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**Nilpotent elements and ideals in  
alternative loop rings**

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# Nilpotent elements and ideals in alternative loop rings

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## Abstract

We show that if the torsion subloop of an RA loop  $L$  is central, then there exist no nonzero nilpotent elements in the loop algebra over any commutative ring, with some obvious exceptions. We also study the nilpotency of augmentation ideals and prove that integral loop rings contain no nontrivial idempotent ideals.

## 1 Introduction

Let  $R$  be a commutative (and associative) ring with unity and let  $L$  be a loop. The *loop algebra* of  $L$  over  $R$  is defined in a way similar to that of a group algebra; i.e., as the free  $R$ -module with basis  $L$ , with a multiplication induced distributively from the operation in  $L$ . For basic facts and notation on alternative loop algebras, we refer the reader to [4]. A loop  $L$  is called an RA (ring alternative) loop if the loop algebra of  $L$  over any ring with no 2-torsion is alternative but not associative.

The question of deciding when an alternative loop algebra  $RL$  contains no nonzero nilpotent elements is completely decided in the case when  $R = K$ , is a field [4, Theorem XIII.2.1]). When  $T = T(L)$ , the set of torsion elements of  $L$ , is central then  $KL$  has no nonzero nilpotent elements regardless of the field of coefficients. On the other hand, if  $T$  is not central, then this fact depends heavily on the structure of  $K$ . In the first section of this paper we

show that, in the case when  $T$  is central, there exist no nonzero nilpotent elements in the loop algebra over any commutative ring, with some obvious exceptions.

We also study the nilpotency of augmentation ideals and finish this paper showing that integral loop rings contain no nontrivial idempotent ideals.

## 2 Nilpotent elements

We recall that, in a loop  $L$  we can define the commutator of two elements in a way which is similar to their definition in groups. We also define the associator of three elements.

**Definition 2.1** *Given elements  $a, b, c$  in a loop  $L$ , the commutator  $(a, b)$  and associator  $(a, b, c)$  are the elements (uniquely) defined by the following equations:*

$$ab = ba(a, b) \quad (\text{loop commutator})$$

$$(ab)c = [a(bc)](a, b, c) \quad (\text{loop associator})$$

The *commutator subloop* is the subloop generated by the set of all commutators and the *associator subloop* is the subloop generated by all associators. The *nucleus* and *centre* of  $L$  are the subloops  $\mathcal{N}(L)$  and  $\mathcal{Z}(L)$ , respectively, defined by

$$\mathcal{N}(L) = \{x \in L \mid (a, b, x) = (a, x, b) = (x, a, b) = 1, \text{ for all } a, b \in L\},$$

$$\mathcal{Z}(L) = \{x \in \mathcal{N}(L) \mid (a, x) = 1 \text{ for all } a \in L\}.$$

We recall the following result which describes the structure of RA loops. ([4, Theorem IV.1.8]).

**Theorem 2.2** *A loop  $L$  is RA if and only if it is not commutative and, for any two elements  $a$  and  $b$  of  $L$  which do not commute, the subloop of  $L$  generated by its centre together with  $a$  and  $b$  is a group  $G$  such that*

- (i) *for any  $u \notin G$ ,  $L = G \cup Gu$  is the disjoint union of  $G$  and the coset  $Gu$ ;*
- (ii)  *$G$  has a unique nonidentity commutator  $s$ , which is necessarily central and of order 2;*

(iii) the map

$$g \mapsto g^* = \begin{cases} g & \text{if } g \text{ is central} \\ sg & \text{otherwise} \end{cases}$$

is an involution of  $G$  (i.e., an antiautomorphism of order 2);

(iv) multiplication in  $L$  is defined by

$$\begin{aligned} g(hu) &= (hg)u \\ (gu)h &= gh^*u \\ (gu)(hu) &= g_0h^*g \end{aligned}$$

where  $g, h \in G$  and  $g_0 = u^2$  is a central element of  $G$  such that  $g_0^* = g_0$ .

