RT-MAT 2000-04

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Março 2000

Esta é uma publicação preliminar ("preprint").

ON STRONG CHAINS OF UNCOUNTABLE FUNCTIONS

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Abstract: For functions $f, g : \omega_1 \to \omega_1$, where ω_1 is the first uncountable cardinal, we write that $f \ll g$ if and only if $\{\xi \in \omega_1 : f(\xi) \geq g(\xi)\}$ is finite. We prove the consistency of the existence of a well-ordered increasing $\ll$-chain of length ω_2, solving a problem of A. Hajnal. The methods involve previously developed by us *forcing with side conditions in morasses* which is a variation on Todorcevic's *forcing with models as side conditions.* The paper is self contained and requires from the reader the knowledge of Kunen's textbook and some basic experience with proper forcing and elementary submodels.

1. Introduction.

It is natural to consider generalizations of standard orders in $\wp(\omega)$ or ω^ω modulo finite sets. One natural way is to consider similar orders in $\wp(\kappa)$ or κ^κ modulo sets of cardinality smaller than κ. This usually leads to similar diagonal or forcing constructions as far as the question of the existence of well-ordered chains is concerned.

Another way is to look at orders modulo finite sets. Let us look at a few definitions in

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[1] Some of the research leading to this paper was supported by NSF of USA Grant DMS-9505098 held by the author at Auburn University, AL, USA.

[2] Some of the research leading to this paper was done while the author was visiting Ohio University at Athens , OH, USA from September 1997 till March 1998.

[3] Some of the research leading to this paper was done at Universidade de São Paulo, where the author is working since 1998.

We would like to thank the set-theory groups from these universities for their hospitality.

the case of $\kappa = \omega_1$ beyond which we do not stray in this paper. $\mathcal{F} \subseteq [\omega_1]^{\omega_1}$ is called *strongly almost disjoint* if $a \cap b$ is finite for each $a, b \in \mathcal{F}$. Similar definition holds for strongly almost disjoint family of functions from ω^{ω_1}. We say that $\{X_\alpha : \alpha < \lambda\} \subseteq [\omega_1]^{\omega_1}$ is a *strong chain* of type λ whenever $X_\alpha - X_\beta$ is finite and $X_\beta - X_\alpha$ is uncountable for $\alpha < \beta < \lambda$. For orders in $\omega_1^{\omega_1}$ there are several natural definitions. If $f, g \in \omega_1^{\omega_1}$, we may consider sets

$$\text{“} =_{f,g} \text{”} = \{\xi : f(\xi) = g(\xi)\}$$

$$\text{“} >_{f,g} = \text{”} \{\xi : f(\xi) > g(\xi)\}$$

We say that $f \ll g$ if and only if both $=_{f,g}$ and $>_{f,g}$ are finite and we say that $f \leq^* g$ if and only if $>_{f,g}$ is finite. Clearly one could also consider orders whose definitions mention countable sets. We will not do this in this paper.

Let us mention a few results about the existence of some families. Baumgartner ([B1]) has proved that there is a c.c.c. forcing which adds a strongly almost disjoint family in $[\omega_1]^{\omega_1}$ of cardinality ω_2. It also follows from his results that arbitrarily large strongly almost disjoint families in $[\omega_1]^{\omega_1}$ consistently exist.

Zapletal ([Z]) proved that it is consistent that arbitrarily large strongly almost disjoint families live in ω^{ω_1}, and he used Todorcevic's result ([T4]) to note that the existence of a c.c.c. forcing adding strongly almost disjoint family in ω^{ω_1} of size ω_2 is equivalent to the failure of Chang's conjecture.

In paper [Ko2], we proved that Chang's conjecture implies the nonexistence of strong chains of type ω_2 in $[\omega_1]^{\omega_1}$. This result implies that if Chang's Conjecture is consistent, then there is a model of ZFC in which there is a strongly almost disjoint family in $[\omega_1]^{\omega_1}$ of size ω_2 but where strong chains of length ω_2 do not exist.

We also proved in [Ko2] that such strong chains consistently exist, and for this we used a c.c.c. forcing constructed with the help of Todorcevic's ρ-function which incorporates Jensen's $\square_{\omega_1}$ principle (see [T2]). Note that strong chains give rise to $<^*$-chains in $\omega_1^{\omega_1}$ by taking the characteristic functions.

In this paper we go further and we prove the following

THEOREM 1.1 *It is consistent that there exists an $\ll$-chain of type ω_2.*

This answers a question of A. Hajnal, we thank S. Todorcevic for communicating this question to us. For this we do not use a c.c.c. forcing, and we prove that such chains cannot be added from e.g., L by a c.c.c. forcing. Namely we have the following:

THEOREM 1.2 *Assume CH. There is no forcing P which satisfies the c.c.c. such that in V^P there is an $\ll$-chain of type ω_2.*

PROOF: Let $C = (\dot{f}_\alpha : \alpha < \omega_2)$ be the P-names for the elements of the chain. Using CH find $M \prec H(\omega_3)$ such that $|M| = \omega_1$, $M \cap \omega_2 = \alpha \in \omega_2$, $C \in M$ and $[M]^\omega \subseteq M$. In particular we have $\omega_1 \subseteq M$ and $cf(\alpha) = \omega_1$.

By induction on $\xi < \omega_1$ we construct strictly increasing $(\beta_\xi : \xi < \omega) \subseteq \omega_1$, $(\alpha_\xi : \xi < \omega_1) \subseteq \alpha$ and $(d_\xi : \xi < \omega_1) \subseteq \omega_1$, $(p_\xi : \xi < \omega_1) \subseteq P$ and $(F_\xi : \xi < \omega_1) \subseteq [\omega_1]^{<\omega}$ such that:

1) $\forall 0 < \xi < \omega_1 \ P \Vdash \forall \eta \leq \xi \ \dot{f}_{\alpha_0}(\beta_\eta) + d_\xi \leq \dot{f}_\alpha(\beta_\eta), \dot{f}_{\alpha_\xi}(\beta_\eta)$.
2) $\forall \xi < \omega_1 \ p_\xi \Vdash \dot{f}_\alpha(\beta_\xi) = \dot{f}_{\alpha_0}(\beta_\xi) + \check{d}_\xi$,
3) $\forall \xi < \omega_1) \ p_\xi \Vdash \{\beta : \dot{f}_{\alpha_\xi}(\beta) \geq \dot{f}_\alpha(\beta)\} \subseteq \dot{F}_\xi$.

Suppose we are done with 1), 2) 3) for all $\eta < \xi$. Let us construct β_ξ, α_ξ, d_ξ, p_ξ, F_ξ. By the c.c.c. of P, there is $\gamma < \omega_1$ such that

$$P \Vdash \{\beta : \dot{f}_{\alpha_0}(\beta) \geq \dot{f}_\alpha(\beta)\} \subseteq \check{\gamma}$$

Pick any $\beta_\xi > \beta_\eta, \gamma$ for $\eta < \xi$. Let d_ξ be the least countable ordinal such that there is $p \in P$ such that $p \Vdash \dot{f}_\alpha(\beta_\xi) = \dot{f}_{\alpha_0}(\beta_\xi) + \check{d}_\xi$. This implies that

$$1 \Vdash \dot{f}_\alpha(\beta_\xi) \geq \dot{f}_{\alpha_0}(\beta_\xi) + \check{d}_\xi.$$

Using ω_1, $[M]^\omega \subseteq M$ we have $(d_\eta : \eta \leq \xi)$ and $(\beta_\eta : \eta \leq \xi)$ in M, so by the elementarity and the inductive assumption, there is $\alpha_\xi \in M$, $\alpha_\xi > \alpha_\eta$ for $\eta < \xi$ such that

$$\forall \eta \leq \xi \ 1 \Vdash \dot{f}_{\alpha_\xi}(\beta_\eta) \geq \dot{f}_{\alpha_0}(\beta_\eta) + \check{d}_\eta.$$

Let $p_\xi \leq p$ and $F_\xi \in [\omega_1]^{<\omega}$ be such that 3) is satisfied. This completes the construction. W.l.o.g. we can assume that F_ξ's form a Δ-system with root $\Delta \in [\omega_1]^{<\omega}$. By thining out we can w.l.o.g. assume that $\beta_\eta \notin F_\xi$ for $\eta < \xi < \omega_1$. Using the c.c.c. for P, find $\eta < \xi < \omega_1$ such that there is $q \in P$ with $q \leq p_\eta, p_\xi$. We have

$$1 \geq q \Vdash \dot{f}_{\alpha_\xi}(\beta_\eta) \geq \dot{f}_{\alpha_0}(\beta_\eta) + \check{d}_\eta.$$

$$p_\eta \geq q \Vdash \dot{f}_\alpha(\beta_\eta) = \dot{f}_{\alpha_0}(\beta_\eta) + \check{d}_\eta.$$

So we may conclude by 3) that

$$p_\xi \geq q \Vdash \beta_\eta \in \check{F}_\xi$$

a contradiction. □

Using the fact that $<^*$-chains were added in [Ko2] by a c.c.c. forcing over some models of CH and using theorem 1.1, we obtain:

THEOREM 1.3. *It is consistent that there is an $<^*$-chain of type ω_2 (or equivalently a strong chain in $(P(\omega_1)/Fin, \subseteq^*)$) but there is no $\ll$-chain of type ω_2.*

Our chain of functions is naturally bounded by one function. This is related to weak Chang's Conjecture (see e.g. [DL]).
The forcing techniques we use here are the fusion of techniques from [Ko2] and [Ko3]. Note that e.g., CH implies the nonexistence of strongly almost disjoint families or strong chains etc., because they yield objects of size ω_2 within $[\omega]^\omega$.
All these results suggest the development of cardinal invariants of $\omega_1^{\omega_1}$ or $[\omega_1]^{\omega_1}$ in the spirit of cardinal invariants p, t, b, d, etc. A thrilling flavour of this theory would be the link of the new cardinal invariants with set-theoretic principles of various consistency strength. As in the case of the strong chains in $[\omega_1]^{\omega_1}$, one can wonder if long $\ll$-chains may give rise to interesting combinatorial or topological constructions as it is in the case of chains of various kinds in the usual orders on $[\omega]^\omega$ or ω^ω; see e.g. [T3], [vD], [N] [Sch]. Note here that strongly almost disjoint families have been used in Boolean algebras or topology, see e.g., [BS] or [R].

The paper is organized as follows: in section 2 we outline facts about Velleman's simplified morasses which are used in the following sections. We look at morasses in question as families living in $[\omega_2]^\omega$ and so all the "morass structure" can be expressed in an intuitive language of relations $\in$ and $\subseteq$. This approach due to Velleman, we believe, practically makes Velleman's morasses an object as simple and natural as the ordinals.
In section 3, we outline the idea of the method of *forcing with side conditions in morasses.* In section 4, we perform the main forcing construction.
The notation and terminology is fairly standard and follow [K] and [B2], in particular we use $[\lambda]^\kappa$ for the family of all subsets of λ of cardinalities κ. $[\alpha, \beta)$ and $(\alpha, \beta]$ denote the intervals with respect to the usual order on ordinals whenever α, β are ordinals. $ordtp(X)$ denotes the ordertype of the set of ordinals X. $H(\kappa)$ denotes the collection of all sets of hereditary cardinality less than κ.

