Smooth Automorphism Group Schemes

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The most naive way to understand a finite-dimensional associative algebra is to find a basis and analyze its multiplication table. In the modern incarnation, one considers the scheme of all associative n-dimensional algebras over the field k as a subscheme of affine space $\mathrm{Hom}_k(k^n \otimes k^n, k^n)$. Then $GL_n(k)$ acts on the k-rational points of the scheme so that orbits can be interpreted as isomorphism classes of n-dimensional algebras.

The stabilizer of a point can be identified with the automorphism group (scheme) of the corresponding algebra. The geometry at a point seems to behave particularly well when the automorphism group scheme is smooth. For example, Gabriel ([Ga], 2.4) proves that if the algebra A corresponds to the point μ then $\mathrm{H}^2(A)$ is isomorphic to the tangent space at μ in the entire scheme modulo the tangent space at μ in its GL_n-orbit. This result requires smoothness of the stabilizer, as first explicitly pointed out in [Maz].

The automorphism group scheme is automatically smooth when $\mathrm{char}\, k = 0$ by the classical characterization of cocommutative connected Hopf algebras. The situation in positive characteristic has been more mysterious. The main contribution of this paper is to provide a simple, user-friendly reformulation of smoothness. We prove that the automorphism group scheme of A is smooth if and only if every k-derivation of A is integrable. Here we mean that D is *integrable* if it is a member of a sequence of k-endomorphisms of A,

$$D^{(0)} = I,\ D^{(1)} = D,\ D^{(2)}, D^{(3)}, \ldots \quad \text{such that} \quad D^{(m)}(ab) = \sum\nolimits_{i+j=m} D^{(i)}(a)D^{(j)}(b)$$

for all $a, b \in A$. The notion of integrability (which also appears in the literature under the name "higher derivations") is far from new although we believe that this application is novel. The proof of our criterion is essentially Hopf algebraic and found in the first section.

The second section reviews known properties of integrable derivations. It also includes a generalization of the well known fact that a derivation of a finite-dimensional algebra over a field of characteristic zero sends the Jacobson radical into itself.

Next, a particular class of examples is studied. Using our criterion, we present a clumsy but algorithmically tractable description of those commutative monomial algebras whose automorphism group scheme is smooth. We obtain both expected results (e.g., smoothness follows when relations "avoid" the characteristic) and bizarre examples.

In the fourth and last section, we explain a result which appears in the original work

1 Sweedler's theorem

We begin by deriving a transparent, intrinsic condition on a finite-dimensional algebra which is equivalent to its having a smooth automorphism group scheme. Until further notice, we let H denote a commutative affine Hopf algebra over the field k. If H represents an affine group scheme then the scheme is smooth precisely when H is reduced, i.e., when H has no nonzero nilpotent elements. It is well known that H is always reduced when char$k = 0$ ([Wa]). In case the characteristic of k is positive and k is perfect, Sweedler ([Sw]) has found a characterization of reduced Hopf algebras which we wish to apply. This result depends on the analysis of a certain k-coalgebra, the *hyperalgebra*, associated with H.

Let ϵ_H be the augmentation map for H and let $\mathcal{M}$ be its kernel. $\mathbf{H}yp(H)$ is the subcoalgebra of the dual H° consisting of all linear functionals which vanish on some power of $\mathcal{M}$. It is possible to prove that $\mathbf{Hyp}(H)$ is the irreducible component of H° containing ϵ_H ([Abe], p.198). Suppose C is a coalgebra with counit ϵ_C. Given $d \in C$, an *infinite sequence of divided powers lying over* d is a sequence $d_0, d_1, d_2, \ldots$ of elements in C such that

$$\Delta d_n = \sum_{i=0}^{n} d_i \otimes d_{n-i} \text{ for all } n \text{ , } \epsilon_C(d_n) = 0 \text{ for } n > 0 \text{ , and } \epsilon_C(d_0) = 1$$

