Horn Semantics and Craig Interpolation

PETER ARNDT[*]
University of Düsseldorf, Düsseldorf, Germany
HUGO LUIZ MARIANO[†]
University of São Paulo, São Paulo, Brazil
DARLLAN CONCEIÇÃO PINTO[‡]
Federal University of Bahia, Salvador, Brazil

Keywords: Horn Semantics, Craig Interpolation, Filter Pairs

In this work we consider semantics of a logic in a class of first order structures axiomatized by universal Horn sentences, a *Horn class*. We give conditions on such a semantics which ensure that an amalgamation property for the Horn class implies Craig interpolation for the logic. This generalizes the well-known result that for an algebraizable logic the amalgamation property for the associated class of algebras implies Craig interpolation.

Let κ be a regular cardinal. Let Σ be a signature consisting of function symbols and Σ^+ an expansion of Σ by relation symbols. A κ-Horn theory is a theory axiomatized by universal strict basic κ-Horn sentences (sentences of the form $\forall \vec{x}\colon \bigwedge_{i\in I} P_i(\vec{x}) \to P(\vec{x})$, with P_i, P atomic formulas over Σ^+ not equivalent to $\perp$ and $|I| < \kappa$). A κ-Horn class is a class of Σ^+-structures axiomatized by a κ-Horn theory.

Let L be a logic over a signature Σ. Recall that an algebraic semantics for a logic L is a translation from formulas of L to sets of equations over the signature of L (i.e. atomic formulas of the first order language associated to Σ), commuting with substitution, and such that inference in the logic under this translation corresponds exactly to inference in the equational logic in a quasivariety $\mathbf{K}$.

This situation has been abstracted into the notion of *filter pair* in [1], [2], [3]: A filter pair is a functor $G\colon \Sigma\text{-Str} \to \kappa\text{-AlgLat}$ together with a natural transformation to the power set functor $i\colon G \to \wp$, which objectwise preserves infima and κ-directed suprema. In the case of algebraic semantics for the functor one takes $G := Co_{\mathbf{K}}(\mathrm{Fm}) := \{\theta \mid \mathrm{Fm}/\theta \in \mathbf{K}\}$.

For a κ-Horn theory $\mathbb{T}$, we define a lattice of atomic Horn formulas that will replace the congruence lattice from algebraic semantics: For a Σ-structure A let

- $G^{\mathbb{T}}(A) := \{(\theta, S) \mid \theta$ is a Σ-congruence on A and S an interpretation of $\mathfrak{R}$ on A/θ s.t. the resulting Σ^+-structure on A/θ is a $\mathbb{T}$-model$\}$
- We define an order on $G^{\mathbb{T}}(A)$ by declaring $(\theta, S) \leq (\theta', S')$ iff $\theta \subseteq \theta'$ and the induced quotient map $q_{\theta\theta'} : A/\theta \twoheadrightarrow A/\theta'$ is a homomorphism of Σ^+-structures for the interpretations S, S'.

Horn Semantics arises by replacing the congruence lattice with the above lattice of atomic formulas of an expansion Σ^+ of Σ.

[*]peter.arndt@hhu.de
[†]hugomar@ime.usp.br
[‡]darllan@ufba.br

Theorem Let τ be a set of *atomic* Σ^+-formulas with at most one free variable, such that $|\tau| < \kappa$. The collection of maps $i^\tau = (i^\tau_A)_{A \in \Sigma - Str}$, defined by

$$\begin{array}{rcl} i^\tau_A : G^{\mathbb{T}}(A) & \to & (\mathcal{P}(A), \subseteq) \\ (\theta, S) & \mapsto & \{a \in A \mid \forall \varphi(x) \in \tau :\ A/(\theta, S) \vDash \varphi(a)\} \end{array}$$

is a natural transformation and for any $A \in \Sigma - Str$, i^τ_A preserves arbitrary infima and κ-directed suprema. In other words, $(G^{\mathbb{T}}, i^\tau)$ is a κ-filter pair.

Such a filter pair is called *Horn filter pair* and a *κ-Horn Semantics* for a logic L is a Horn filter pair whose image over the formula algebra is the lattice of theories of L. It is an *equivalent Horn Semantics* if the natural transformations are injective.

Some examples that demonstrate the potential of the approach are:

- For $\Sigma^+ = \Sigma$ a Horn semantics is precisely an algebraic semantics, and an equivalent Horn semantics corresponds precisely to an algebraizable logic.
- For $\Sigma^+ = \Sigma \cup \{F\}$ an expansion of the signature with a unary relation symbol one can define an equivalent Horn semantics corresponding to matrix semantics.
- For $\Sigma^+ = \Sigma \cup \{\leq\}$ an expansion of the signature with an inequality symbol, and a Horn theory demanding that this be an order relation, a Horn semantics is precisely an order algebraic semantics, and an equivalent Horn semantics corresponds precisely to an order algebraizable logic in the sense of [4]

We shall say that a class $\mathbf{K}$ of Σ^+-structures has the *atomic amalgamation property* if given $A, B, C \in \mathbf{K}$ and maps $i_B \colon A \to B$, $i_C \colon A \to C$ that preserve and reflect the validity of atomic formulas (atomic embeddings), there exist a Σ^+-structure $D \in \mathbf{K}$ and atomic embeddings $e_B : B \to D$, $e_C : C \to D$ such that $e_B \circ i_B = e_C \circ i_C$.

Using the formalism of filter pairs, we can prove a general Craig Interpolation result:

Theorem Let $(G^{\mathbb{T}}, i^\tau)$ be a Horn semantics for a logic L. Suppose that the filter pair $(G^{\mathbb{T}}, i^\tau)$ has the "theory lifting property". If $\mathbf{K} := \mathrm{Mod}(\mathbb{T})$ has the atomic amalgamation property, then the logic L associated to $(G^{\mathbb{T}}, i^\tau)$ has the Craig entailment property.

The "theory lifting property" is a technical condition, satisfied by every filter pair presenting an equivalent Horn Semantics, but also in other cases.

In the talk we will review the notion of filter pair and explain the above results with examples.

References

[1] Arndt, P.; Mariano H.L., Pinto D.C., Finitary Filter Pair and propositional logics . *South American Journal of Logics*, 4(2):257–280, 2018.

[2] Arndt, P.; Mariano H.L., Pinto D.C., Filter Pairs and natural extension of logics. *Archive for Mathematical Logic*, 62:113–145, 2023.

[3] Arndt, P.; Mariano H.L., Pinto D.C., Congruence Filter pairs, equational filter pairs and adjoints. *Journal of Logic and Computation*, https://doi.org/10.1093/logcom/exae002, to appear 2025.

[4] Raftery, J.G., Order algebraizable logics. *Annals of Pure and Applied Logic*, 164:251–283, 2013.