2. Facts about Velleman's simplified morasses.

In the definition below, we will use the following notation: $\mathcal{F}|X = \{Y \in \mathcal{F} : Y \subset X\}$ and $X < Y$ if and only if $\alpha < \beta$ for all ordinals $\alpha \in X$ and $\beta \in Y$.

DEFINITION 2.1. ([V1, V2]) *Let κ be a regular cardinal. A simplified $(\kappa, 1)$-morass is a family $\mathcal{F} \subseteq [\kappa^+]^{<\kappa} (= \{X \subseteq \kappa^+ : |X| < \kappa\})$ which satisfies the following conditions.*

1) $\mathcal{F}$ is well-founded with respect to inclusion.

2) $\mathcal{F}$ is locally small i.e. $\forall X \in \mathcal{F}\ |\mathcal{F}|X| < \kappa$

3) $\mathcal{F}$ is homogenous i.e. if $X, Y \in \mathcal{F}$, $rank(X) = rank(Y) = \alpha$, then X, Y have the same order type (denoted θ_α) and if f_{XY} is the unique order preserving mapping from X onto Y, then $\mathcal{F}|Y = \{f''(Z) : Z \in \mathcal{F}|X\}$.

4) $\mathcal{F}$ is directed i.e. $\forall\ X, Y \in \mathcal{F}\ \exists Z \in \mathcal{F}$ s.t. $X, Y \subseteq Z$

5) $\mathcal{F}$ is locally almost directed, i.e.,

a) $\mathcal{F}|X$ is directed or

b) $\exists\ X_1, X_2 \in \mathcal{F}$ s.t. $rank(X_1) = rank(X_2)$ & $X = X_1 * X_2$

where $X = X_1 * X_2$ means that $X = X_1 \cup X_2$, $X_1 \cap X_2 < X_1 - X_2 < X_2 - X_1$, $\mathcal{F}|X = \mathcal{F}|X_1 \cup \mathcal{F}|X_2 \cup \{X_1, X_2\}$

6) $\mathcal{F}$ covers κ^+ i.e. $\bigcup \mathcal{F} = \kappa^+$.

If $\forall X \in \mathcal{F}\ (rank(X) \neq 0 \Rightarrow X = \bigcup \mathcal{F}|X)$, then we say that $\mathcal{F}$ is a neat simplified $(\kappa, 1)$-morass.

Thus, a $(\kappa, 1)$-morass is in particular a directed set of size κ^+ with initial fragments of cardinalities less than κ. The morasses were introduced by R. Jensen (see [D]). Their intention is to provide an order along which inductive constructions of directed systems of structures can be carried out. In some situation we encounter problems with handling initial fragments of constructions if they have size κ. In the language of [T2], a morass can be named a stepping-up tool, it unables us to step-up properties of κ, obtained by the usual induction, to κ^+, since the initial fragments of the constructions are of sizes less than κ. In the above sense every well-founded directed set of size κ^+ with initial fragments of sizes less than κ is a stepping-up tool. Additional strength and the essence of a morass as well as other nontrivial stepping-up frameworks is hidden in *coherence properties* of the framework.

The existence of a morass is a principle with enormous variety of consequeces (see [Ka] for classical "non-generic" ones) such as the existence of a κ-Kurepa tree or a κ^+-Aronszajn tree, often a κ^+-Souslin tree, weak $\Box_\kappa$. Acctually, D. Velleman has proved that the existence of a $(\kappa, 1)$-morass is equivalent to a a forcing axiom for a certain wide class of forcings (see [V2]).

In this note we will need several properties of Velleman's simplified morasses, for convenience of the reader let us prove them within the formalism which we are using.

THE MAIN LEMMA 2.2. (implicit in [V1] and [V2]) *Let $\mathcal{F}$ be a $(\kappa,1)$-morass. Let $X, Y \in \mathcal{F}$, $rank(X) = rank(Y)$, $\alpha \in X \cap Y$, then $X \cap \alpha = Y \cap \alpha$.*

PROOF: By induction on $rank(Z)$ such that $X, Y \subseteq Z \in \mathcal{F}$. If 5 a) holds for $\mathcal{F}|Z$, then $\exists Z_1 \subset Z$ such that $X, Y \subseteq Z_1 \subset Z$, so by inductive hypothesis we are done.

If 5 b) holds i.e., $Z = Z_1 * Z_2$, so say $X \subseteq Z_1$, $Y \subseteq Z_2$, (otherwise we are done by inductive hypothesis), then we have $\alpha \in Z_1 \cap Z_2$, since $\alpha \in X \cap Y$. Consider $f''_{Z_1,Z_2}(X) \subseteq Z_2$, then by homogeneity $rank(f''_{Z_1,Z_2}(X)) = rank(X) = rank(Y)$. We know also that $\alpha \in f''_{Z_1,Z_2}(X)$, since by 5b) and the homogeneity $f_{Z_1,Z_2}|Z_1 \cap Z_2$ is an identity. Now, by inductive hypothesis for Z_2, we obtain that $f''_{Z_1,Z_2}(X) \cap \alpha = Y \cap \alpha$, but again since $f_{Z_1,Z_2}|Z_1 \cap Z_2 = Id_{Z_1 \cap Z_2}$, we have $f''_{Z_1,Z_2}(X) \cap \alpha = X \cap \alpha$, so $Y \cap \alpha = X \cap \alpha$. $\Box$

DENSITY LEMMA 2.3. ([V2]): *Let $\mathcal{F}$ be a $(\kappa, 1)$-morass. Then the following conditions are satisfied:*

a) $\forall Y \in \mathcal{F}\ \forall X \in (\mathcal{F}|Y)\ \forall rank(X) < \alpha < rank(Y)\ \exists Z \in \mathcal{F}\ rank(Z) = \alpha,\ X \subseteq Z \subseteq Y$

This implies that $ht(\mathcal{F}) \leq \kappa$,since $\mathcal{F}$ is locally small.

b) $\forall X \in \mathcal{F}\ \forall \alpha < ht(\mathcal{F})[rank(X) < \alpha \Rightarrow \exists Z \in \mathcal{F}(rank(Z) = \alpha,\ X \subseteq Z)]$

PROOF: First we prove part b). Fix $X \in \mathcal{F}$ take $\alpha < ht(\mathcal{F})$ and take $Y \in \mathcal{F}$ of minimal rank such that $X \subseteq Y$, $rank(Y) \geq \alpha$. There is such a Y, since $\mathcal{F}$ is directed (take any element $Y' \in \mathcal{F}$, $rank(Y') \geq \alpha$ and find $Y \in \mathcal{F}$ such that $X, Y' \subseteq Y$).

If $rank(Y) = \alpha$ we are done. We will prove that $rank(Y) > \alpha$ give rise to a contradiction. We apply 5) of definition 1.1. to Y.

If $\mathcal{F}|Y$ is directed, then there is $Z \in \mathcal{F}|Y$, $rank(Z) = \alpha$ as $rank(Y) > \alpha$. take $Z' \in \mathcal{F}|Y$ such that $X, Z \subseteq Z'$, now $rank(Z') \geq \alpha$ and this contradicts the minimality of the rank of Y.

If $Y = Y_1 * Y_2$, then $rank(Y) = rank(Y_i) + 1$, so $rank(Y_i) \geq \alpha$ and $X \in \mathcal{F}|Y_1$ or $X \in \mathcal{F}|Y_2$. This also contradicts the minimality of the rank of Y. This completes the proof of part b).

Fix $\alpha < ht(\mathcal{F})$, $X, Y \in \mathcal{F}$ and $X \subseteq Y$ such that $rank(X) < \alpha < rank(Y)$. Using the part b), find $Z_1, Z_2 \in \mathcal{F}$ such that $X \subseteq Z_1 \subseteq Z_2$ and $rank(Z_1) = \alpha$ and $rank(Z_2) = rank(Y)$. Consider f_{Z_2Y}, we get $f_{Z_2Y}(X) \subseteq f_{Z_2Y}(Z_1)$ and $f_{Z_2Y}(Z_1) \in \mathcal{F}$ and $rank(f_{Z_2Y}(Z_1) = \alpha$. It is enough to prove that $f_{Z_2Y}(X) = X$, i.e., $f_{Z_2Y}(\alpha) = \alpha$ for all α's in X. But if $\alpha \in X$ and $\alpha \in Z_2 \cap Y$, the main lemma implies that $ordtp(\alpha \cap Z_2) = ordtp(\alpha \cap Y)$ as f_{Z_2Y} is order preserving, so $f_{Z_2Y}(\alpha) = \alpha$. $\Box$

DEFINITION 2.4. *Let $\mathcal{F}$ be $(\kappa, 1)$-morass. Let $\alpha \in \kappa^+$. The sequence $(\mathcal{F}_\xi(\alpha))_{\xi<\kappa}$ is called the α-sequence (with respect to $\mathcal{F}$) iff for all $\xi \in \kappa$ we have $\mathcal{F}_\xi(\alpha) = X_\xi \cap \alpha$, where $X_\xi \in \mathcal{F}$ is such that rank$(X_\xi) = \xi$, $\alpha \in X_\xi$, if there exists such an X_ξ and otherwise $\mathcal{F}_\xi(\alpha) = \emptyset$*

LEMMA 2.5. *Let $\mathcal{F}$ be $(\kappa, 1)$-morass. Let $\alpha \in \kappa^+$, then the α-sequence is a uniquely defined non-decreasing (continuous if $\mathcal{F}$ is neat) sequence in $[\alpha]^{<\kappa}$, with its union equal to α.*

PROOF: The uniqueness follows from the main lemma. Now let $\xi < \xi' < \kappa$. If $\mathcal{F}_\xi(\alpha) \neq \emptyset$, then by the density lemma there is $Z \in \mathcal{F}$ of rank ξ' such that $\mathcal{F}_\xi(\alpha) \subseteq Z$ and the main lemma implies that $Z \cap \alpha = \mathcal{F}_{\xi'}(\alpha)$, so $\mathcal{F}_{\xi'}(\alpha) \supseteq \mathcal{F}_\xi(\alpha)$. Directedness and covering of κ^+ imply that the union is equal to α. □

Note that in particular a $(\kappa, 1)$-morass is a cofinal family $\mathcal{F}$ in $[\kappa^+]^{<\kappa}$ which is a union of at most κ many subfamilies $\mathcal{F}_\alpha$ for $\alpha < \kappa$ (i.e, $\mathcal{F}_\alpha$ consists of elements of rank α) such that for every two $f, f' \in \mathcal{F}_\alpha$ we have $f \cap f' < f - f', f' - f$ by the main lemma.

Note that the families $\mathcal{F}_\alpha$'s cannot be Δ-systems, i.e, have the property that there is $\Delta_\alpha \in [\kappa^+]^{<\kappa}$ such that for each $f, f' \in \mathcal{F}_\alpha$ we have $\Delta_\alpha = f \cap f'$. This follows from a general fact that no regular λ can carry a cofinal family that is a union of less than λ Δ-systems. The proof of this fact, goes as follows: Suppose $\mathcal{F} = \bigcup_{\alpha<\lambda} \mathcal{F}_\alpha$, where each $\mathcal{F}_\alpha$ is a Δ-system with a root Δ_α. Find a bound $\delta < \lambda$ of all Δ_α'. Now, $\delta + 1$ belongs to at most one element of each $\mathcal{F}_\alpha$, take ξ a bound for this family. Now there is no $d \in \mathcal{F}$ which contains $\delta + 1, \xi + 1$, hence $\mathcal{F}$ is not cofinal. Note that for λ singular the situation is opposite.

