THE UNIVERSAL ENVELOPING ALGEBRA OF A TRAIN ALGEBRA OF RANK 3[1]

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Abstract

We prove that for every representation of a finite-dimensional train algebra of rank 3, the algebra generated by the representation of the nil ideal is finite-dimensional and nilpotent. We use these results to prove that the universal enveloping algebra of a finite-dimensional train algebra of rank 3 is finite-dimensional.

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1. INTRODUCTION.

In what follows, F denotes an arbitrary infinite field of characteristic not 2. Let A be a not necessarily associative F-algebra. This F-algebra is *baric* when there exists a nonzero algebra homomorphism $w : A \to F$. In this case, w is called *the weight function of* A. If $a \in A$ is such that $w(a) = 1$ then we have a direct sum decomposition $A = Fa \oplus N$, where $N = Ker(w)$ is an ideal of A of codimension one. A commutative baric algebra A is called a *train algebra* if there exist elements $\gamma_1, \cdots, \gamma_{t-1} \in F$ such that every element $x \in A$ satisfies the equation

$$x^t + \gamma_1 w(x) x^{t-1} + \cdots + \gamma_{t-1} w(x)^{t-1} x = 0 \tag{1}$$

where the powers x^i of x, defined recursively by $x^1 = x$ and $x^i = x^{i-1}x$ for $i \geq 2$, are called the *principal powers of* x. The smallest t such that (1) holds identically in A is called the *rank* of A and the corresponding equation (1), which is unique, is called the *train equation of* A. Such structures were introduced by I. M. Etherington in 1939, aiming for an algebraic treatment of genetic problems, see [4].

2. PRELIMINAIRES.

In this paper we are interested in train algebras of rank 3. See [2], [3] and [4] for more details about these algebras. It is easy to see that in equation (1) we always have $1 + \gamma_1 + \cdots + \gamma_{t-1} = 0$. In particular, calling $\gamma_2 = \gamma$, train algebras of rank 3 must satisfy the train equation

$$x^3 - (1+\gamma) w(x) x^2 + \gamma w(x)^2 x = 0 \tag{2}$$

We will assume from now on that $2\gamma \neq 1$. This ensures the existence of at least one idempotent of weight one in A, see [4]. Let $A = Fe \oplus A_{1/2} \oplus A_\gamma$ be the Peirce decomposition of A relative to the idempotent e, where $A_{1/2} = \{x \in N \ / \ ex = 1/2x\}$ and $A_\gamma = \{x \in N \ / \ ex = \gamma x\}$ are the proper subspaces corresponding to the proper values $\frac{1}{2}$ and γ of the linear operator $L_e : N \to N$ defined by $L_e(x) = ex$. The following inclusions hold (see Costa and Suazo [3]):

$$A_{1/2}^2 \subseteq A_\gamma, \ A_{1/2}A_\gamma \subseteq A_{1/2}, \ A_\gamma^2 = 0 \qquad (3)$$

Moreover, $A_\gamma \neq 0$. Otherwise, A must satisfy the identity $x^2 - w(x)x = 0$ and so A will have rank 2.

We remark that $N = Ker(w)$ satisfies $x^3 = 0, \ \forall \ x \in N$. It is an ideal sometimes called *the nil ideal* of A. Linearizing the identity $x^3 = 0$, we get the Jacobi's identity

$$x(yz) + y(zx) + z(xy) = 0 \qquad (4)$$

A *Jordan algebra* is a commutative not necessarily associative algebra A satisfying the identity $(x^2y)x = x^2(yx)$. The following result is well known.

Lemma 1. *Every commutative algebra satisfying the identity $x^3 = 0$ is a Jordan algebra.*

An algebra A over F is called *power-associative* in case the subalgebra of A generated by every element x in A is associative. This is equivalent to the fact that the principal powers of every element x in A, satisfy the identities $x^i x^j = x^{i+j}, \ \forall \ i,j = 1,2 \cdots$. It is known [8] that every Jordan algebra is power-associative. An element x in a power-associative algebra A is called *nilpotent* in case there is an integer r such that $x^r = 0$. An algebra (an ideal) consisting only of nilpotent elements is called a *nilalgebra* (*nilideal*). A nonassociative algebra A is called *nilpotent* in case the descending chain of subalgebras $A^1 = A$, $A^t = \sum_{r+s=t} A^r A^s$ reaches $0 : A^t = 0$ for some t. The smallest t such that $A^t = 0$ is called the *index of nilpotence* of A.