The loop described by this theorem shall be denoted by  $L(G, *, g_0)$ .

**Proposition 2.3** *Let  $L = L(G, *, g_0)$  be an RA loop. Then, one and only one of the following conditions holds:*

- (i) *If  $T(L)$  is central in  $L$ , then we have that  $T(L) = T(G)$  and  $g_0 \notin T(G)$ .*
- (ii) *If  $T(L)$  is not central in  $L$ , then there exists a subgroup  $H$  in  $L$  and a non central element  $z \in T(L)$ , such that  $T(L) = T(H) \cup T(H)z$ .*

**Proof.** If  $T(L)$  is central in  $L$ , we have that  $T(L) \subset \mathcal{Z}(L) = \mathcal{Z}(G) \subset G$ . Consequently,  $T(L) = T(G)$  and, in this case,  $g_0 \notin T(G)$  (otherwise, we would have that  $u \in T(L)$ ).

To prove (ii), we notice that if  $T(L)$  is non central in  $L$ , then there exists  $z \in T(L)$  such that  $z \notin \mathcal{Z}(L) = N(L)$  so, there exist elements  $x, y \in L$  such that  $(x, y, z) \neq 1$ . Consequently,  $L = L(H, *, h_0)$ , where  $H = \langle \mathcal{Z}(L), x, y \rangle$  and  $h_0 = z^2$ . We claim that  $T(L) = T(H) \cup T(H)z$ . In fact, since  $L = H \cup Hz$  for each element  $w \in T(L)$ , we have that, either  $w = h \in H$ , implying  $w \in T(L) \cap H = T(H)$ , or  $w = hz \in Hz$  and thus

$$w^2 = \begin{cases} h_0h^2, & \text{if } h \in \mathcal{Z}(L), \\ h_0sh^2, & \text{if } h \notin \mathcal{Z}(L). \end{cases}$$

As  $w \in T(L)$  and  $h_0, s \in \mathcal{Z}(T(L))$ , we have that  $h \in (T(L) \cap H) = T(H)$ , so  $w \in T(H)z$ . Thus,  $T(L) \subset T(H) \cup T(H)z$ . We note that  $T(H) \subset T(L)$ ; so, to prove the opposite inclusion, it will suffice to consider the case when

$w \in T(H)z$ . In this case  $w = gz$ , for some  $g \in T(H)$ . Since  $L' = \{1, s\} \subset N(L)$ , we have that  $w^n = s^k g^n z^n$  for some  $k \in \{0, 1\}$  and thus  $w \in T(L)$ .  $\square$

We shall need the following technical lemmas.

**Lemma 2.4** *Let  $L = L(G, *, g_0)$  be an RA loop. Given an element  $x \in RL$  of the form  $x = (1 - s)(\alpha + \beta u)$ ,  $\alpha, \beta \in RG$ , we have that:*

$$(i) \quad x^2 = 2(1 - s)[(\alpha^2 + g_0\beta^*\beta) + \beta(\alpha + \alpha^*)u],$$

$$(ii) \quad n(x) = 2(1 - s)(\alpha\alpha^* - g_0\beta\beta^*).$$

(iii) *If, moreover, we have that  $\alpha^* = s\alpha$  then*

$$x^2 = 2(1 - s)(\alpha^2 + g_0\beta^*\beta).$$

(iv) *If  $g_0 = s$ ,  $\alpha = x_1a + x_2b + x_3ab$  and  $\beta = x_4 + x_5a + x_6b + x_7ab$ , where  $a, b$  are non central elements in  $G$  such that  $a^2 = b^2 = s$ , and  $x_i \in R[\mathcal{Z}(G)]$ , then we have that*