Note that there is no family $\mathcal{F} \subseteq [\lambda]^{<\kappa}$, satisfying definition 2.1. for $\lambda > \kappa^+$. To see this, suppose that $\lambda > \kappa^+$ and consider κ^+-sequence, as defined in definition 2.4., by lemma 2.5. it covers κ^+, but κ^+ is regular, so it cannot be covered by this sequence. In [Ko1] we have generalized the notion of simplified morass to the notion of (κ, λ)-semimorass which is a family in $[\lambda]^{<\kappa}$ and exsits consistently for any cardinal λ bigger than κ.

In this paper we will be interested just in the case of $\kappa = \omega_1$, thus our morasses will be subfamilies of $[\omega_2]^\omega$. We will morover require that the morass we will be denoting by $\mathcal{F}$ is a stationary coding set (see [Zw]). This means that $\mathcal{F}$ is stationary subset of $[\omega_2]^\omega$ and that there is a one-to-one function $c : \mathcal{F} \to \omega_2$ such that

$$\forall X, Y \in \mathcal{F} \quad X \subset Y \;\Rightarrow c(X) \in Y.$$

The forcing proof of the existence of neat morasses which are stationary coding sets which is based on a proof of Velleman from [V2] can be obtained from the corresponding proof for semimorasses in [Ko1] (Theorem 3, Section 2). Let's note two simple facts about stationary coding sets which we learned from S. Todorcevic.

FACT 2.6. (folklore) *Suppose that $\mathcal{F} \subseteq [\omega_2]^\omega$ is a stationary coding set and $\mathcal{F} \in M \prec H(\omega_3)$, $|M| = \omega$, $M \cap \omega_2 \in \mathcal{F}$. If $X \in \mathcal{F}$ and $X \subset M$, then $X \in M$.*

PROOF: Suppose $X \in \mathcal{F}$ and $X \subset M$. As $\mathcal{F} \in M \prec H(\omega_3)$, we have that M *thinks* that $\mathcal{F}$ is a stationary coding set, so there is $c : \mathcal{F} \to \omega_2$ witnessing this fact in M. In particular $\alpha = c(X) \in M \cap \omega_2$, so M thinks that $c^{-1}(\alpha)$ is a countable subset of ω_2, hence $X \in M \cap [\omega_2]^\omega$ as required.

FACT 2.7. (folklore) *Suppose that* $\mathcal{F} \in M \prec H(\omega_3)$, $|M| = \omega$, $X = M \cap \omega_2 \in \mathcal{F}$. *Then* $rank(X) = M \cap \omega_1$.

PROOF: Put $rank(X) = \delta$. If $\delta \in M$, then we would have in M an element Y of $\mathcal{F}$ of rank δ and this would give rise by homogeneity to an isomorphism between $\mathcal{F}|Y$ and $\mathcal{F}|X$ which would contradict well-foundedness of $\mathcal{F}$.
In M there are all ordinals less than δ and so there are also elements of $\mathcal{F}$ of all ranks less than δ. They are included in $M \cap \omega_2 = X$, so $rank(X)$ is at least δ.

The fact below is crucial in our method of forcing with side condition in morasses which we introduced in [Ko3] and which is outlined in the context of this paper in the following section.

FACT 2.8. *Suppose that* $\mathcal{F} \subseteq [\omega_2]^\omega$ *is a stationary coding set and* $\mathcal{F} \in M \prec H(\omega_3)$, $|M| = \omega$, $M \cap \omega_2 = X_0 \in \mathcal{F}$. *Let* $Y \in \mathcal{F}$, $rank(Y) < M \cap \omega_1 = \delta$. *Then there is* $Z(Y) \in M$ *such that*
1) $Y \cap X_0 \subseteq Z(Y)$.
2) $rank(Z(Y)) = rank(Y)$.

Note that by the main lemma, it follows that $Z(Y)$ is an end-extension of $X_0 \cap Y$.
PROOF: By density lemma, find $Y' \supseteq Y$, $Y' \in \mathcal{F}$ such that $rank(Y') = rank(X_0) = \delta$. Now use the isomorphism $f_{Y'X_0}$ to find a copy $Z(Y)$ of Y below X_0. Note that $Y \cap X_0 \subseteq Y' \cap X_0$ and $f_{Y'X_0}$ is constant on $Y' \cap X_0$ so $Y \cap X_0 = Y \cap M \subseteq Z(Y)$. Now use fact 2.6 to conclude that $Z(Y) \in \mathcal{F}|X_0$ implies $Z(Y) \in M$.

Now we will consider some notions of distance. For every $\alpha_1 < \alpha_2 < \omega_2$ such that there is $X \in \mathcal{F}$, $rank(X) \leq \beta$ and $\alpha_1, \alpha_2 \in X$ we can define $D_\beta(\alpha_1, \alpha_2)$ by putting it equal to $ordtp(X \cap (\alpha_1, \alpha_2])$. This "$\beta$-distance" behaves very nicely and was used in a diguise provided by ρ-function in [Ko2]. It turns out that for the purposes of this paper we need a more subtle and complex notion which still is a version of the above notion.

DEFINITION 2.9 *Suppose* $A \in [\mathcal{F}]^{<\omega}$ *and* $\beta \in \omega_1$, *then we say that* $\alpha_1 < \alpha_2 < \omega_2$ *are* β-*connected in* A *if and only if there is a sequence* $W_1, ..., W_k$ *of elements of* A *of ranks less or equal to* β *and if there is a sequence of ordinals* $\gamma_1, ..., \gamma_k$ *such that* $\alpha_1 = \gamma_1 < ... < \gamma_k < \alpha_2$ *and such that* $\alpha_1 = \gamma_1 \in W_1$, $\alpha_2 \in W_k$ *and* $\gamma_i \in W_i \cap W_{i-1}$ *for* $1 < i \leq k$. *We will say that* α *is* β-*connected in* A *with itself.*
If $\alpha \in \omega_2$, *then by* $A_\beta(\alpha)$ *we denote the set* $X \cap \alpha$ *for* $X \in A$ *of maximal rank less or equal to* β *such that* $\alpha \in X$. *If there is no such* X, *we put* $A_\beta(\alpha) = \emptyset$.

FACT 2.10 *Suppose that* $A \in [\mathcal{F}]^{<\omega}$, $\beta < \omega_1$ *and* $\alpha_1, \alpha_2, \alpha_3 \in \omega_2$; $\alpha_1 < \alpha_2 < \alpha_3$.
a) If α_1, α_2 *are* β-*connected in* A *and* α_2, α_3 *are* β-*connected in* A, *then* α_1, α_3 *are* β-*connected in* A *as well.*
b) If α_1, α_3 *are* β-*connected in* A *and* α_2, α_3 *are* β-*connected in* A, *then* α_1, α_2 *are* β-*connected in* A *as well.*

PROOF: a) follows directly from the definition of being β-connected in A.
For b) let $W_1, ...W_k$, $\gamma_1, ...\gamma_k$ witness the fact that α_1, α_3 are β-connected in A and let

$W'_1, ...W'_m, \gamma'_1, ...\gamma'_m$ witness the fact that α_2, α_3 are β-connected in A. The proof of the lemma is by induction on $k+m$.
Let Z be the one of the elements W_k or W'_m whichever has biggest rank or any of these elements if they have the same ranks. By the main lemma we have that $\gamma_k, \gamma'_m \in Z$.
Case 1. $\gamma_k = \gamma'_m$. Then the fact follows directly from the inductive assumption.
Case 2. $\gamma_k < \gamma'_m$. $W_1, ...W_{k-1}, Z, \gamma_1, ...\gamma_k$ witness the fact that α_1 and γ'_m are β connected in A (and this takes care of the case when $\alpha_2 = \gamma'_m$) and $W'_1, ...W'_{m-1}, \gamma'_1, ...\gamma'_{m-1}$ witness the fact that α_2, γ'_m are β-connected in A. So we can use the inductive assumption to conclude that α_1, α_2 are β-connected in A
Case 3. $\gamma'_m < \gamma_k$. $W'_1, ...W'_{m-1}, Z, \gamma'_1, ...\gamma'_m$ witness the fact that α_2 and γ_k are β connected in A and $W_1, ...W_{k-1}, \gamma_1, ...\gamma_{k-1}$ witness the fact that α_1, γ_k are β-connected in A. So we can use the inductive assumption to conclude that α_1, α_2 are β-connected in A.

DEFINITION 2.11 *Suppose $A \in [\mathcal{F}]^{<\omega}$, $\beta < \omega_1$ and $\alpha_1, \alpha_2 \in \omega_2$; $\alpha_1 < \alpha_2$. We define $d_{A,\beta}(\alpha_1, \alpha_2)$ by induction, i.e., given that $d_{A,\beta}(\cdot, \alpha)$ are defined for all $\alpha < \alpha_2$ we define $d_{A,\beta}(\alpha_1, \alpha_2)$ by induction on $\alpha_1 < \alpha_2$ by*

$$d_{A,\beta}(\alpha_1, \alpha_2) = \sup\{d_{A,\beta}(\alpha_1, \gamma) + ordtp(A_\beta(\alpha_2) - \gamma) :$$

$$\alpha_1 \leq \gamma < \alpha_2; \gamma \in A_\beta(\alpha_2); \alpha_1, \gamma \text{ are } \beta\text{-connected in } A\}$$

FACT 2.12 *Suppose that $A \in [\mathcal{F}]^{<\omega}$, $\beta < \omega_1$ and $\alpha_1, \alpha_2 \in \omega_2$; $\alpha_1 < \alpha_2$.*
Then $d_{A,\beta}(\alpha_1, \alpha_2) \neq 0$ if and only if α_1, α_2 are β-connected in A.

PROOF: Note that if α_1, α_2 are ordinals which are not β-connected, then by fact 2.10. a) the set in the definition of $d_{A,\beta}(\alpha_1, \alpha_2)$ is empty. Conversely if α_1, α_2 are β-connected in A, then there is $\gamma \in A_\beta(\alpha_2)$ such that $\alpha_1 \leq \gamma < \alpha_2$ and α_1, γ are β-connected in A and $\gamma \in A_\beta(\alpha_2)$. Then $ordtp(A_\beta(\alpha_2) - \gamma) \neq 0$ and so $d_{A,\beta}(\alpha_1, \alpha_2) \neq 0$.

