., u_p, t_1, ..., t_{n-p})$ for x such that, for each $(u_1, ..., u_p)$ fixed, the coordinates $(t_1, ..., t_{n-p})$ parametrize a leaf of D_λ. Then, we have

$$(2.24)\qquad A_N\left(\frac{\partial x}{\partial t_i}\right) = \lambda\frac{\partial x}{\partial t_i}\ ,\ i = 1, 2, ..., n-p\ .$$

Since $\lambda > 1$, $r = \coth^{-1}(\lambda)$ is well defined. Consider the focal set of x relative to λ given by $g = \alpha x + \beta N$, where N is a unit vector field normal to x. Clearly $\lambda = \frac{\alpha}{\beta}$. Since $\frac{\partial \lambda}{\partial t_i} = 0$, it follows that $\frac{\partial r}{\partial t_i} = 0$ and by (2.24), we obtain $\frac{\partial g}{\partial t_i} = 0$, that is $g = g(u_1, ..., u_p)$. The fact that g is a submanifold of $\mathbb{H}^{n+1}$ follows from [CR]. Now, we have

$$\left\langle N, \frac{\partial g}{\partial u_j} \right\rangle_{-1} = \alpha \frac{\partial r}{\partial u_j}, \ \langle N, g \rangle_{-1} = \beta$$

and taking an orthonormal frame $\{b_1, ..., b_p\}$ tangent to g, we will obtain $< N, b_i >_{-1} = < \alpha \nabla r, b_i >_{-1}$. Then, in $\mathbb{L}^{n+2}$, N can be written in the form $N = -\beta g + \alpha \nabla r + \gamma \xi$, where ξ is a unit vector normal to g in $\mathbb{H}^{n+1}$. Since $< N, N >_{-1} = 1$ we can choose $\gamma = -\sqrt{1 - |\nabla r|^2}$. Finally

$$N = -\beta g + \alpha \nabla r - \sqrt{1 - |\nabla r|^2}\, \xi$$

and

(2.25) $$x = \alpha g - \beta \nabla r + \beta \sqrt{1 - |\nabla r|^2}\, \xi .$$

Now, we take the bundle Λ^1 over the focal submanifold g given above. We know that for each $(u_1, ..., u_p)$ fixed, $(t_1, ..., t_{n-p})$ parametrizes a leaf of D_λ; this is equivalent to the fact that for each $y \in L^p$, ξ_y describes a leaf Σ_λ^{n-p} of D_λ. Then, x given locally by (2.25) can be view locally as the map $\psi : \Lambda^1 - \mathbb{H}^{n+1}$ given by (1.6).

3. Proofs of the results about types I, II and III.

3.1. Proof of Theorem 1.12.

In type I 2-SPES the function r is constant, which simplifies the expressions for P, T and U and, consequentely, UQ^{-1}, see (2.19). By bringing these simplified expressions into (2.17) and (2.18), we obtain

(3.2) $$nH = \frac{(n\alpha^2 - 2)\alpha + (n\alpha^2 - 1)\beta \operatorname{tr} A_\xi + n\alpha\beta^2 \det A_\xi}{\beta(\alpha^2 + \alpha\beta \operatorname{tr} A_\xi + \beta^2 \det A_\xi)},$$

(3.3) $$K = \frac{c^2\alpha^{n-2}\beta^2 - c\alpha^{n-1}\beta \operatorname{tr} A_\xi + \alpha^n \det A_\xi}{\beta^{n-2}(\alpha^2 + \alpha\beta \operatorname{tr} A_\xi + \beta^2 \det A_\xi)}.$$

Observe that in the regular points (y, ξ_y) we have

$$\alpha^2 + \alpha\beta \operatorname{tr} A_\xi + \beta^2 \det A_\xi \neq 0 .$$

Now it is not difficult to conclude from (3.2) and (3.3) that (1.13) holds if and only if $\operatorname{tr} A_\xi = 0$ for all $(y, \xi_y) \in \Lambda^1$, that is, if and only if g is minimal, which concludes the proof.