$$x^2 = 2(s - 1)\left(\sum_{i=1}^7 x_i^2\right).$$

**Proof.** Let  $x \in RL$  be as described above. Then:

$$x^2 = 2(1 - s)[(\alpha^2 + g_0\beta^*\beta) + \beta(\alpha + \alpha^*)u],$$

and

$$n(x) = xx^* = 2(1 - s)(\alpha\alpha^* - g_0\beta\beta^*).$$

If, in addition, we have that  $\alpha^* = s\alpha$ , then

$$\begin{aligned} x^2 &= 2(1 - s)[(\alpha^2 + g_0\beta^*\beta) + \beta(\alpha + \alpha^*)u] \\ &= 2(1 - s)[(\alpha^2 + g_0\beta^*\beta) + (1 + s)\beta\alpha.u] \\ &= 2(1 - s)(\alpha^2 + g_0\beta^*\beta). \end{aligned}$$

If we also have that  $g_0 = s$ ,  $\alpha = x_1a + x_2b + x_3ab$  and  $\beta = x_4 + x_5a + x_6b + x_7ab$  where  $a$  and  $b$  are non central elements in  $G$  such that  $a^2 = b^2 = s$  and  $x_i \in R\mathcal{Z}(G)$ , then:

$$\begin{aligned} (1 - s)\alpha^2 &= (1 - s)(x_1a + x_2b + x_3ab)^2 \\ &= (1 - s)[s(x_1^2 + x_2^2 + x_3^2) + (1 + s)(x_1x_2ab + x_1x_3b + x_2x_3a)] \\ &= (s - 1)(x_1^2 + x_2^2 + x_3^2). \end{aligned}$$

In a similar way, we have that:

$$(1-s)\beta^*\beta = (1-s)(x_4^2 + x_5^2 + x_6^2 + x_7^2),$$

and thus

$$(1-s)g_0\beta^*\beta = (1-s)s\beta^*\beta = (s-1)(x_4^2 + x_5^2 + x_6^2 + x_7^2).$$

Substituting in the expression for  $x^2$  above, we obtain:

$$x^2 = 2(s-1) \left( \sum_{i=1}^7 x_i^2 \right).$$

□

**Lemma 2.5** *Let  $L = L(G, *, g_0)$  be an RA loop and let  $R$  be a ring with no 2-torsion such that  $R[\mathcal{Z}(G)]$  contains no nonzero nilpotent elements and let  $X$  be a transversal of  $\langle s \rangle$  in  $\mathcal{Z}(G)$ . If  $x \in RL$  is of the form*

$$x = (1-s)[(x_0 + x_1a + x_2b + x_3ab) + (x_4 + x_5a + x_6b + x_7ab)u],$$

with  $x_i \in RX$   $i = 0, 1, \dots, 7$  and we have that  $x^2 = 0$ , then  $x_0 = 0$ .

**Proof.** Let  $x \in RL$  be as stated; set:

$$\alpha = x_0 + x_1a + x_2b + x_3ab$$

$$\beta = x_4 + x_5a + x_6b + x_7ab.$$

Since  $x^2 = 0$  we have that  $n(x)^2 = 0$  so, by part (ii) of Lemma 2.4, we have that

$$\begin{aligned} n(x) &= 2(1-s)(\alpha\alpha^* - g_0\beta\beta^*) = \\ &= 2(1-s)[(x_0^2 + x_1^2sa^2 + x_2^2sb^2 + x_3a^2b^2) - \\ &\quad - g_0(x_4^2 + x_5^2sa^2 + x_6^2sb^2 + x_7a^2b^2)] \end{aligned}$$

thus  $n(x) \in R[\mathcal{Z}(G)]$ , so  $n(x) = 0$ . From part (i) of Lemma 2.4 we have that  $0 = x^2 = 2(1-s)[(\alpha^2 + g_0\beta\beta^*) + \beta(\alpha + \alpha^*)u]$ . so we obtain the following equations.

$$0 = 2(1-s)(\alpha^2 + g_0\beta\beta^*)$$

$$0 = 2(1-s)(\alpha\alpha^* - g_0\beta\beta^*).$$

Hence

$$0 = 2(1 - s)\alpha(\alpha + \alpha^*),$$

where  $4(1 - s)x_0^2 = 0$ , so  $2((1 - s)x_0)^2 = 0$ . Since 2 is not a zero divisor in  $R$  and  $R[\mathcal{Z}(G)]$  contains no nonzero nilpotent elements, we have that  $(1 - s)x_0 = 0$ . But  $x_0 \in RX$ , so  $x_0 = 0$ .  $\square$

We are now ready to prove the following.