FACT 2.13 *Suppose that $A \in [\mathcal{F}]^{<\omega}$, $\beta < \omega_1$ and $\alpha_1, \alpha_2, \alpha_3 \in \omega_2$; $\alpha_1 < \alpha_2 < \alpha_3$ and ordinals α_1, α_2 are β-connected in A as well as ordinals α_2, α_3 are β-connected in A. Then the following hold:*
1) If $\alpha' < \alpha_2$ and there is $X \in A$, $rank(X) \leq \beta$; $\alpha', \alpha_2 \in X$, then

$$ordtp(A_\beta(\alpha_2) - \alpha') \leq d_{A,\beta}(\alpha', \alpha_2)$$

2) For any $Y \in A$ such that $\alpha', \alpha_2 \in Y$, $\alpha' < \alpha_2$ and $rank(Y) \leq \beta$ we have

$$ordtp(Y \cap [\alpha', \alpha_2)) \leq d_{A,\beta}(\alpha', \alpha_2)$$

3)

$$d_{A,\beta}(\alpha_1, \alpha_3) = d_{A,\beta}(\alpha_1, \alpha_2) + d_{A,\beta}(\alpha_2, \alpha_3)$$

PROOF: For 1) put $\gamma = \alpha'$ in the definition of $d_{A,\beta}(\alpha', \alpha)$. For 2) note that $Y \cap [\alpha', \alpha_2) \subseteq A_\beta(\alpha_2) - \alpha'$ and use 1).

3) Is proved by induction on α_3; Suppose 3) holds for all triples of ordinals whose maximal ordinal is less than α_3. Before moving to the main body of the proof of the inductive step of 3) we prove the following two claims:

CLAIM 1: Suppose $\alpha_1 \leq \gamma_1 < \gamma_2$ and $\gamma_1, \gamma_2 \in A_\beta(\alpha_3)$ and α_1, γ_1 are β-connected in A and α_1, γ_2 are β-connected in A, then

$$d_{A,\beta}(\alpha_1, \gamma_1) + ordtp(A_\beta(\alpha_3) - \gamma_1) \leq d_{A,\beta}(\alpha_1, \gamma_2) + ordtp(A_\beta(\alpha_3) - \gamma_2)$$

PROOF:

$$d_{A,\beta}(\alpha_1, \gamma_1) + ordtp(A_\beta(\alpha_3) - \gamma_1) =$$

$$= d_{A,\beta}(\alpha_1, \gamma_1) + ordtp(A_\beta(\alpha_3) \cap [\gamma_1, \gamma_2)) + ordtp(A_\beta(\alpha_3) \cap [\gamma_2, \alpha_3)) \leq$$

(by 2))

$$d_{A,\beta}(\alpha_1, \gamma_1) + d_{A,\beta}(\gamma_1, \gamma_2) + ordtp(A_\beta(\alpha_3) - \gamma_2)$$

which completes the proof of claim 1 by the inductive assumption.

CLAIM 2: Suppose $\alpha_2 \leq \gamma < \alpha_3$ and $\gamma \in A_\beta(\alpha_3)$. Then γ, α_2 are β-connected in A and γ, α_1 are β-connected in A.
PROOF: It follows from 2.10 and the assumptions about $\alpha_1, \alpha_2, \alpha_3$ and γ.

So now, let's go to the main body of the inductive step for 3).
By the definition 2.11 we have

$$d_{A,\beta}(\alpha_1, \alpha_3) = \sup\{d_{A,\beta}(\alpha_1, \gamma) + ordtp(A_\beta(\alpha_3) - \gamma) :$$

$$\alpha_1 \leq \gamma < \alpha_3; \gamma \in A_\beta(\alpha_3); \alpha_1, \gamma \text{ are } \beta\text{-connected in } A\}$$

As α_2, α_3 are β-connected in A and so let $W_1, ...W_k$, $\gamma_1, ...\gamma_k$ witness this fact. Then

$$\alpha_1 \leq \gamma_k < \alpha_3; \gamma_k \in A_\beta(\alpha_3)$$

and α_1, γ_k are β-connected in A by claim 2. This together with claim 1 means that actually

$$d_{A,\beta}(\alpha_1, \alpha_3) = \sup\{d_{A,\beta}(\alpha_1, \gamma) + ordtp(A_\beta(\alpha_3) - \gamma) :$$

$$\alpha_2 \leq \gamma < \alpha_3; \gamma \in A_\beta(\alpha_3); \alpha_1, \gamma \text{ are } \beta\text{-connected in } A\}$$

Now using claim 2 and the fact that α_1, α_2 are β-connected in A we conclude that actually

$$d_{A,\beta}(\alpha_1, \alpha_3) = \sup\{d_{A,\beta}(\alpha_1, \gamma) + ordtp(A_\beta(\alpha_3) - \gamma) :$$

$$\alpha_2 \leq \gamma < \alpha_3; \gamma \in A_\beta(\alpha_3); \alpha_2, \gamma \text{ are } \beta\text{-connected in } A\}$$

Now use the inductive assumption to get

$$d_{A,\beta}(\alpha_1, \gamma) = d_{A,\beta}(\alpha_1, \alpha_2) + d_{A,\beta}(\alpha_2, \gamma)$$

and also use the fact that $\sup(\alpha + A) = \alpha + \sup A$ (observe that $\sup(A + \alpha)$ may not be equal to $\sup A + \alpha$) and conclude that

$$d_{A,\beta}(\alpha_1, \alpha_3) = d_{A,\beta}(\alpha_1, \alpha_2) + \sup\{d_{A,\beta}(\alpha_2, \gamma) + ordtp(A_\beta(\alpha_3) - \gamma) :$$

$$\alpha_2 \leq \gamma < \alpha_3; \gamma \in A_\beta(\alpha_3); \alpha_2, \gamma \text{ are } \beta\text{-connected in } A\} =$$
$$= d_{A,\beta}(\alpha_1, \alpha_2) + d_{A,\beta}(\alpha_2, \alpha_3)$$

which completes the proof of the fact.

DEFINITION 2.14. *Suppose $\beta \in \omega_1$. We call $\xi \in \omega_1$ a β-number if an only if for every countable $M \prec H(\omega_3)$ such that $\mathcal{F} \in M$, we have that $\beta \in M$, implies $\xi \in M$.*

REMARK 2.15. Note that β-numbers are bounded in ω_1, form an initial segment of ω_1, finite sums of β-numbers are β-numbers, if $\beta' < \beta$ then β'-numbers are β-numbers. Also θ_β which is the order type of any set of $\mathcal{F}$ of rank β is a β-number which is well defined by the homogeneity of $\mathcal{F}$.

FACT 2.16 *For every $\beta < \omega_1$ and $A \in [\mathcal{F}]^{<\omega}$ and every $\alpha_1 < \alpha_2 < \omega_2$ the number $d_{A,\beta}(\alpha_1, \alpha_2)$ is a β-number.*

PROOF: First note that using the main lemma it is easy to conclude that if α_1, α_2 are β-connected in some A then $\alpha_1 \in \mathcal{F}_\beta(\alpha_2)$. Now is easy to prove by induction that

$$d_{A,\beta}(\alpha_1, \alpha_2) \leq ordtp(\mathcal{F}_\beta(\alpha_2) \cap [\alpha_1, \alpha_2)) \leq \theta_\beta$$

The last number is a β-number by remark 2.15. This completes the proof of the fact.

DEFINITION 2.17. *Suppose $\xi_1 < \xi_2 < \omega_1$ and $\beta < \omega_1$, we write $\xi_1 <^\beta \xi_2$ if and only if $\xi_1 + \eta < \xi_2$ for any β-number $\eta < \omega_1$.*

3. Forcing with side-conditions in morasses

In [Ko3] we introduced a version of Todorcevic's method of forcing with models as side conditions which utilizes elements of (semi)morasses as side conditions. In fact a natural way of looking at morasses is to see them as families similar to $\{M \cap \omega_2 : M \prec H(\omega_3)\}$ with some extra properties (which actually make it impossible for a morass to include a club set unlike the set above). To use elements of a morass $\mathcal{F}$ as side conditions means to use forcings P whose conditions are of the form (p, A) where p is a finite condition of a natural forcing adding the structure in question and A is a finite subset of $\mathcal{F}$. The order is given by the forcing order on the first coordinate and the inverse inclusion on the second coordinate. In addition we require the existence of some natural projections of p onto the elements of A as a part of the definition of the forcing notion.

In the case of forcings described above, special combinatorial properties of morasses allow us to perform many manouvers with ease as well as the definitions are simplified. This method seems equivalent to the variant of Todorcevic's method when one employs matrices of models (see [T1] section 4., for an example with detailed definitions). Instead of a more complicated forcing that adds a version of morass and the structure in question "in one blow" we factor this forcing into one adding a morass (or we actually just assume the existence of it) and another simple forcing employing the morass. The price we need to pay for this convenience is that P is not proper (unlike Todorcevic's forcings,) but only

$\mathcal{F}$-proper, i.e., There is a club $C \subseteq [\omega_2]^\omega$ such that for models $M \prec H(\omega_3)$ such that $M \in \mathcal{F} \cap C$ and $p \in P \cap M$, there are (P, M)-generic conditions stronger than p. As $\mathcal{F}$ may be assumed to be stationary, $\mathcal{F}$-properness implies the preservation of ω_1 (proof like for proper forcings, see [B2]). The preservation of bigger cardinals follows from the ω_2-chain condition.

To illustrate the method and show the crucial use of combinatorial properties of a morass which is a stationary coding set, let us consider the following forcing notion P whose conditions are of the form (a_p, f_p, A_p) where $a_p \in [\omega_2]^{<\omega}$, $f_p : a_p \to \omega_1$, $A_p \in [\mathcal{F}]^{<\omega}$ and we require that $f_p(\alpha) < rank(X)$ whenever $\alpha \in a_p \cap X$ for $X \in A_p$. We consider P with the coordinatewise inverse inclusion as the order.

P adds an $f : \omega_2 \to \omega_1$ which is unbounded on any uncountable subset of ω_2 belonging to the ground model. Note that such functions cannot be added by any c.c.c. forcing and on the other hand our P in a sense approximates f with finite approximatitions. As a warming up argument before the main argument of this paper of the next section let us prove the following facts.

FACT 3.1. *P is an $\mathcal{F}$-proper notion of forcing and hence preserves ω_1.*

PROOF: Let $M \prec H(\omega_3)$ be such that $X_0 = M \cap \omega_2 \in \mathcal{F}$, $P, \mathcal{F} \in M$. By the stationarity of $\mathcal{F}$ we can easily find such an M. Let $p_0 \in M$. We will show that there is $p \leq p_0$ which is (P, M)-generic.

Define p as follows $a_p = a_{p_0}$, $f_p = f_{p_0}$, $A_p = A_{p_0} \cup \{X_0\}$. Clearly $p \in P$.

Now to prove that p is a (P, M)-generic, consider $q \leq p$ and a dense $D \in M$. We may w.l.o.g. assume that $q \in D$.

Define $q|M = (a_q \cap M, f_q|M, A_q \cap M)$. Introduce notation $\delta = M \cap \omega_1 = rank(M \cap \omega_2)$ (the second equality follows from fact 2.7).

Note that $A_q \cap M = A_{q|M} = \{X \in A_q : X \subset X_0\}$. This follows from the fact that $\mathcal{F}$ is a stationary coding set i.e., fact 2.6.

Now we would like to reflect q to M, i.e., find $s \leq q|M$, $s \in M$, having some properties of q (in particular $s \in D$) and use these properties to prove that s and q are compatible which would finish the proof of the fact.

Condition s will satisfy a formula $\phi(\sigma)$ with parameters from M. We will obtain the existence of s by the elementarity of M and the fact that $\phi(q)$ holds in $H(\omega_3)$. Let us write parts of the conjuction which forms ϕ. Before this, we need to define some parameters from M.