Proposition 1.(Albert) Every finite-dimensional Jordan nilalgebra over a field of characteristic $\neq 2$ is nilpotent.

Proof. See [8, Corollary, p. 92].

Let A be an algebra which belongs to a class C of commutative algebras over F and M be a vector space over F. Following Eilenberg [5], a linear map $\mu : A \to End(M)$ is called *a representation on A in the class C* if the split null extension $\overline{A} = A \oplus M$ of M, with multiplication given by $(a+m)(b+n) = ab + \mu(a)(n) + \mu(b)(m)$, $\forall a, b \in A$, $m, n \in M$, belongs to the class C.

We denote by I_M the identity function defined on M. It is easy to prove the following result:

Lemma 2. *If A is a train algebra of rank 3, a linear map $\mu : A \to End(M)$ is a representation on A if and only if for every $x \in A$ we have*

$$2\mu(x)^2 + \mu(x^2) - 2(1+\gamma)w(x)\mu(x) + \gamma w(x^2)I_M = 0 \tag{5}$$

3. UNIVERSAL ENVELOPING ALGEBRA.

Let A be a train algebra of rank 3. An *enveloping algebra* of A is a pair (B, π) consisting of an associative algebra B with identity element 1_B and a representation $\pi : A \to B$; that is, a linear map satisfying

$$2\pi(a)^2 + \pi(a^2) = 2(1+\gamma)w(a)\mu(a) - \gamma w(a^2)1_B \tag{6}$$

for all $a \in A$. In particular, if $\mu : A \to End(M)$ is a representation, then $(End(M), \mu)$ is an enveloping algebra of A. A *universal enveloping algebra* of A is an enveloping algebra (U, ϕ) of A such that if (B, π) is any other enveloping of A, then there exists a unique algebra homomorphism $\widehat{\pi} : U \to B$ such that $\widehat{\pi} \circ \phi = \pi$. Thus, any representation $\mu : A \to End(M)$ lifts to a representation $\widehat{\mu} : U \to End(M)$.

Standard arguments show that if U exists, it is unique up to isomorphism (compare [6, Thm. 1, p. 152]), and so we refer to U as *the* universal enveloping algebra of A. The proof of the existence of such universal enveloping follows standard arguments. We begin with the tensor algebra $T(A)$ on A and let J be the ideal in $T(A)$ generated by all $2a \otimes a + a^2 - 2(1+\gamma)w(a)\pi(a) + \gamma w(a^2)1_B$, $a \in A$. Define $U(A) := T(A)/J$ and let $\phi : T(A) \to U(A)$ be the canonical homomorphism. Restricting ϕ to $A \subseteq T(A)$ we obtain a linear map $\phi : A \to U(A)$. We claim that $(U(A), \phi)$ is the universal enveloping of A. Indeed, let (B, π) be an enveloping algebra of A. As $T(A)$ is associative and freely generated by A, the map π extends to an homomorphism of $T(A)$ to B, which we denote also by π and which sends 1 to 1. The fact that (B, π) is an enveloping of A forces all $2a \otimes a + a^2 - 2(1+\gamma)w(a)\pi(a) + \gamma w(a^2)1_B$, $a \in A$, to lie in $Ker(\pi)$, so π induces a homomorphism $\widehat{\pi} : U(A) \to B$ such that $\widehat{\pi} \circ \phi = \pi$.

It is proved in [7] that for a finite-dimensional Jordan algebra A, the universal enveloping algebra $U(A)$ is finite-dimensional. For a nonzero Lie algebra A, $U(A)$ is always infinite-dimensional but is Noetherian when the

Lie algebra is finite-dimensional. For Bernstein algebras, some conditions on A are required to guarantee that its universal enveloping algebra is finite-dimensional or Noetherian (see [1]). Here we establish some properties of $U(A)$ for train algebras of rank 3.

We will say that the map μ is *r-dimensional* if M is a r-dimensional vector space.