**Theorem 2.6 .** *Let  $L$  be an RA loop such that  $T(L)$  is central in  $L$ . Then  $RL$  contains no nonzero nilpotent elements if and only if  $R$  contains no nonzero nilpotent elements and, for every  $g \in T(L)$  we have that  $o(g)$  is not a zero divisor in  $R$ .*

**Proof.** Since  $RL$  contains no nonzero nilpotent elements we have that also  $RT(L)$  contains no nonzero nilpotent elements and, as  $T(L)$  is an abelian group, the "only if" part of the statement follows directly from [6, Proposition 3.2]

To prove the converse, assume that there exists an element  $x \in RL$  such that  $x^2 = 0$ . We can assume, without loss of generality, that  $L$  is finitely generated since we can restrict ourselves to work with the subloop generated by the support of  $x$ . Since  $L/\langle s \rangle$  is an abelian group and for each element  $\bar{g} \in T(L/\langle s \rangle)$ , we have that  $o(\bar{g})$  is not a zero divisor in  $R$ , again by [6, Proposition 3.2], it follows that  $R(L/\langle s \rangle)$  is a ring without nonzero nilpotent elements. So, we have that  $x \in \Delta_R(L/\langle s \rangle)$  and thus  $x$  is of the form:

$$(1) \quad x = (1 - s)(\alpha + \beta u), \quad \text{where } \alpha, \beta \in R(G).$$

By (i) of lemma 2.4 we have that

$$(2) \quad x^2 = 2(1 - s)[(\alpha^2 + g_0\beta^*\beta) + \beta(\alpha + \alpha^*)u].$$

Let  $a, b \in G$  be non central elements and let  $X$  be a transversal of  $\langle s \rangle$  in  $\mathcal{Z}(G)$ . Then, we can assume that  $\alpha, \beta$  are of the form:

$$\begin{aligned} \alpha &= x_0 + x_1a + x_2b + x_3ab \\ \beta &= x_4 + x_5a + x_6b + x_7ab, \end{aligned}$$

with  $x_i \in RX$ ,  $i = 0, \dots, 7$ .

Due to the hypothesis that  $R$  contains no nonzero nilpotent elements and that  $o(g)$  is not a zero divisor in  $R$ , for every  $g \in T(G)$ , using once again [6, Proposition 3.2], we see that  $R\mathcal{Z}(G)$  contains no nonzero nilpotent elements. Hence, by lemma 2.5, we have that  $x_0 = 0$ .

Thus,  $\alpha^* = s\alpha$  and, by (iii) of lemma 2.4, we have that

$$(3) \quad 0 = x^2 = 2(1-s)(\alpha^2 + g_0\beta^*\beta).$$

Since  $G/T(G)$  is a finitely generated abelian group, it is ordered. Let  $\mathcal{Q}$  be a transversal of  $T(G)$  in  $G$ . Then, there exists a positive integer  $m$  and a set  $Y = \{q_1, q_2, \dots, q_m\} \subset \mathcal{Q}$  such that  $q_i < q_{i+1}$ ,  $i = 1, \dots, m-1$ ,  $g_0 \in Y$ ,

$$\alpha = \sum_{i=1}^m \alpha_i q_i \quad \text{and} \quad \beta = \sum_{i=1}^m \beta_i q_i,$$

where  $\alpha_i, \beta_i \in RT(G) = RT(L) \subset R\mathcal{Z}(L) = R\mathcal{Z}(G)$ .