3.4. For type II, the surface $g : L^2 \to Q_c^{n+1}$ is given by (1.11) wich, for each ambient, takes the particular form

$$g(s,t) = a(t)\gamma(s) + b(t)v(s) \tag{3.5}$$

where $(a(t), b(t)) = (1, t)$, $(\cos t, \sin t)$ or $(\cosh t, \sinh t)$ according to $c = 0$, 1 or -1.

We claim that the function $r : L^2 \to \mathbb{R}$ of a 2-SPES of type II does not depend on the variable s, that is $r = r(t)$. In fact, taking an orthonormal frame $\{b_1, b_2\}$ tangent to g, given by $b_1 = \dfrac{\partial g}{\partial s} \cdot \left\| \dfrac{\partial g}{\partial s} \right\|^{-1}$ and $b_2 = \dfrac{\partial g}{\partial t}$, we obtain

$$\begin{cases} b_1 = \dot{\gamma}(s), \\ b_2 = a'(t)\gamma(s) + b'(t)v(s), \end{cases} \tag{3.6}$$

where $'$ denotes the derivative with respect to the variable t. By Theorem 1.9 we have that $\nabla r(q) \in B(q)$, $\forall q \in L$. Then $\alpha(\nabla r, b_1) = 0$, that is

$$b_1(r)\alpha(b_1, b_1) + b_2(r)\alpha(b_2, b_1) = 0 . \tag{3.7}$$

Taking an orthonormal frame $\{\xi_1, \ldots, \xi_{n-1}\}$ normal to g, we have

$$\begin{cases} \alpha(b_1, b_1) = \dfrac{1}{a(t) + \rho(s)b(t)} \displaystyle\sum_{k=1}^{n-1} < \tilde{\gamma}, \xi_k >_c \xi_k , \\ \\ \alpha(b_i, b_j) = 0. \quad \text{for } i, j \neq 1 , \ \forall c . \end{cases} \tag{3.8}$$

By substituting (3.8) in (3.7) and recalling that $\alpha(b_1, b_1) \neq 0$ (g does not have umbilic points), we conclude that $r = r(t)$.

3.9. Remark. By (3.8) and the relation between the second fundamental form α and the Weingarten operator A_ξ, for each unit vector ξ, normal to the immersion g, we obtain the matrix of A_ξ given by

$$[A_\xi] = \begin{bmatrix} \dfrac{< \ddot{\gamma}(s), \xi >_c}{a(t) + \rho(s)b(t)} & 0 \\ & \\ 0 & 0 \end{bmatrix} .$$

Writing $H_{ij} = <\ \mathrm{Hess}_r b_i, b_j >$ and using (2.2) and (2.5) we obtain, after calculations,

$$(3.10) \qquad \begin{cases} H_{11} &= \dfrac{a'(t) + \rho(s)b'(t)}{a(t) + \rho(s)b(t)}\, b_2(r)\,, \\ & \\ H_{22} &= b_2(b_2(r)), \quad H_{12} = H_{21} = 0\,; \end{cases}$$

$$(3.11) \begin{cases} P_{11} &= \alpha - \beta\dfrac{a'(t) + \rho(s)b'(t)}{a(t) + \rho(s)b(t)}\, b_2(r) - \beta\dfrac{\sqrt{1 - b^2(r)}}{a(t) + \rho(s)b(t)} < \ddot{\gamma}, \xi >_c\,, \\ & \\ P_{22} &= \alpha(1 - b_2^2(r)) - \beta(b_2(b_2(r)), \quad P_{12} = P_{21} = 0; \end{cases}$$

$$(3.12) \begin{cases} T_{11} &= -\dfrac{\alpha\left[(a'(t) + \rho(s)b'(t))b_2(r) + \sqrt{1 - b^2(r)} < \ddot{\gamma}, \xi >_c\right]}{a(t) + \rho(s)b(t)} - c\beta\,, \\ & \\ T_{22} &= -\left[\alpha b_2(b_2(r)) + c\beta(1 - b_2^2(r))\right], \; T_{12} = T_{21} = 0\,. \end{cases}$$