Theorem 1. *Let $A = Fe \oplus A_{1/2} \oplus A_\gamma$ be a train algebra of rank 3. Then:*

1.- If $b \in A_\gamma - A_{1/2}^2$ then there exists a 2-dimensional representation $\mu : A \to End(M)$ such that $\mu(b) \neq 0$.

2.- If $b \in A_{1/2} - A_{1/2}A_\gamma$ then there exists a 2-dimensional representation $\mu : A \to End(M)$ such that $\mu(b) \neq 0$.

Proof. (1) Assume $b \in A_\gamma - A_{1/2}^2$. Let $A = Fe \oplus A_{1/2} \oplus Fb \oplus A_{1/2}^2$ where $A_\gamma = Fb \oplus A_{1/2}^2$. For $a = \alpha e + u + \beta b + w$, with $\alpha, \beta \in F$, $w \in A_{1/2}^2$, $u \in A_{1/2}$ define $\mu : A \to End(M)$ by

$$\mu(\alpha e + u + \beta b + w) = \begin{pmatrix} \frac{1}{2}\alpha & 0 \\ \beta & \frac{1}{2}\alpha \end{pmatrix}.$$

As $a^2 = \alpha^2 e + (2\alpha u + 2\beta bu + 2wu) + 2\alpha\beta\gamma b + (u^2 + 2\gamma w) = \alpha^2 e + u' + 2\alpha\beta\gamma b + w'$ where $u' \in A_{1/2}, w' \in A_{1/2}^2$, we have:

$$2\mu(a)^2 + \mu(a^2) - 2(1+\gamma)w(a)\mu(a) + \gamma w(a^2) I_M =$$

$$\begin{pmatrix} \frac{1}{2}\alpha^2 & 0 \\ 2\alpha\beta & \frac{1}{2}\alpha^2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\alpha^2 & 0 \\ 2\alpha\beta\gamma & \frac{1}{2}\alpha^2 \end{pmatrix}$$

$$-2(1+\gamma)\alpha\begin{pmatrix}\frac{1}{2}\alpha & 0\\ 2\beta & \frac{1}{2}\alpha\end{pmatrix}+\gamma\alpha^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=0$$

and so μ is a representation of A such that $\mu(b)\neq 0$.

(2) Here we proceed similarly. Assume $b\in A_{1/2}-A_{1/2}A_\gamma$. Let $A=Fe\oplus Fb\oplus Z$ where $Z=A'_{1/2}\oplus A_\gamma$, $A'_{1/2}\supseteq A_{1/2}A_\gamma$ and $A_{1/2}=Fb\oplus A'_{1/2}$.

Taking $\mu: A\to End(M)$ by $\mu(\alpha e+\beta b+z)=\begin{pmatrix}\frac{1}{2}\alpha & 0\\ \beta & \alpha\gamma\end{pmatrix}$

$\forall\ \alpha,\beta\in F,\ z\in Z$, then:

$$2\mu(a)^2+\mu(a^2)-2(1+\gamma)w(a)\mu(a)+\gamma w(a^2)I_M=$$

$$\begin{pmatrix}\frac{1}{2}\alpha^2 & 0\\ \alpha\beta+2\alpha\beta\gamma & 2\alpha^2\gamma^2\end{pmatrix}+\begin{pmatrix}\frac{1}{2}\alpha^2 & 0\\ \alpha\beta & \frac{1}{2}\alpha^2\gamma\end{pmatrix}$$

$$-2(1+\gamma)\alpha\begin{pmatrix}\frac{1}{2}\alpha & 0\\ \beta & \alpha\gamma\end{pmatrix}+\gamma\alpha^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=0$$

so μ is a representation of A such that $\mu(b)\neq 0$. □

Corollary 1. *Let A be a train algebra of rank 3. Suppose that $A_{1/2}=S_{1/2}\oplus A_\gamma A_{1/2}$, where $S_{1/2}$ is any complementary subspace of $A_\gamma A_{1/2}$, and similarly assume $A_\gamma=S_\gamma\oplus A^2_{1/2}$. Then the mapping $\phi: A\to U(A)$, as explained above, restricted to the subspace $Ke\oplus S_{1/2}\oplus S_\gamma$, is a monomorphism.*