Hence, we have that

$$\alpha^2 = \sum_{i,j=1}^m \alpha_i \alpha_j q_i q_j \quad \text{and} \quad g_0 \beta^* \beta = \sum_{i,j=1}^m \beta_i \beta_j g_0 q_i^* q_j = \sum_{i,j=1}^m \beta_i \beta_j g_0 h_i q_i q_j$$

where

$$h_i = \begin{cases} 1 & \text{if } q_i \in \mathcal{Z}(G) \\ s & \text{if } q_i \notin \mathcal{Z}(G). \end{cases}$$

Since either  $q_1^2 < g_0 q_1^2$  or  $q_1^2 > g_0 q_1^2$ , we obtain from equation (3) that either  $((1-s)\alpha_1)^2 = 0$  or  $((1-s)\beta_1)^2 = 0$ . Thus, it follows that either  $(1-s)\alpha_1 = 0$  or  $(1-s)\beta_1 = 0$ . Without loss of generality we can assume that  $(1-s)\alpha_1 = 0$ . In this case, we obtain again from equation (3) that also  $(1-s)\beta_1 = 0$ . Repeating this process a finite number of times we obtain  $(1-s)\alpha_i = 0$  and  $(1-s)\beta_i = 0$ ,  $i = 1, 2, \dots, m$ .

Consequently  $x = 0$ . □

### 3 Nilpotency of augmentation ideals

In this section, we study necessary and sufficient conditions for augmentation ideals of the loop ring  $RL$  of an RA loop  $L$  to be nilpotent. We take into account the fact that if  $L$  is an RA loop and  $R$  is a ring of characteristic 2, then  $RL$  is still an alternative ring [3].

**Lemma 3.1** *Let  $L = L(G, *, g_0)$  be an RA loop. Then, the augmentation ideal  $\Delta(L)$  of  $RL$  is nilpotent if and only if the augmentation ideal  $\Delta(G)$  of  $RG$  is nilpotent.*

**Proof.** Necessity is obvious. To prove sufficiency, assume that  $\Delta(G)$  is nilpotent. Then, by [6, Lemma I.2.21] we have that  $G$  is a finite  $p$ -group and that  $R$  is a ring of characteristic  $p^m$  for some rational prime  $p$  and some positive integer  $m \geq 1$ .

Since  $s \in G$  and  $o(s) = 2$  it follows that  $G$  is a finite 2-group and  $\text{char}(R) = 2^m$ , for some  $m \geq 1$ . Hence, also  $L/\langle s \rangle$  is a finite 2-group. Again, [6, Lemma I.2.21] shows that  $\Delta_R(L/\langle s \rangle)$  is nilpotent of a certain index  $r$ .

Since  $RL/\Delta_R(L:\langle s \rangle) \cong R(L/\langle s \rangle)$ , it follows that any product of  $r$  elements of  $\Delta_R(L)$  with any order of parentheses belongs to  $\Delta_R(L:\langle s \rangle)$ .

As  $\Delta_R(L:\langle s \rangle) = RL(1-s)$ ,  $(1-s)$  is central in  $RL$  and  $(1-s)^{m+1} = 2^m(1-s)$ , we have that

$$\begin{aligned} \Delta(L)^{2(m+1)r} &\subset (\Delta_R(L:\langle s \rangle))^{m+1} \subset ((1-s)RL)^{(m+1)} \\ &\subset (1-s)^{(m+1)}RL = 2^m RL = 0 \end{aligned}$$

So  $\Delta(L)$  is nilpotent, as desired.  $\square$

As an immediate consequence, we get the following.