Now, the formulas (2.20) to (2.22) take the simpler form

$$(3.13) \qquad \det P = P_{11}P_{22}\,,$$

$$(3.14) \qquad \mathrm{tr}[UQ^{-1}]_{E.E} = (T_{11}P_{22} + T_{22}P_{11}) \cdot (\det P)^{-1}\,,$$

$$(3.15) \qquad \det[UQ^{-1}]_{E,E} = (T_{11}T_{22}) \cdot (\det P)^{-1}\,.$$

Finally, by (2.17) and (2.18), we obtain

$$(3.16) \qquad nH = (n-2)\lambda + (T_{11}P_{22} + T_{22}P_{11}) \cdot (\det P)^{-1}\,,$$

$$(3.17) \qquad K = \lambda^{n-2}T_{11} \cdot T_{22} \cdot (\det P)^{-1}\,.$$

3.18. Proof of Theorem 1.14.

At this point, it is an easy consequence of the above formulas. In fact, if r is a constant, then (3.16) and (3.19) become

$$(3.19)\quad nH = \frac{[(n-2)\alpha^2 - 2c\beta^2\alpha^2]\,[a(t) + \rho(s)b(t)] - (\alpha^3\beta - c\beta^3\alpha) < \ddot{\gamma}, \xi >_c}{\beta\alpha[a(t) + \rho(s)b(t)] - \beta^2 < \ddot{\gamma}, \xi >_c}$$

$$(3.20)\qquad K = \frac{c\alpha^{n-1}\beta < \ddot{\gamma}, \xi >_c + c^2\alpha^{n-2}\beta^2(a(t) + \rho(s)b(t))}{\alpha^2\beta^{n-2}(a(t) + \rho(s)b(t)) - \alpha\beta^{n-1} < \ddot{\gamma}, \xi >_c}$$

and from this we immediately get the desired Weingarten relation

3.21. Proposition *Let $\psi : \Lambda^1 \to Q_c^{n+1}$ be a 2-SPES of type II in Q_c^{n+1}, $c \neq 0$. Then $K = 0$ if and only if the function $r = r(t)$ is a solution of the equation*

$$\alpha(r)b_2(b_2(r)) + c\beta(r)(1 - b_2^2(r)) = 0\,, \quad 1 - b_2^2(r) > 0$$

and satisfy (1.4). *In particular, if $c = 0$, $r = r(t)$ is a linear function of the variable t, satisfying* (1.4).

Proof: It follows easily from (3.12) and (3.17), if we observe that $T_{11} \neq 0$ and $\lambda \neq 0$.

3.22. Considerations about type III; the proofs of Theorems 1.16, 1.19 and 1.20.

In the type III 2-SPES, $g : L^2 \to Q_c^{n+1}$ is an open piece of a 2-dimensional totally geodesic submanifold of Q_c^{n+1}. So g can be given by the intersection of Q_c^{n+1} with a 3-dimensional vector subspace P^3 of $\mathbb{R}^{n+2}$ if $c > 0$, or $\mathbb{L}^{n+2}$ if $c < 0$. If $c = 0$, g is a piece of a 2-plane, which can be assumed through the origin and generated by two vectors v_1 and v_2. If $c \neq 0$, we can assume that P^3 is generated by $\{p, v_1, v_2\}$, where p is the vertex of $\mathbb{H}^{n+1}$ $(c < 0)$ or any point of S^{n+1} $(c > 0)$, in a way that $< p, p >_c = c$ in any case, and $\{v_1, v_2\}$ is an orthonormal basis in $T_pQ_c^{n+1}$.

To give g in coordinates, let $\gamma(u_1) = \exp_p u_1 v_1$ be the geodesic of Q_c^{n+1}, through p, in the direction of v_1. Precisely,

$$(3.23)\qquad \gamma(u_1) = \begin{cases} p + u_1 v_1 & (c = 0)\,, \\ \cos u_1 p + \sin u_1 v_1 & (c = 1)\,, \\ \cosh u_1 p + \sinh u_1 v_1 & (c = -1)\,. \end{cases}$$

Observe that $p = 0$ if $c = 0$. Since $< v_2, v_1 >_c = 0$, v_2 is normal to $\gamma(u_1)$ at the point p. Let $V_2 = V_2(u_1)$ be the parallel transport of v_2 along γ, and in this case it is easy to see that $V_2 = v_2$ (note that $v_2 \in T_{\gamma(u_1)}Q_c^{n+1}$ and $< v_2, \dot{\gamma}(u_1) > = 0$).