DEFINITION 3.2.
a) Let $X_1, ...X_m$ *and* $m \in \omega$ *be such that* $\{X \in A_q : rank(X) < \delta\} = \{X_1, ..., X_m\}$.
b) Let $Y_i = X_i \cap X_0$.
(Note that $Y_i \in M$ *for* $i \in [m]$*. This follows from lemma 2.8.and the main lemma)*
c) $\phi(\sigma)$ *is the conjunction of the following formulas i), ii), iii) :*
i) $\sigma \in D$.
ii) $\sigma \leq q|M$.
iii) there is $Z \in \mathcal{F}$ *such that*
  1) For every $i \in [m]$ *we have* $Y_i \subseteq Z$
  2) $a_{q|M} \subseteq Z$.
  3) $(a_\sigma - a_{q|M}) \cap Z = \emptyset$.

FACT 3.3.$\phi(q)$ *holds in* $H(\omega_3)$.

PROOF: Clauses i) and ii) are clear and iii) is witnessed by $Z = X_0$.

Now we continuue the proof of the fact that P is $\mathcal{F}$-proper. Let s be such a condition of P that M satisfies $\phi(s)$ (usually ϕ is much more complicated). We will note that s and q are compatible by constructing their amalgamation r. Let $a_r = a_s \cup a_q$, $f_r = f_s \cup f_q$, $A_r = A_s \cup A_q$. The only nontrivial condition which we need to check is whether $f_r(\alpha) < rank(X)$ whenever $\alpha \in a_r \cap X$ for $X \in A_r$. So consider $\alpha \in a_r$ and $X \in A_r$. There are two nontrivial cases: first if $\alpha \in a_q - a_s$ and $X \in A_s$ which cannot take place because $A_s \subset M$ and $a_q - a_s$ is disjoint from M; the second case if $\alpha \in a_s - a_q$ and $X \in A_q$. Note that the fact that s satisfies clause iii) implies that this can happen only if $rank(X) \geq \delta$, but $f_s(\alpha) < \delta$ because $s \in M$ which completes the proof.

FACT 3.4 *P is an* ω_2*-cc notion of forcing.*

PROOF: Let $(p_\xi : \xi < \omega_2)$ be a sequence of conditions of P. We will show that there are two compatible conditions among the elements of this sequence.
We may w.l.o.g. assume that for every $\xi < \omega_2$ the set A_{p_ξ} has a maximal element denoted by X_ξ. This follows from the directedness of $\mathcal{F}$. We may also w.l.o.g. assume that all X_ξ's have the same rank $\alpha < \omega_1$ and that $f_{X_{\xi_1}, X_{\xi_2}}$ lifts up to an isomorphism between p_{ξ_1} and p_{ξ_2} for every $\xi_1, \xi_2 \in \omega_2$. This follows from the homogeneity and local smallness of $\mathcal{F}$ as well as from the fact that a_{p_ξ}'s and A_{p_ξ}'s are finite. Now we claim that actually any two conditions from the sequence are compatible. Consider two of them p, q with maximal elements X_p and X_q of A_p and A_q respectivelly. Put $a_r = a_p \cup a_q$, $f_r = f_q \cup f_p$ and finally put $A_r = A_p \cup A_q$.
Now it is enough to check that it is a condition which follows from the isomorphism of p and q.

As a corollary we obtain the following

FACT 3.5. *P preserves cardinals.*

4. Forcing construction

$\mathcal{F}$ will denote an $(\omega_1, 1)$-morass which is a stationary coding set.

DEFINITION 4.1. *We define forcing notion P. Conditions p of P are of the form*

$$p = (a_p, b_p, F_p, A_p),$$

where $0 \in a_p \in [\omega_2]^{<\omega}, b_p \in [\omega_1]^{<\omega}$, $A_p \in [\mathcal{F}]^{<\omega}$, $F_p = \{f_p^\alpha : \alpha \in a_p\}$, *and* $f_p^\alpha : b_p \to \omega_1$, *and for each* $\beta \in b_p$ *we have* $f_p^0(\beta) = 0$ *and*

**) $\forall \beta \in b_p \forall \alpha \in a_p$ $f_p^\alpha(\beta)$ is a β-number.*

***) $\forall \beta \in b_p \forall \alpha_1 < \alpha_2; \alpha_1, \alpha_2 \in a_p$, if $d_{A_p,\beta}(\alpha_1, \alpha_2) \neq 0$, then*

$$f_p^{\alpha_2}(\beta) \geq f_p^{\alpha_1}(\beta) + d_{A_p,\beta}(\alpha_1, \alpha_2)$$

$p \leq q$ if and only if $a_p \supseteq a_q$, $b_p \supseteq b_q$, $A_p \supseteq A_q$, $f_p^\alpha \supseteq f_q^\alpha$ for all $\alpha \in a_q$ and

****) $\forall \beta \in b_p - b_q \forall \alpha_1 < \alpha_2; \alpha_1, \alpha_2 \in a_q \quad f_p^{\alpha_2}(\beta) > f_p^{\alpha_1}(\beta)$.*

Recall Shelah's notion of (P, M)-generic condition (see [B2]): $p \in P$ is called a (P, M)-generic if and only if for every $D \in M$ which is dense in P we have that $D \cap M$ is predense below p. Most of this section we devote to the proof of the following

STATEMENT 4.2. Suppose $P, \mathcal{F}, p \in M$; $M \prec H(\omega_3); M \cap \omega_2 \in \mathcal{F}$; Then for any $p \in P \cap M$ the condition $p_0 = (a_p, b_p, F_p, A_p \cup \{M \cap \omega_2\})$ is a (P, M)-generic which extends p.

PROOF: Take $q \leq p_0, D \in M$ dense in P. We may w.l.o.g. assume that $q \in D$. Define $q|M = (a_q \cap M, b_q \cap M, \{f_\alpha^p | M : \alpha \in a_q \cap M\}, A_q \cap M)$. Introduce notation $\delta = M \cap \omega_1 = rank(M \cap \omega_2)$ (the second equality follows from fact 2.7).
$X_0 = M \cap \omega_2 \in \mathcal{F}$ (the membership follows from the assumption).
Note that $A_q \cap M = A_{q|M} = \{X \in A_q : X \in M, X \subset X_0\}$. This follows from the fact that $\mathcal{F}$ is a stationary coding set i.e., fact 2.6. Also as *) is satisfied for $q \in P$, we may conclude that $f_{q|M}^\alpha(\beta) \in M$, for $\beta, \alpha \in M$.
Clearly by the above we have that $q|M \in M$, now let us see that $q|M \in P$. For this, the only nontrivial clause of the definition 4.1. which needs to be checked is **). If $\beta \in b_{q|M}$ $\alpha_1, \alpha_2 \in a_{q|M}$ and $d_{A_{q|M},\beta}(\alpha_1, \alpha_2) \neq 0$, then $d_{A_q,\beta}(\alpha_1, \alpha_2) \neq 0$ by fact 2.12. Now as $A_{q|M} \subseteq A_q$ we have $(A_{q|M})_\beta(\alpha) \subseteq (A_q)_\beta(\alpha)$, so it is easy to prove by induction that $d_{A_{q|M},\beta}(\alpha_1, \alpha_2) \leq d_{A_q,\beta}(\alpha_1, \alpha_2)$ and consequently that

$$f_{q|M}^{\alpha_2}(\beta) = f_q^{\alpha_2}(\beta) \geq f_q^{\alpha_1}(\beta) + d_{A_q,\beta}(\alpha_1, \alpha_2) =$$

$$= f_{q|M}^{\alpha_1}(\beta) + d_{A_q,\beta}(\alpha_1, \alpha_2) \geq f_{q|M}^{\alpha_1}(\beta) + d_{A_{q|M},\beta}(\alpha_1, \alpha_2)$$

which completes the proof that **) holds for $q|M$ and so that $q|M$ is a condition of P.

Now we would like to reflect q to M, i.e., find $s \leq q|M$, $s \in M$, having some properties of q (in particular $s \in D$) and use these properties to prove that s and q are compatible which would finish the proof of the statement.
Condition s will satisfy a formula $\phi(\sigma)$ with parameters from M. We will obtain the existence of s by the elementarity of M and the fact that $\phi(q)$ holds in $H(\omega_3)$. Let us write parts of the conjuction which form ϕ. Before this we need to define some parameters from M.

DEFINITION 4.3.
a) Let $X_1, \ldots X_m$ *and* $m \in \omega$ *be such that* $\{X \in A_q : rank(X) < \delta\} = \{X_1, \ldots, X_m\}$.
b) Let $r_i = rank(X_i)$ *for* $i \in [m]$.
c) Let $Y_i = X_i \cap X_0$.
(Note that $Y_i \in M$ *for* $i \in [m]$. *This follows from lemma 2.8. and the main lemma)*
d) Let $\beta_0 = \max\{r_1, \ldots r_m, \max(b_{q|M}) + 1\}$.
e) Let $\{\xi_1, \ldots, \xi_l\} = (a_q - M) \cap (supM \cap \omega_2)$.
f) Let $\eta_i = \min(M - \xi_i)$ *for* $i \in [l]$.
g) Let $\nu_j = \max(\{\sup(Y_i \cap \eta_j) : i \in [m]\}$ *for* $j \in [l]$.
(Note that as Y_i*'s are countable in* $H(\omega_3)$ *and they belong to* M*, they are countable in* M*, and so* ν_j*s are in* M *and they are smaller than corresponding* η_j*s.)*

h) $\phi(\sigma)$ *is the conjunction of the following formulas i), ii), iii) :*
i) $\sigma \in D$.
ii) $\sigma \leq q|M$.
iii) There is $Z \in A_\sigma, rank(Z) > \beta_0$ *such that:*
  1) For every $X \in A_\sigma$ *such that* $rank(X) \leq \beta_0$ *there is* $i \in [m]$ $X \cap Z = Y_i$.
  2) For every $i \in [m]$ *there is* $X \in A_\sigma$ *such that* $rank(X) \leq \beta_0$ *and* $X \cap Z = Y_i$.
  3) $a_{q|M} \subseteq Z$.
  4) $(a_\sigma - a_{q|M}) \cap Z = \emptyset$.
  5) $(b_\sigma - b_{q|M}) \cap (rank(Z)) = \emptyset$.
  6) For all $j \in [l]$ *we have* $\nu_j, \eta_j \in Z$ *and there is* $\zeta_j \in Z$ *such that*

$$\nu_j < \zeta_j < \eta_j$$

  and

$$0 <^{\beta_0} ordtp(Z \cap [\nu_j, \zeta_j)).$$

FACT 4.4. $\phi(q)$ *holds in* $H(\omega_3)$.
PROOF: Clause i) is clear from our assumption on q.
To see ii) first note that we already proved that $q|M$ is a condition of P. We need to check only ***). So let $\beta \in b_q - b_{q|M}$, $\alpha_1, \alpha_2 \in a_{q|M}$. As $X_0 \in A_q$ and $\delta \leq \beta$, by **) for q we conclude from fact 2.13 2) that

$$f^{\alpha_2}_{q|M}(\beta) \geq f^{\alpha_1}_{q|M}(\beta) + ordtp(X_0 \cap [\alpha_1, \alpha_2)) > f^{\alpha_1}_{q|M}(\beta)$$

as required in ***).