**Corollary 3.2** *Let  $L = L(G, *, g_0)$  be an RA loop. Then, the augmentation ideal  $\Delta(L)$  of  $RL$  is nilpotent if and only if  $L$  is a finite 2-loop and  $R$  is a ring of characteristic  $2^m$ , for some  $m \geq 1$ .*

**Lemma 3.3** *Let  $H$  be a subgroup of an RA loop  $L$ . Then there exists a subgroup  $G$  of  $L$  such that  $h \in G$ ,  $L = L(G, *, u^2)$  and for every positive integer  $n$ , we have that*

$$\Delta_R(L:H)^n = \Delta_R(G:H)^n + \Delta_R(G:H)^n u.$$

**Proof.** It follows from [4, Corollary IV.2.3], that if  $H$  is non abelian then  $G = H \cup \mathcal{Z}(L)$  contains  $H$  and  $L = L(G, *, u^2)$ , where  $u$  is any element in  $L \setminus G$ . On the other hand, if  $H$  is abelian but non central, there exists an element  $g \in L$  which does not commute with some element in  $H$ . Since two elements which commute do associate with every third element ([4, Theorem IV.1.b (iv)]) it follows that  $H_1 = \langle g, H \rangle$  is a non commutative group and we can obtain  $G$  as before. Finally, if  $H$  is central taking any

two non commuting elements  $x, y \in L$  we have that  $H_1 = \langle x, y, H \rangle$  is not commutative and, once again, we can obtain a group  $G$  as before.

To prove that  $\Delta_R(L : H)^n = \Delta_R(G : H)^n + \Delta_R(G : H)^n u$  we shall proceed by induction on  $n$ . Since  $\Delta_R(L : H) = \Delta_R(G : H) + \Delta_R(G : H)u$ , (see [4, Lemma VI.1.1]) the statement holds for  $n = 1$ , so we now assume that it holds for every positive integer  $k$ ,  $1 \leq k < n$ . Let  $x \in \Delta_R(L : H)^n$ ; then there exist elements  $y \in \Delta_R(L : H)^r$  and  $z \in \Delta_R(L : H)^{t-r}$  for some positive integer  $r$ ,  $1 \leq r \leq n-1$ , such that  $x = yz$ . Thus, by our hypothesis, there exist elements  $y_1, y_2 \in \Delta_R(G : H)^r$  and  $z_1, z_2 \in \Delta_R(G : H)^{t-r}$  such that

$$\begin{aligned} y &= y_1 + y_2 u \\ z &= z_1 + z_2 u, \end{aligned}$$

so

$$x = yz = (y_1 z_1 + z_2^* y_2) + (y_2 z_1^* + z_2 y_1)u.$$

Let  $X$  be a transversal of  $H$  in  $G$ . Given an element  $\alpha \in \Delta_R(G : H)$  it is of the form:

$$\alpha = \sum_{i,j} r_{i,j} x_i (h_j - 1) \text{ with } r_{i,j} \in R, \ x_i \in X \text{ and } h_j \in H.$$

Hence

$$\alpha^* = \sum_{i=1}^n r_{i,j} (h_j^* - 1) x_i^* \in \Delta_R(G : H).$$

Similarly, if  $w \in \Delta_R(G : H)^n$  for some positive integer  $n$ , then also  $w^* \in \Delta_R(G : H)^n$ .

The argument above shows that  $z_2^*, z_1^* \in \Delta_R(G : H)^{t-r}$  so we have that  $x \in \Delta_R(L : H)^n$ . Thus  $\Delta_R(L : H)^n \subset \Delta_R(G : H)^n + \Delta_R(G : H)^n u$  and the opposite inclusion is obvious.  $\square$

Now we are ready to prove the following.

**Theorem 3.4** *Let  $H$  be a normal subloop of an RA loop  $L$ . Then, the augmentation ideal  $\Delta_R(L : H)$  is nilpotent if and only if  $H$  is a finite 2-subloop of  $L$  and  $R$  is a ring of characteristic  $2^m$ , for some  $m \geq 1$ .*

**Proof.** If  $\Delta_R(L : H)$  is nilpotent, then also  $\Delta_R(H) \subset \Delta_R(L : H)$  is nilpotent, so Corollary 3.2 shows that  $H$  and  $R$  are as stated.