Finally, taking $g(u_1, u_2) = \exp_{\gamma(u_1)} u_2 v_2$, we get

$$(3.24)\quad g(u_1,u_2) = \begin{cases} p + u_1 v_1 + u_2 v_2 & (c = 0)\,, \\ \cos u_2 \cos u_1 p + \cos u_2 \sin u_1 v_1 + \sin u_2 v_2 & (c = 1)\,, \\ \cosh u_2 \cosh u_1 p + \cosh u_2 \sinh u_1 v_1 + \sinh u_2 v_2 & (c = -1)\,. \end{cases}$$

3.25. Sketch of the proof of Theorem 1.16.

Since $A_\xi \equiv 0$ for all ξ, (2.2) and (2.5) take the simpler form (1.15). From the definition of b_i, $i = 1,2$ and (3.24), we obtain

(3.26)

$$\begin{cases} b_i = v_i\ , i = 1,2 & (c = 0)\,, \\ b_1 = -\sin u_1 p + \cos u_1 v_1 & (c = 1)\,, \\ b_2 = -\sin u_2 \cos u_1 p - \sin u_2 \sin u_1 v_1 + \cos u_2 v_2 & (c = 1)\,, \\ b_1 = \sinh u_1 p + \cosh u_1 v_1 & (c = -1)\,, \\ b_2 = \sinh u_2 \cosh u_1 p + \sinh u_2 \sinh u_1 v_1 + \cosh u_2 v_2 & (c = -1)\,. \end{cases}$$

Now following (3.10) to (3.12), we can compute by (1.15), the expressions of H_{ij}, P_{ij} and T_{ij} for the type III, and the desired result follows from formulas (2.17) and (2.21).

3.27. Proof of Theorem 1.19.

By putting the expression of $g(u_1, u_2)$ in (3.24) for $c = 0$ into (1.6), we obtain

$$(3.28)\qquad \psi(y, \xi_y) = u_1 b_1 + u_2 b_2 - r(u_1) r'(u_1) b_1 - r(u_1)\sqrt{1 - r'^2(u_1)}\ \xi_y\,.$$

Taking $\{\xi_1, \xi_2, \ldots \xi_{n-1}\}$ an orthonormal frame, normal to the plane spanned by $\{b_1, b_2\}$, we can write $\xi_y = \sum_{k=1}^{n-1} \phi_k \xi_k(y)$ where $\phi_k = \phi_k(t_1, t_2, \ldots, t_{n-2})$ is an orthogonal parametrization of the $n-2$ unit sphere. Observe that for each value of the variable u_2, we obtain the same rotation hypersurface M^{n-1} in the euclidean space spanned by $\{b_1, \xi_1, \xi_2, \ldots, \xi_{n-1}\}$. Then the envelope is a cylinder over a rotation hypersurface. It is not difficult to see that $n-1$ eigenvalues of the second fundamental form of M^n are the same as the eigenvalues of M^{n-1}, the remainder being zero. Then $H = \frac{n-1}{n} H'$, where H' is the mean curvature of M^{n-1} and the proof follows by observing, once again, that minimal hypersurfaces of rotation in $\mathbb{R}^n$ are catenoids.

Remarks. 1. It would be interesting and desirable to obtain a result similar to Theorem 1.19 for the cases $c = \pm 1$.