For iii) we claim that $Z = X_0$ witnesses iii) for q. The subclauses 1)-5) are clear. To see that 6) holds, take $\alpha \in a_q - a_{q|M} = a_q - M$ such that $\alpha < sup(M)$. Then there is $j \in [l]$ such that $\alpha = \xi_j$. $M \cap \eta_j$ is bounded in η_j by ξ_j and so there is no cofinal in η_j countable sequence included in M, hence η_j has uncountable cofinality in M. Also $\nu_j \in M$ and $\nu_j < \eta_j$. So take e.g. $\zeta_j = \nu_j + \alpha'$ where α' is any countable ordinal in M which is bigger than all β_0-numbers (the existence of such a number follows from the elementarity of M).

So now, using the elementarity of M, find s such that $s \in M$ and $\phi(s)$ holds in M. Now our aim is to prove that s and q are compatible in P. So we need to define $r \leq s, q$. We put $a_r = a_s \cup a_q, b_r = b_q \cup b_s$, $A_r = A_s \cup A_q$, $f_r^\alpha(\beta) = f_s^\alpha(\beta)$ for $\alpha \in a_s$, $\beta \in b_s$, $f_r^\alpha(\beta) = f_q^\alpha(\beta)$ for $\alpha \in a_q$, $\beta \in b_q$ (these agree on the common part $q|M$) and we need to define $f_r^\alpha(\beta)$ for $\langle\alpha,\beta\rangle \in (a_s - a_q) \times (b_q - b_s) \cup (a_q - a_s) \times (b_s - b_q)$ which is the only freedom we have in defining the extension r.

For $\alpha \in a_s - a_q$, $\beta \in b_q - b_s$ we define $f_r^\alpha(\beta)$ separately for each $\beta \in b_q - b_s$ and by induction. Suppose that we already defined it for $(a_s - a_q) \cap \alpha$ for $\alpha \in a_s - a_q$, then we put

$$D1) \qquad f_r^\alpha(\beta) = \max\{f_r^{\alpha'}(\beta) + d_{A_r,\beta}(\alpha',\alpha) : d_{A_r,\beta}(\alpha',\alpha) \neq 0,\ \alpha' \in a_r \cap \alpha\}$$

Note that by definition 4.1, we have always $\alpha' = 0 \in a_r$ and also $0, \alpha \in X_0$ and so $d_{A_r,\beta}(\alpha',\alpha) > 0$ for some $\alpha' \in a_r \cap \alpha$.

For $\alpha \in a_q - a_s$, $\beta \in b_s - b_q$ we define $f_r^\alpha(\beta)$ separately for each $\beta \in b_s - b_q$ and by induction. Suppose that we already defined it for $(a_q - a_s) \cap \alpha$ for $\alpha \in a_q - a_s$, then we put $f_r^\alpha(\beta)$ to be the maximum of the following two numbers:

$$D2) \qquad \max\{f_r^{\alpha'}(\beta) + d_{A_r,\beta}(\alpha',\alpha) : d_{A_r,\beta}(\alpha',\alpha) \neq 0,\ \alpha' \in a_r \cap \alpha\}$$

$$D3) \qquad \max\{f_r^{\alpha'}(\beta) + 1 : \alpha' \in a_q \cap \alpha\}$$

Now we need to prove that r is a condition of P which extends both s and q. The only nontrivial clauses of the definition 4.1. to be checked are *), **) and ***). In the sequel we will refer to these clauses as simply *), **) and **). Before moving to following parts of the proof of statement 4.2, we will prove some lemmas.

LEMMA 4.5. *Suppose* $\alpha_1 < \alpha_2 < \omega_2$; $\beta < \omega_1$; $\alpha_1, \alpha_2, \beta \in M$. *Then* α_1, α_2 *are* β*-connected in* A_s *if and only if* α_1, α_2 *are* β*-connected in* A_r.

PROOF: Let $W_1, ..., W_k$, $\gamma_1, ..., \gamma_k$ be as in definition 2.9. witnessing the fact that α_1, α_2 are β-connected in A_r. Since $\alpha_2 \in M$ and $\beta \in M \cap \omega_2 \in \mathcal{F}$, by the main lemma and the fact that s satisfies clause 4.3. h) iii) 1)-2) of ϕ, we conclude that all γ_i's are in M and there are $V_i \in A_s$ such that $V_i \cap \gamma_i = W_i \cap \gamma_i$ and $\gamma_i \in V_i$ and $rank(V_j) = rank(W_j)$. It follows that α_1, α_2 are β-connected in A_s. The opposite implication is clear as $A_s \subseteq A_r$.

LEMMA 4.6. *Suppose* $\alpha_1 < \alpha_2 < \omega_2; \beta \in b_s \cap b_q$, $\alpha_2 \in (\omega_2 - M) \cup Z$. *Then* α_1, α_2 *are* β*-connected in* A_q *if and only if* α_1, α_2 *are* β*-connected in* A_r.

PROOF: Let $W_1, ..., W_k, \gamma_1, ..., \gamma_k$ be as in definition 2.9. witnessing the fact that α_1, α_2 are β-connected in A_r. If there is no i such that $\gamma_i \in M$, then all W_i's are not in M and so are in A_q which proves that α_1, α_2 are β-connected in A_q. If $\alpha_2 \in Z$, then an argument similar to the one from the proof of 4.5. works. Otherwise take maximal i such that $\gamma_i \in M$ and assume $\alpha_2 \notin Z$. By the maximality and the fact that $\alpha_2 \notin M$, $W_j \in A_q - A_s$ for $j \geq i$. So, we conlude that $\gamma_i \in Z$. By the main lemma and the fact that $rank(W_i) \leq \beta \in b_q \cap b_s < \beta_0 < rank(Z)$ (see 4.3. d) we conclude that all γ_j's for $j \leq i$ are in Z. By clause 4.3. h) iii), 1)-2) this means that there are $V_i \in A_q$ such that $V_j \cap \gamma_j = W_j \cap \gamma_j$ and $\gamma_j \in V_j$ and $rank(V_j) = rank(W_j)$. This implies that α_1 and α_2 are β-connected in A_q.
Again as $A_q \subseteq A_r$, the opposite implication is clear.

LEMMA 4.7. *Suppose* $\alpha_1 < \alpha_2 < \omega_2$; $\delta \leq \beta < \omega_1$. *Then* α_1, α_2 *are* β*-connected in* A_q *if and only if* α_1, α_2 *are* β*-connected in* A_r.

PROOF: Let $W_1, ..., W_k, \gamma_1, ..., \gamma_k$ be as in definition 2.9. witnessing the fact that α_1, α_2 are β-connected in A_r. As $s \in M$, we have $W_i \subseteq X_0 = M \cap \omega_2 \in A_q$ for $W_i \in A_s$. Thus by replacing W_i's from $A_r - A_q = A_s - A_q$ by X_0 we obtain that α_1, α_2 are β-connected in A_q.
Again as $A_q \subseteq A_r$, the opposite implication is clear.

LEMMA 4.8. *Suppose* $\alpha < \omega_2; \beta < \omega_1; \alpha, \beta \in M$. *Then*

$$(A_r)_\beta(\alpha) = (A_s)_\beta(\alpha)$$

PROOF: (see definition 2.9) If $X \in A_r - A_s$ and $rank(X) < \delta$, then $rank(X) < \beta_0$. If $\alpha \in X$, then by 4.3. h) iii) 1)-2), there is $Y \in A_s$ such that $Y \cap M = X \cap M$ and so $Y \cap \alpha = X \cap \alpha$ by the main lemma. This proves that $(A_r)_\beta(\alpha) \subseteq (A_s)_\beta(\alpha)$. The other inclusion is clear.

LEMMA 4.9. *Suppose* $\alpha < \omega_2; \beta \in b_q \cap b_s$ *and* $\alpha \in (\omega_2 - M) \cup Z$. *Then*

$$(A_r)_\beta(\alpha) = (A_q)_\beta(\alpha)$$

PROOF: Note that $\beta < \beta_0$ (by 4.3. d) and by 4.3. h) iii) 1)-2) , for each $X \in A_s$ of rank less or equal than β such that $\alpha \in X$ (in this case $\alpha \in Z$, as $A_s \in M$) there is $Y \in A_q$ of the same rank such that $\alpha \in Y$, and hence $(A_r)_\beta(\alpha) \subseteq (A_s)_\beta(\alpha)$. The other inclusion is clear.

LEMMA 4.10. *Suppose* $\alpha < \omega_2; \delta \leq \beta < \omega_1$. *Then*

$$(A_r)_\beta(\alpha) = (A_q)_\beta(\alpha)$$

PROOF: Note that any $X \in A_r - A_q$ is in A_s and so is included in $X_0 \in A_q$ and so $(A_r)_\beta(\alpha) \subseteq (A_q)_\beta(\alpha)$. The other inclusion is clear.

LEMMA 4.11. *Suppose* $\alpha_1 < \alpha_2 < \omega_2; \beta < \omega_1; \alpha_1, \alpha_2, \beta \in M$. *Then*

$$d_{A_r,\beta}(\alpha_1, \alpha_2) = d_{A_s,\beta}(\alpha_1, \alpha_2).$$

PROOF: By induction on α_2. Suppose the equality is true for pairs of ordinals from ω_2 with the bigger ordinal less than α_2. Note that by lemma 4.8. and the assumptions we have that $(A_r)_\beta(\alpha_2) \subseteq M$, so by lemmas 4.8. and 4.5. the set

$$\{\gamma : \gamma \in (A_r)_\beta(\alpha_2) \ \& \ \alpha_1, \gamma \text{ are } \beta\text{-connected in } A_r\}$$

is equal to the set

$$\{\gamma : \gamma \in (A_s)_\beta(\alpha_2) \ \& \ \alpha_1, \gamma \text{ are } \beta\text{-connected in } A_s\}.$$

Also for γ's from the above set by the inductive assumption and lemma 4.8. we have

$$d_{A_r,\beta}(\alpha_1, \gamma) + ordtp((A_r)_\beta(\alpha_2) - \gamma) = d_{A_s,\beta}(\alpha_1, \gamma) + ordtp((A_s)_\beta(\alpha_2) - \gamma)$$

Now using definition 2.9. we conclude the statement of the lemma.

LEMMA 4.12. *Suppose* $\alpha_1 < \alpha_2 < \omega_2; \beta \in b_q \cap b_s;\ \alpha_2 \in (\omega_2 - M) \cup Z$. *Then*

$$d_{A_r,\beta}(\alpha_1, \alpha_2) = d_{A_q,\beta}(\alpha_1, \alpha_2).$$

PROOF: By induction on α_2. Suppose the lemma is true for pairs of ordinals from ω_2 with the bigger ordinal less than α_2. Note that $(A_r)_\beta(\alpha_2) \subseteq (\omega_2 - M) \cup Z$, so by lemmas 4.9. and 4.6. the set

$$\{\gamma : \gamma \in (A_r)_\beta(\alpha_2) \ \& \ \alpha_1, \gamma \text{ are } \beta\text{-connected in } A_r\}$$

is equal to the set

$$\{\gamma : \gamma \in (A_q)_\beta(\alpha_2) \ \& \ \alpha_1, \gamma \text{ are } \beta\text{-connected in } A_q\}$$

also for γ's from the above set by the inductive assumption and lemma 4.9. we have

$$d_{A_r,\beta}(\alpha_1, \gamma) + ordtp((A_r)_\beta(\alpha_2) - \gamma) = d_{A_q,\beta}(\alpha_1, \gamma) + ordtp((A_q)_\beta(\alpha_2) - \gamma)$$

Now using definition 2.9. we conclude the statement of the lemma.