Conversely, assume that  $H \triangleleft L$  is a  $p$ -subloop and that  $R$  is a ring of characteristic  $2^m$ , for some  $m \geq 1$ . It follows again from Corollary 3.2 that  $\Delta_R(H)$  is nilpotent.

We shall consider first the case when  $H$  is associative. Then, the previous lemma shows there exists a group  $G \subset L$  containing  $H$  such that  $L = L(G, *, u^2)$  and

$$\Delta_R(L : H)^n = \Delta_R(G : H)^n + \Delta_R(G : H)^n u.$$

Since  $\Delta_R(G : H)$  is nilpotent, it follows that  $\Delta_R(L : H)$  is also nilpotent of the same nilpotency index.

Now we consider the case when  $H$  is not associative. Then, there exist elements  $a, b, u \in H$  such that  $(a, b, u) = s$  so  $L = L(G, *, g_0)$  where  $g_0 = u^2$  and  $G = \langle \mathcal{Z}(L), a, b \rangle$ .

Let  $X$  be a transversal of  $H$  in  $L$ . Since  $a, b, u \in H$  we can take  $X \subset \mathcal{Z}(L)$ . Every element  $w \in \Delta_R(L : H)$  can be written in the form

$$w = \sum_{i,j}^n r_{i,j} x_i (h_i - 1) \text{ with } r_{i,j} \in R, x_i \in \mathcal{Z}(L) \text{ and } h_i \in H,$$

An element  $\alpha \in \Delta_R(L : H)^n$ , is a product of the form  $\alpha = \prod_{i=1}^n w_i$  with a certain order of parentheses. Hence, using distributivity, we see that it is a sum of terms of the form

$$\alpha_h = \prod_{i=1}^n x_i (h_i - 1)$$

where  $(\prod_{i=1}^n)$  represents a product, with a certain order of parentheses.

As  $x_i \in \mathcal{Z}(L)$  it follows that  $\alpha = (x_1 x_2 \cdots x_n) \prod_{i=1}^n (h_i - 1)$ , where  $\prod_{i=1}^n (h_i - 1) \in \Delta_R(H)^n$ . Since  $\Delta_R(H)$  is nilpotent, the result follows.  $\square$

## 4 Idempotent Ideals

In the study of group rings, the existence of idempotent ideals in  $\mathbb{Z}G$  depends on the solvability of the given group. It was shown by T. Akasaki [1], [2] that a finite group  $G$  is solvable if and only if  $\mathbb{Z}G$  contains no idempotent ideals which are contained in the augmentation ideal. This result was improved by K.W. Roggenkamp [5] and extended by P.F. Smith [7, Theorem 2.2], who proved that if  $G$  is a polycyclic group, then  $\mathbb{Z}G$

contains no nontrivial idempotent ideals. We shall obtain a similar result for RA loops.

**Lemma 4.1** *Let  $L$  be an RA loop and let  $I$  be an idempotent ideal of the loop ring  $\mathbb{Z}L$  contained in  $\Delta_{\mathbb{Z}}(L : \langle s \rangle)$ . Then  $I = 0$ .*

**Proof.** Assume that  $L$  is an RA loop and that  $I$  is an idempotent ideal contained in  $\Delta_{\mathbb{Z}}(L : \langle s \rangle)$ ,

Set  $x \in I$ . Then, for every positive integer  $n \geq 1$ , there exist elements  $x_1, x_2, \dots, x_n \in I$  such that  $x = \prod_{i=1}^n x_i$ , where the product is taken with some choice of parentheses.