2. It is instructive to see what happens with the envelope of a line ($p = 1$) in $\mathbb{R}^3$. Taking an orthonormal frame $\{b_1, \xi_1, \xi_2\}$, in such a way that the line is parametrized by $c(s) = sb_1$, the local parametrization of the 1-parameter envelope is

$$\psi(s, \xi(s,t)) = sb_1 - r(s)r'(s)b_1 - r(s)\sqrt{1 - r'^2(s)}\ \xi(s,t)$$

where $\xi(s,t) = \cos t\xi_1 + \sin t\xi_2$. Then

$$\psi(s, \xi_s) = \left(s - r(s)r'(s), -r(s)\sqrt{1 - r'^2(s)}\ \cos t, -r(s)\sqrt{1 - r'^2(s)}\ \sin t\right)$$

which is a rotation surface in $\mathbb{R}^3$. The conclusion is that the minimal envelope of a line in $\mathbb{R}^3$ is a catenoid. The main result in [B] is a generalization to greater dimensions $n \geq 4$, of this observation, since minimal conformally flat hypersurfaces (wich are treated in [B]) can be seen as **1-parameter** envelopes. We point out that our Theorem 1.19 can be seen as an extension of the above result for **2-parameter** envelopes.

For the proof of Theorem 1.20 we will need the following

3.29. Lemma. *Let $g : L^2 \to Q_c^{n+1}$ be an isometric immersion with $n \geq 4$. If there is a point $y \in L^2$ such that for all unit vector $\xi \in (T_yL)^\perp$ we have* $\det A_\xi = k$ *(constant), then* $k = 0$.

3.30. Proof. Since $\dim(T_yL)^\perp = n - 1 \geq 3$, there exist a unit vector $\xi_0 \in (T_yL)^\perp$, which is normal to the first normal space of g. Therefore $\det A_{\xi_0} \geq 0$, and consequently $k \geq 0$.

If $k > 0$, take $\xi, \eta \in (T_yL)^\perp$ any orthonormal pair of vectors. Choose an orthonormal basis $\{b_1, b_2\} \in T_yL$ such that

$$\begin{cases} A_\xi(b_1) = \alpha b_1\ , & A_\xi(b_2) = \beta b_2\ , \\ A_\eta(b_1) = \gamma b_1 + \delta b_2\ , & A_\eta(b_2) = \delta b_1 + \mu b_2\ . \end{cases}$$

From the hypothesis, we have $\alpha\beta = k > 0$ and $\gamma\mu = k + \delta^2 > 0$.

Now, taking $\xi' = \frac{\sqrt{2}}{2}\xi + \frac{\sqrt{2}}{2}\eta \in (T_yL)^\perp$, we obtain

$$\begin{cases} A_{\xi'}(b_1) = \frac{\sqrt{2}}{2}(\alpha+\gamma)b_1 + \frac{\sqrt{2}}{2}\delta b_2\,, \\ A_{\xi'}(b_2) = \frac{\sqrt{2}}{2}\delta b_1 + \frac{\sqrt{2}}{2}(\beta+\mu)b_2\,. \end{cases}$$

So $k = \det A_{\xi'} = k + \frac{1}{2}(\alpha\mu + \beta\gamma)$, which implies $\alpha\mu + \beta\gamma = 0$. It follows that $k\mu^2 + \beta^2\gamma\mu = 0$ and $\gamma\mu = -\frac{k\mu^2}{\beta^2} < 0$, which is a contradiction. Then $k = 0$.

3.31. Proof of Theorem 1.20.

Since the proof of (ii) is similar, we will only present here the proof of (i).

Suppose there exists a 2-SPES $\psi : \Lambda^1 \to Q_c^{n+1}$ with *constant* H (the proof for *constant* K is similar). By (3.4), if $nH = k$, we obtain

$$(n\alpha^2 - 2)\alpha - k\beta\alpha^2 + (n\alpha^2 - 1 - k\alpha\beta^2)\mathrm{tr}A_\xi + (n\alpha - k)\beta^2 \det A_\xi = 0$$

for all regular point (y, ξ_y). Since A_ξ is a 2×2 matrix, we have $\mathrm{tr}A_{-\xi} = -\mathrm{tr}A_\xi$ and $\det A_{-\xi} = \det A_\xi$, then putting $-\xi$ in the place of ξ in the above expression, and adding up the results we get that $\det A_\xi$ is constant, for all $\xi_y \in (T_yL)^\perp$. Now, by Lemma 3.29 we obtain $\det A_\xi = 0$, $\forall\xi$, and the Gauss equation of $g : L^2 \to Q_c^{n+1}$ reduces to $K_g = c$. Since these surfaces are excluded from type I, the result follows.

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