LEMMA 4.13. *Suppose* $\alpha_1 < \alpha_2 < \omega_2;\ \delta \leq \beta < \omega_1$. *Then*

$$d_{A_r,\beta}(\alpha_1, \alpha_2) = d_{A_q,\beta}(\alpha_1, \alpha_2).$$

PROOF: By induction on α_2. Suppose the equality is true for pairs of ordinals from ω_2 with the bigger ordinal less than α_2. Note that by lemmas 4.10. and 4.7. the set

$$\{\gamma : \gamma \in (A_r)_\beta(\alpha_2) \ \& \ \alpha_1, \gamma \text{ are } \beta\text{-connected in } A_r\}$$

is equal to the set

$$\{\gamma : \gamma \in (A_q)_\beta(\alpha_2) \ \& \ \alpha_1, \gamma \text{ are } \beta\text{-connected in } A_s\}$$

also for γ's from the above set by the inductive assumption and lemma 4.10. we have

$$d_{A_r,\beta}(\alpha_1, \gamma) + ordtp((A_r)_\beta(\alpha_2) - \gamma) = d_{A_q,\beta}(\alpha_1, \gamma) + ordtp((A_q)_\beta(\alpha_2) - \gamma)$$

Now using definition 2.9. we conclude the statement of the lemma.

LEMMA 4.14. *Suppose* $\alpha_1 < \alpha_2 < \omega_2$. *If* $\beta < \delta$ *and* $\alpha_1 \in a_q - a_s$ *and* $\alpha_2 \in a_s$, *then* α_1, α_2 *are not* β*-connected in* A_r.

PROOF: Suppose the opposite. Let $W_1, ..., W_k$, $\gamma_1, ..., \gamma_k$ be as in definition 2.9. witnessing the fact that α_1, α_2 are β-connected in A_r. It follows from the main lemma that $\alpha_1 \in (\mathcal{F})_\beta(\alpha_2) \subseteq M$. But this contradicts the fact that $\alpha_1 \notin M$.

LEMMA 4.15. *Suppose* $\alpha_1 < \alpha_2 < \omega_2$. *If* $\beta \in b_s \cap b_q$ *and* $\alpha_1 \in a_s - a_q$ *and* $\alpha_2 \in a_q - a_s$, *then* α_1, α_2 *are not* β*-connected in* A_r.

PROOF: Suppose the opposite. Let $W_1, ..., W_k$, $\gamma_0, ..., \gamma_k$ be as in definition 2.9. witnessing the fact that α_1, α_2 being β-connected in A_r. Note that if $\gamma_i \notin M$, then $X_i \notin M$ and so $X_i \in A_q - A_s$ and consequently $rank(X_i) < \beta_0$.
Take maximal i such that $\gamma_i \in M$. As $\alpha_1 = \gamma_1 \in M$ and $\alpha_2 \notin M$, we have that such an i exists and $X_i \notin M$ as X_i contains the next element in the sequence $\gamma_1, ..., \gamma_k, \alpha_2$ after γ_i. Now this implies that $\gamma_i \in Z$ which in turn implies by the main lemma that all γ_j for $j < i$ are in Z as well. In particular $\alpha_1 \in Z$ which contradicts the choice of α_1 and 4.3. h) iii) 4).

LEMMA 4.16. *Suppose that* $\alpha_1 < \alpha_2 < \omega_2$, $\alpha_1, \alpha_2 \notin M$ *and* $\beta < \delta$. *Then* $d_{A_r,\beta}(\alpha_1, \alpha_2) = d_{A_r,\beta_0}(\alpha_1, \alpha_2)$.

PROOF: First note as in the proof of lemma 4.14 that if $\alpha_1 < \gamma$ and α_1, γ are β-connected then $\gamma \notin M$. Secondly note that this implies that if $W_1, ..., W_k$, $\gamma_0, ..., \gamma_k$ are as in definition 2.9. witnessing the fact that α_1, γ are β-connected in A_r, then all γ_is are not in M and all X_is are not in M and consequently $rank(X_i) \leq \beta_0$ for all $i \leq k$ by 4.3. a), b) d). In other words α_1, γ are β-connected in A_r if and only if α_1, γ are β_0-connected in A_r, now the proof is by induction on $\alpha_2 \notin M$. Using the definition 2.11 we have

$$d_{A_r,\beta}(\alpha_1, \alpha_2) = \sup\{d_{A_r,\beta}(\alpha_1, \gamma) + ordtp((A_r)_\beta(\alpha_2) - \gamma) :$$

$$\alpha_1 \leq \gamma < \alpha_2; \gamma \in (A_r)_\beta(\alpha_2); \alpha_1, \gamma \text{ are } \beta\text{-connected in } A_r\}$$

Using the inductive assumption and the fact that $(A_r)_\beta(\alpha_2) = (A_r)_{\beta_0}(\alpha_2)$ (this follows again from 4.3. a) b) d)) this number is clearly equal to

$$\sup\{d_{A_r,\beta_0}(\alpha_1, \gamma) + ordtp((A_r)_{\beta_0}(\alpha_2) - \gamma) :$$

$$\alpha_1 \leq \gamma < \alpha_2; \gamma \in (A_r)_\beta(\alpha_2); \alpha_1, \gamma \text{ are } \beta_0\text{-connected in } A_r\}$$

which is $d_{A_r,\beta_0}(\alpha_1,\alpha_2)$.

Now we are ready to start proving the main part of 4.2. So let's start with *). It follows from the definition of $f_\alpha^r(\beta)$'s (see D1), D2), D3)) and the fact that $d_{A_r,\beta}(\alpha',\alpha)$ is a β-number (see fact 2.15) and finite sums of β-numbers are β-numbers (see remark 2.16).

So now let us check **). For this we will need to consider many cases.

Case 1. $\beta \in b_q \cap b_s$.

Case 1.1. $\beta \in b_q \cap b_s$ and $\alpha_1,\alpha_2 \in a_s$ or $\alpha_1,\alpha_2 \in a_q$.
This follows from the fact that s and q are conditions of P and from lemmas 4.11, and 4.12.

Case 1.2. $\beta \in b_q \cap b_s$ and $\alpha_1 \in a_s - a_q$, $\alpha_2 \in a_q - a_s$ or $\alpha_1 \in a_q - a_s$, $\alpha_2 \in a_s - a_q$.
This follows from lemmas 4.14 and 4.15.

Case 2. $\beta \in b_s - b_q$.

Case 2.1. $\beta \in b_s - b_q$, $\alpha_1 < \alpha_2$; $\alpha_1,\alpha_2 \in a_s$.
This follows from lemma 4.11 and the fact that s is a condition of P.

Case 2.2. $\beta \in b_s - b_q$, $\alpha_1 < \alpha_2$; $\alpha_2 \in a_q - a_s$
It follows from the construction (see D2)) of $f_r^{\alpha_2}(\beta)$.

Case 2.3. $\beta \in b_s - b_q$, $\alpha_1 < \alpha_2$; $\alpha_1 \in a_q - a_s, \alpha_2 \in a_s$.
This follows from lemma 4.14.

Case 3. $\beta \in b_q - b_s$.

Case 3.1. $\beta \in b_q - b_s$, $\alpha_1 < \alpha_2$; $\alpha_1,\alpha_2 \in a_q$.
This follows from lemma 4.13. and the fact that q is a condition of P.

Case 3.2. $\beta \in b_q - b_s$, $\alpha_1 < \alpha_2$; $\alpha_2 \in a_s - a_q$.
This case follows from the construction of $f_r^{\alpha_2}(\beta)$.

Case 3.3. $\beta \in b_q - b_s$, $\alpha_1 < \alpha_2$; $\alpha_1 \in a_s - a_q$; $\alpha_2 \in a_q$.
Suppose that we have proved **) for all $\alpha' \in (a_q \cup a_s) \cap \alpha_1$ and α_2 and β.
We may w.l.o.g. assume that α_1 and α_2 are β-connected in A_r.
Note that $f_r^{\alpha_1}(\beta)$ is defined according to the construction (see D1)) i.e.,

$$f_r^{\alpha_1}(\beta) = \max\{f_r^{\alpha'}(\beta) + d_{A_r,\beta}(\alpha',\alpha_1) : d_{A_r,\beta}(\alpha',\alpha_1) \neq 0 : \alpha' \in a_r \cap \alpha_1\}$$

Note that the above set of α's is nonempty as 0 is its member since $0, \alpha_1 \in X_0$. So let α' be the ordinal giving the maximal value which gave $f_r^{\alpha_1}(\beta)$. By the inductive assumption we have

$$f_r^{\alpha_2}(\beta) \geq f_r^{\alpha'}(\beta) + d_{A_r,\beta}(\alpha',\alpha_2)$$

As α' and α_1 as well as α_1 and α_2 are β-connected in A_r it follows from lemma 2.13 (3) that

$$d_{A_r,\beta}(\alpha',\alpha_1)+d_{A_r,\beta}(\alpha_1,\alpha_2)=d_{A_r,\beta}(\alpha',\alpha_2)$$

and so

$$f_r^{\alpha_2}(\beta)\geq f_r^{\alpha'}(\beta)+d_{A_r,\beta}(\alpha',\alpha_2)=$$
$$=f_r^{\alpha'}(\beta)+d_{A_r,\beta}(\alpha',\alpha_1)+d_{A_r,\beta}(\alpha_1,\alpha_2)=$$
$$=f_r^{\alpha_1}(\beta)+d_{A_r,\beta}(\alpha_1,\alpha_2)$$

as required in **).

So now we need to check ***).

Case 1. $\beta\in b_q-b_s$, $\alpha_1<\alpha_2$; $\alpha_1,\alpha_2\in a_s$.
Note that in this case we have $\alpha_1,\alpha_2\in X_0$ and so ***) follows from **).

Case 2. $\beta\in b_s-b_q$, $\alpha_1<\alpha_2$; $\alpha_1,\alpha_2\in a_q$.

Case 2.1. $\beta\in b_s-b_q$, $\alpha_2\in a_q-a_s$, $\alpha_1<\alpha_2$.
This case follows directly from the construction of $f_r^{\alpha_2}(\beta)$ (see D3)).

Case 2.2. $\beta\in b_s-b_q$, $\alpha_2\in a_q\cap a_s$ and $\alpha_1\in a_s\cap a_q$ and $\alpha_1<\alpha_2$.
Note that then $\alpha_1,\alpha_2\in Z$ and $Z\in A_s$ and $rank(Z)<\beta$ (by 4.3. 5)) as well as $\alpha_1,\alpha_2\in a_s$, and so as in case 1, we conclude ***) from **) for s.