Proof. Assume for $\phi : A \to U(A)$, that $x = \zeta e + s + t \in \ker \phi$, where $s \in S_{1/2}$ and $t \in S_\gamma$. Then x is in the kernel of every representation of A. Now if $s \neq 0$, we may find a representation $\mu : A \to End(M)$ as in Th. 1, part (2), so that $\mu(x) = \mu(\zeta e + s) \neq 0$. Consequently, $s = 0$ and $\zeta = 0$ must hold. But then if $x = t \neq 0$, we may construct a representation as in Th. 1, part (1), for which $\mu(t) \neq 0$. Consequently $x = 0$ and the restriction of ϕ to $Ke \oplus S_{1/2} \oplus S_\gamma$ is a monomorphism. □

Recall the universal property of $U(A)$: if (B, π) is any other enveloping of A, then there exists a unique algebra homomorphism $\widehat{\pi} : U \to B$ such that $\widehat{\pi} \circ \phi = \pi$, where $\phi : A \to U(A)$ is the canonical linear map called the universal representation of A.

Lemma 3. *If $\pi : A \to B$ is a representation of a train algebra of rank 3, then for every $a, b \in N$, we have*

$$\pi(a)\pi(b) + \pi(b)\pi(a) + \pi(ab) = 0 \text{ and } \pi(a)^3 = 0 \tag{7}$$

Proof. Since π is a representation of A, relation (6) implies that $2\pi(a)^2 + \pi(a^2) = 0$, $\forall\, a \in N$. Linearizing this identity and canceling out by the factor 2 we have $\pi(a)\pi(b) + \pi(b)\pi(a) + \pi(ab) = 0\ \forall\ a, b \in N$. Moreover, using $a^3 = 0$ and the relation $2\pi(a)^2 + \pi(a^2) = 0$ we have that $0 = \pi(a^3) = \pi(a^2a) = -\pi(a^2)\pi(a) - \pi(a)\pi(a^2) = 2\pi(a)^2\pi(a) + \pi(a)\pi(a)^2 = 4\pi(a)^3$. □

Proposition 2. *Let A be a finite-dimensional train algebra of rank 3. Then for every representation $\pi : A \to B$ on A, the subalgebra $< \pi(N) >$ of B, generated by $\pi(N)$, is finite-dimensional.*

Proof. Let $\{n_1, \cdots, n_p\}$ be a basis of N. Then $\pi(N)$ is the subspace generated by $\pi(n_1), \cdots, \pi(n_p)$ and consequently the subalgebra $< \pi(N) >$

$= \{ \sum_{finite} \alpha\pi(n_{i1})^{\alpha_{i1}} \cdots \pi(n_{ik})^{\alpha_{ik}}$ / not all $\alpha_{ij} = 0\}$. Since $\pi(n)^3 = 0$, for each element $n \in N$, we have that $0 \leq \alpha_{ij} \leq 2$. By relation (7) we have $\pi(n_{ij})\pi(n_{ik}) = -\pi(n_{ik})\pi(n_{ij}) - \pi(n_{ij}n_{ik})$ and $n_{ij}n_{ik} = \sum_{l=1}^{p} \beta_{il}^{jk} n_l$. Therefore $< \pi(N) >$ is described by $< \pi(N) > = \{\sum \alpha\pi(n_1)^{\alpha_1} \cdots \pi(n_p)^{\alpha_p}$ / not all $\alpha_j = 0, \ 0 \leq \alpha_j \leq 1\}$. Thus dim $< \pi(N) > \leq 2^p - 1$. □

Remark : Lemma 1 and Proposition 1 imply that N is nilpotent.