Since  $I \subset \Delta_{\mathbb{Z}}(L : \langle s \rangle)$ , there exist element  $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{Z}L$  such that  $x_i = (1 - s)\alpha_i$ ,  $1 \leq i \leq n$ . Thus,

$$x = \prod_{i=1}^n x_i = \prod_{i=1}^n (1 - s)\alpha_i = 2^{m-1} \left( \prod_{i=1}^n \alpha_i \right)$$

This shows that  $2^{m-1}$  divides the coefficients of  $x$ . Since  $m$  is arbitrary, it follows that  $x = 0$ .

**Lemma 4.2** *Let  $L = L(G, *, g_0)$  be a finite RA loop and let  $I$  be an idempotent ideal of  $\mathbb{Z}L$  such that  $I \subset \Delta_{\mathbb{Z}}(L : G)$ . Then  $I = \{0\}$ .*

**Proof.** Let  $I$  be an idempotent ideal of  $\mathbb{Z}L$  contained in  $\Delta_{\mathbb{Z}}(L : G)$ .

We consider the following subset of  $G$ :

$$J = \{\beta \in \mathbb{Z}G : \exists (\delta \in \mathbb{Z}G) / \beta + \delta u \in I\}$$

It is easy to see that  $J$  is an ideal of  $\mathbb{Z}G$  and that  $J \subset \Delta_{\mathbb{Z}}(G)$ . We claim that it is an idempotent ideal of  $\mathbb{Z}G$ .

In fact, set  $\beta \in \mathbb{Z}G$ . Then, there exists an element  $\delta \in \mathbb{Z}G$  such that  $z = (\beta + \delta u) \in I$ . Multiplying on the right by  $u$ , we see that  $(\beta^* u + \delta g_0) \in I$ . Hence,  $g_0 \delta \in J$  and thus  $\delta \in J$ .

Also,  $(\beta^* u + \delta g_0)u \in I$  so  $(\beta^* g_0 + g_0 \delta u) \in I$  which shows that  $g_0 \beta^* \in J$ ; hence  $\beta^* \in J$ .

Since  $(\beta + \delta u) \in I$  and  $I$  is an idempotent ideal, there exist elements  $z = z_1 + z_2 u$  and  $w = w_1 + w_2 u$  in  $I$  such that

$$\beta + \delta u = zw = (z_1 + z_2 u)(w_1 + w_2 u) = (z_1 w_1 + g_0 w_2^* z_2) + (w_2 z_1 + z_2 w_1^*)u.$$

so  $\beta = z_1 w_1 + g_0 w_2^* z_2$  where  $z_1, w_1, z_2, w_2^* \in J$ . Consequently  $\beta \in J^2$ , and it follows that  $J$  is an idempotent ideal of  $\mathbb{Z}G$ .

Now, since  $G$  is a finite solvable group it follows from [2, Theorem 2], that  $J = \{0\}$  and thus also  $I = \{0\}$ .  $\square$

**Theorem 4.3** *Let  $L = L(G, *, g_0)$  be finite loop such that  $L/L'$  is of finite rank and let  $I$  be an idempotent ideal of  $\mathbb{Z}L$ . Then, either  $I = 0$  or  $I = \mathbb{Z}L$ .*

**Proof.** Let  $I$  be an idempotent ideal of  $\mathbb{Z}L$ . Since  $L/\langle s \rangle$  is an abelian group of finite rank, it is a polycyclic group. It follows from [7, Theorem 2.2] that  $\mathbb{Z}(L/\langle s \rangle)$  contains no nontrivial idempotent ideals.

Since  $\mathbb{Z}L/\Delta_{\mathbb{Z}}(L : \langle s \rangle) \cong \mathbb{Z}(L/\langle s \rangle)$  and the image  $\bar{I}$  of  $I$  in the quotient is idempotent, we have that  $\bar{I} = \bar{0}$  or  $\bar{I} = \mathbb{Z}(L/\langle s \rangle)$ .

The first equality implies that  $I \subset \Delta_{\mathbb{Z}}(L : \langle s \rangle)$  so Lemma 4.1 shows that  $I = 0$ . The second equality clearly implies  $I = \mathbb{Z}L$   $\square$

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