Case 2.3. $\beta\in b_s-b_q$, $\alpha_1<\alpha_2$, $\alpha_2\in a_q\cap a_s$ and $\alpha_1\in a_q-a_s$.
For this case, we will prove by induction on $\alpha_0\in a_q$, $\alpha_0\leq\alpha_1$ that

$$f_r^{\alpha_0}(\beta)<^{\beta_0}f_r^{\alpha_2}(\beta)$$

(see definition 2.17). Certainly for $\alpha_0=\alpha_1$ this will be sufficient for proving the case.
Let $j\in[l]$ (see definitions 4.3. e)) be such that $\alpha_1=\xi_j$. It exists because $\alpha_1<\alpha_2\in M$.
Then clearly $\nu_j<\zeta_j<\xi_j<\eta_j<\alpha_2$ (see 4.3. f) g) h) iii) 6)).
Suppose we are done below α_0.
If $\alpha_0\in a_q\cap a_s\subseteq Z$, then note that by the definition of ν_j we have $\alpha_0<\nu_j$, as $\alpha_0<\alpha_1$ and so by fact 2.13, 2) and definition 4.3. h) iii) 6) we have

$$d_{A_s,\beta}(\alpha_0,\alpha_2)\geq ordtp(Z\cap[\alpha_0,\alpha_2))\geq ordtp(Z\cap[\nu_j,\zeta_j))>^{\beta_0}0$$

Now use **) for r and lemma 4.11, to conclude that $f_r^{\alpha_0}(\beta)<^{\beta_0}f_r^{\alpha_2}(\beta)$.
Now assume that $\alpha_0\in a_q-a_s$. In this case $f_r^{\alpha_0}(\beta)$ has been defined according to the construction i.e., D2), D3), $f_r^{\alpha_0}(\beta)$ is the maximum of the following two numbers:

$$\max\{f_r^{\alpha'}(\beta)+d_{A_r,\beta}(\alpha',\alpha_0):d_{A_r,\beta}(\alpha',\alpha_0)\neq 0:\alpha'\in a_r\cap\alpha_0\}$$

$$\max\{f_r^{\alpha'}(\beta)+1:\alpha'\in a_q\cap\alpha\}$$

If this is the second number, the inductive assumption and remark 2.15 imply that $f_r^{\alpha_0}(\beta) <^{\beta_0} f_r^{\alpha_2}$.
So we will assume that $f_r^{\alpha_0}(\beta)$ is the first and not the second of the two above numbers. Note that as $0 \in a_q \cap \alpha_0 \subseteq X_0$ the fact that the first number is bigger than the second means that the first one is a nonzero number and in particular, by fact 2.12, that α' and α_0 are β-connected in A_r.
First assume that $\alpha' \notin M$ i.e, $\alpha' \notin a_s$, then by lemma 4.16. we conclude that $d_{A_r,\beta}(\alpha',\alpha_0)$ equals to $d_{A_r,\beta_0}(\alpha',\alpha_0)$ and by fact 2.16 this number is a β_0-number, so the inductive assumption implies $f_r^{\alpha_0}(\beta) <^{\beta_0} f_r^{\alpha_2}(\beta)$.
Now assume that $\alpha' \in a_s$. Let γ_0 be the minimal ordinal such that $\alpha' < \gamma_0 \leq \alpha_0$, $\gamma_0 \notin M$ and both α', γ_0 and γ_0, α_0 are β-connected in A_r. The existence of such a γ_0 follows from the fact that α_0 has the above property.
By fact 2.13. 3) and lemma 4.16 we have that

$$+)\qquad d_{A_r,\beta}(\alpha',\alpha_0) = d_{A_r,\beta}(\alpha',\gamma_0) + d_{A_r,\beta}(\gamma_0,\alpha_0) = d_{A_r,\beta}(\alpha',\gamma_0) + d_{A_r,\beta_0}(\gamma_0,\alpha_0)$$

Now lemma 2.16 implies that the ordinal $d_{A_r,\beta_0}(\gamma_0,\alpha_0)$ is a β_0-number. So we will focus on estimating $d_{A_r,\beta}(\alpha',\gamma_0)$.
By definitions 2.9, 2.11, and the definition of β_0 (see 4.3. a), b) d)) we have

$$d_{A_r,\beta}(\alpha',\gamma_0) \leq \sup\{d_{A_r,\beta}(\alpha',\gamma) + \theta_{\beta_0} :$$

$$\alpha' \leq \gamma < \gamma_0; \gamma \in (A_r)_\beta(\gamma_0); \alpha', \gamma \text{ are } \beta\text{-connected in } A_r\}$$

Where θ_{β_0} is the order type of elements of $\mathcal{F}$ of rank β_0.
Now note that by the minimality of γ_0 we have that whenever γ is such that $\alpha' < \gamma < \gamma_0; \gamma \in (A_r)_\beta(\gamma_0); \alpha', \gamma$ are β-connected in A_r, we have that $\gamma \in M$ (remember that by now we are assuming that $\alpha' \in M$ so case $\alpha' = \gamma$ follows) and actually $\gamma \in Y$ for this $Y \in A_q - A_s$ that $(A_r)_\beta(\gamma_0) = (A_r)_{\beta_0}(\gamma_0) = Y \cap \gamma_0$. It follows that for such a γ we have $\gamma < \nu_j < \zeta_j$ and $\gamma \in Z$ and in particular $\gamma \in A_\beta(\nu_j)$. It follows that α', ν_j are β-connected in A_r (by 4.3.h) iii) 5)) and that

$$++)\qquad d_{A_r,\beta}(\alpha',\nu_j) + \theta_{\beta_0} = \sup\{d_{A_r,\beta}(\alpha',\gamma) + ordtp((A_r)_\beta(\nu_j) - \gamma) :$$

$$\alpha' \leq \gamma < \nu_j; \gamma \in (A_r)_\beta(\nu_j); \alpha', \gamma \text{ are } \beta\text{-connected in } A_r\} + \theta_{\beta_0} \geq d_{A_r,\beta}(\alpha',\gamma_0)$$

Now, as $\nu_j < \zeta_j < \alpha_2$ are all in Z, by fact 2.13 and lemma 4.3. h) iii) 6), we have

$$+++)\qquad d_{A_r,\beta}(\nu_j,\alpha_2) \geq d_{A_r,\beta}(\nu_j,\zeta_j) \geq ordtp(Z \cap [\nu_j,\zeta_j)) >^{\beta_0} 0$$

Since α', ν_j are β-connected as noted above and +), ++) and +++) hold we obtain that

$$d_{A_r,\beta}(\alpha',\alpha_2) = d_{A_r,\beta}(\alpha',\nu_j) + d_{A_r,\beta}(\nu_j,\alpha_2) >^{\beta_0} d_{A_r,\beta}(\alpha',\nu_j)$$

$$d_{A_r,\beta}(\alpha',\alpha_0) \leq d_{A_r,\beta}(\alpha',\nu_j) + \theta_{\beta_0} + d_{A_r,\beta_0}(\gamma_0,\alpha_0)$$

Which completes the proof of the case and the statement 4.2.

FACT 4.17 P *preserves* ω_1.

PROOF: It follows from statement 4.2. as $\mathcal{F}$-proper forcings preserve ω_1, the proof of this fact is the same as for proper forcings see e.g., [B2].

FACT 4.18 P *satisfies the* ω_2*-c.c.*

PROOF: Let $(p_\xi : \xi < \omega_2)$ be a sequence of conditions of P. We will show that there are two compatible conditions among the elements of this sequence.
We may w.l.o.g. assume that for every $\xi < \omega_2$ the set A_{p_ξ} has a maximal element denoted by X_ξ. This follows from the directedness of $\mathcal{F}$. We may also w.l.o.g. assume that all X_ξ's have the same rank $\alpha < \omega_1$ and that $f_{X_{\xi_1}, X_{\xi_2}}$ lifts up to an isomorphism between p_{ξ_1} and p_{ξ_2} for every $\xi_1, \xi_2 \in \omega_2$. This follows from the homogeneity and local smallness of $\mathcal{F}$ as well as from the fact that a_{p_ξ}'s, b_{p_ξ}'s, F_{p_ξ}'s and A_{p_ξ}'s are finite. We can also assume that all b_{p_ξ}s are the same and are equal to b.
Now we claim that actually any two conditions from the sequence are compatible. Consider two of them p, q with maximal elements X_p and X_q of A_p and A_q respectivelly. Put $a_r = a_p \cup a_q$, $b_r = b$, $F_r = F_q \cup F_p$ and finally put $A_r = A_p \cup A_q$.
Now it is enough to check that r is a condition which follows from the isomorphism of p and q. For this note that unless $\langle \alpha_1, \alpha_2 \rangle \in [a_r]^2$ is already in one of the sets $[a_p]^2$ or $[a_q]^2$, it is not β-connected in A_r for any $\beta \in b$. It follows that **) is trivially satisfied. Also ***) holds trivially as $b_q = b_p = b_r = b$.

FACT 4. 19 P *preserves cardinals.*

FACT 4.20 *For all* $\alpha \in \omega_2$ *and all* $\beta <\in \omega_1$ *the following sets are dense in* P

$$D_{\alpha,\beta} = \{p \in P : \max(a_p) > \alpha,\ \max(b_p) > \beta\}$$

PROOF: To make the checking that the extension is in P a triviality, given $q \in P$ and $\alpha \in \omega_2$ and $\beta \in \omega_1$ define p as follows $a_p = a_q \cup \{\alpha_0\}$, $b_p = b_q \cup \{\beta_0\}$, $A_p = A_q$ where $\alpha_0 > \alpha$ is such that $\alpha_0 > \sup(\bigcup A_p)$ and $\beta_0 > \beta$ is such that $\beta > \max(\{rank(X) : X \in A_p\}, \max(b_q))$. Now define $f_p^{\alpha'}(\beta)$ for $\alpha' \in a_q$ so that **) and ***) are satisfied which can be easily done by induction and making sure that the appropriate values are β_0-numbers; $f_p^{\alpha_0}(\beta)$ can be arbitrary as long as *) is satisfied.

This completes the proof of theorem 1.1.

REMARK 4.21. Note that it was in the proof of Case 2.3. of ***) when we essentially needed the notion $d_\beta(\alpha_1, \alpha_2)$ of β-distance between $\alpha_1, \alpha_2 \in \omega_2$ different than *ordtp*$(X \cap (\alpha_1, \alpha_2])$ for $X \in \mathcal{F}$ of rank β such that $\alpha_1, \alpha_2 \in X$. With the β-distance as above we could have $d_\beta(0, \alpha_1) = d_\beta(0, \alpha_2)$ and $f_{\alpha_2}(\beta) = f_s^0(\beta) + d_\beta(0, \alpha_2)$, which by D2) would imply $f_r^{\alpha_1}(\beta) > f_r^{\alpha_2}(\beta)$ which would give that r is not an extension of q. Note that the distance as above may exist under CH (as morasses which are stationary codings exist in L) so theorem 1.2. explains why the situation as above must occur. Actually the proof of 1.2. is derived from the above counterexample to 2.3.

It would be interesting to axiomatize the other distance that we use as well as associated colorings of pairs from ω_2 into ω_1 colors. This could lead to obtaining stronger properties that these usually listed for ρ (see [T2]).

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