Theorem 2. *If A is a finite-dimensional train algebra of rank 3, then for every representation $\pi : A \to B$ on A, the subalgebra $< \pi(N) >$ of B generated by $\pi(N)$, is nilpotent.*

Proof. Let $\pi : A \to B$ be a representation on A. Then $\pi(N)$ is a vector subspace of B and by relation (7) it is a subalgebra of B^+, where B^+ is a special Jordan algebra with multiplication given by $a \circ b = ab + ba$, for all elements $a, b \in B$. Moreover $(\pi(N), \circ, +)$ is nilpotent. In fact, define the map $\overline{\pi} : N \to < \pi(N) >^+$ by $\overline{\pi}(a) = -\pi(a)$. Relation (7) implies that $\overline{\pi}$ is an algebra homomorphism, that is, $\overline{\pi}(n_1 n_2) = \overline{\pi}(n_1) \circ \overline{\pi}(n_2)$. Therefore $N \ / \ Ker(\overline{\pi}) \simeq \overline{\pi}(N)$. Since N is finite-dimensional and nilpotent, then $\overline{\pi}(N) = (\pi(N), \circ, +)$ is a finite-dimensional nilpotent subalgebra of $< \pi(N) >^+$. Now by [8, Prop. 2, p. 96] we conclude that $< \pi(N) >$ is a nilpotent subalgebra of B. □

Let us consider the free associative algebra $Ass[\phi(a_1), \cdots, \phi(a_n)]$, where $\{a_1, \cdots, a_n\}$ is a basis of A. For every enveloping algebra (B, π) of A, define the homomorphism $I_\pi : Ass[\phi(a_1), \cdots, \phi(a_n)] \to B$ by $I_\pi(\phi(a_i)) = \pi(a_i) \ \forall \ i = 1, \cdots, n$. Let now $I = \cap_\pi I_\pi$; then I is an ideal of $Ass[\phi(a_1), \cdots, \phi(a_n)]$ and $U(A) = Ass[\phi(a_1), \cdots, \phi(a_n)] \ / \ I$.

Theorem 3. *If A is a finite-dimensional train algebra of rank 3, then $U(A)$ is finite-dimensional.*

Proof. Let $\{e, u_1, \cdots, u_m, v_1, \cdots, v_k\}$ be a basis of A. It is clearly sufficient to prove that there exists a positive number t such that any polynomial of $Ass[\phi(e), \cdots, \phi(v_k)]$, with more than t elements from N, belongs to $I = \cap_\pi I_\pi$. We know that the subalgebra of B generated by $\pi(N)$ is nilpotent for every representation $\pi : A \to B$ on A. Moreover, $\dim < \pi(N) > \leq 2^p - 1$ so that the index of nilpotence of $< \pi(N) >$ is $\leq 2^p$, where $p = \dim(N)$. Therefore, there exists a positive number $t = 2^p$ such that for every representation π of A, we have $\pi(n_1)\cdots\pi(n_t) = 0 \quad \forall\, n_i \in N$. Let $q = q(\phi(e), \cdots, \phi(v_k))$ be a polynomial of $Ass[\phi(e), \cdots, \phi(v_k)]$ with more than t elements from N. By the universal property of $U(A)$, there exists a unique algebra homomorphism $\widehat{\pi} : U(A) \to B$ such that $\widehat{\pi} \circ \phi = \pi$, and $\widehat{\pi}(p(\phi(e), \cdots, \phi(v_k))) = p(\widehat{\pi}(\phi(e)), \cdots, \widehat{\pi}(\phi(v_k))) = p(\pi(e), \cdots, \pi(v_k))$. Linearizing the first equality in (7) and canceling out by the factor 2 we have that, for every $a, b \in A$,

$$\pi(a)\pi(b) + \pi(b)\pi(a) + \pi(ab) = (1+\gamma)w(a)\pi(b) + (1+\gamma)w(b)\pi(a) - \gamma w(ab)1_B \tag{8}$$

It follows from (8) that

$$2(\pi(e))^2 = (1+2\gamma)\pi(e) - \gamma 1_B \tag{9}$$

and

$$\pi(e)\pi(u) = (\frac{1}{2}+\gamma)\pi(u) - \pi(u)\pi(e), \ \ \pi(e)\pi(v) = \pi(v) - \pi(v)\pi(e). \tag{10}$$

for every $u \in A_{1/2}, v \in A_\gamma$. The element $\pi(e)$ disappears or is at the end of the polynomial q by relations (9) and (10). So if we have more than

$t = 2^p$ elements of N then $q \in KerI_\pi$, for every representation π, since $\pi(n_1)\cdots\pi(n_t) = 0 \; \forall \; n_i \in N$. Therefore $q \in I = \cap_\pi I_\pi$. □

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