with $d_1 = d$. Equivalently, we may regard an infinite sequence of divided powers in C as a coalgebra morphism from the coalgebra of divided powers $B = kx^{(0)} + kx^{(1)} + \cdots$ to C. Now suppose $d_0, d_1, \ldots$ is an infinite sequence of divided powers in $\mathbf{Hyp}(H)$. Notice that $d_0(1) = 1$ and $d_n(1) = 0$ for $n > 0$. Since d_0 must be group-like, we have $d_0 = \epsilon_H$. Moreover, consider any infinite sequence of divided powers $d_0, d_1, \ldots$ in H° such that $d_0 = \epsilon_H$. If $a, b \in \mathcal{M}$ then $d_1(ab) = d_0(a)d_1(b) + d_1(a)d_0(b) = 0$. Continuing by induction, we see that $d_n(\mathcal{M}^{n+1}) = 0$. Hence the sequence lies in $\mathbf{Hyp}(H)$. Thus we may identify the collection of all infinite divided powers in $\mathbf{Hyp}(H)$ with the subgroup (under convolution) of $\mathbf{coalg}(B, H^\circ)$ consisting of those α with $\alpha(x^{(0)}) = \epsilon_H$. Recall that an element a in a bialgebra is *primitive* when $\Delta a = 1 \otimes a + a \otimes 1$. Since 1 in H° is identified with ϵ_H, we see that any term d_1 belonging to an infinite sequence of divided powers in $\mathbf{Hyp}(H)$ must be primitive. (In this context, a linear functional $d \in H^\circ$ with $d(ab) = \epsilon_H(a)d(b) + \epsilon_H(b)d(a)$ for all $a, b \in H$ is also called an ϵ-derivation.)

Theorem 1 ([Sw]). *Assume H is an affine commutative Hopf algebra over a perfect field k of positive characteristic. Then H is reduced if and only if there is an infinite sequence of divided powers in* $\mathbf{Hyp}(H)$ *lying over each primitive element.*

In order to apply this theorem when H represents the automorphism group scheme of a finite-dimensional k-algebra A, we need to interpret infinite sequences of divided powers intrinsically for A. This will be done in a series of steps which are more or less standard. We begin by reminding the reader that if B is the k-coalgebra of divided powers then B^* can be identified with $k[[t]]$ by sending $f \in B$ to $\sum f(x^{(n)})t^n$.

Lemma 2. $\mathbf{coalg}(B, H^\circ) \simeq \mathbf{alg}(H, B^*)$ *as groups.*

For the remainder of this section, we assume that H represents the automorphism group scheme $\mathbf{Aut}_A$ of A. That is, if R is any commutative k-algebra then $\mathbf{Aut}_A(R) = \mathbf{alg}(H, R)$. Of course, $\mathbf{Aut}_A(k) = \mathrm{Aut}_k(A)$. The action of this automorphism group on A can be described via an H-comodule algebra structure on A: there is a coaction $\lambda : A \to H \otimes A$ making A a left H-comodule so that $\lambda(ab) = \sum a_{(0)}b_{(0)} \otimes a_{(1)}b_{(1)}$ and $\lambda(1) = 1 \otimes 1$ for $a, b \in A$. (See [Mo], section 4.1.) The explicit isomorphism from $\mathbf{alg}(H, k)$ to $\mathrm{Aut}_k(A)$ sends θ to the automorphism $\widetilde{\theta}$ where $\widetilde{\theta}(a) = \sum \theta(a_{(0)})a_{(1)}$. If R is any commutative k-algebra then the group $\mathbf{Aut}_A(R)$ is isomorphic to $\mathrm{Aut}_R(R \otimes_k A)$ under the extension of the comodule algebra action to $R \otimes H$ on $R \otimes A$.

We are particularly interested in the case that $R = B^*$. Since A is finite-dimensional, we have $B^* \otimes A \simeq k[[t]] \otimes A \simeq A[[t]]$. Again, since A is finite-dimensional, a $k[[t]]$-automorphism of $A[[t]]$ is determined by its effect on elements of A.

A k-algebra map $\delta : A \to A[[t]]$ is a *higher derivation* of A provided that for all $a \in A$, the constant term of the power series $\delta(a)$, is simply a. Clearly, higher derivations of A are in one-to-one correspondence with $k[[t]]$-automorphisms of $A[[t]]$ which "preserve constant terms". Alternatively, we may regard a higher derivation as a sequence of linear endomorphisms of A, say $D^{(0)} = I, D^{(1)}, D^{(2)}, \ldots$, such that $D^{(n)}(ab) = \sum_{i+j=n} D^{(i)}(a)D^{(j)}(b)$ for all $a, b \in A$ and $n \geq 0$. (The point is to expand $\delta(c) = \sum_{n=0}^{\infty} D^{(n)}(c)t^n$ for $c \in A$.)

Lemma 3. *There is a one-to-one correspondence between infinite sequences of divided powers in* $\mathbf{Hyp}(H)$ *and higher derivations of A. The map sends $\epsilon_H = d_0, d_1, \ldots$ to $I = D^{(0)}$, $D^{(1)}$, ... where $D^{(m)}(a) = \sum d_m(a_{(0)})a_{(1)}$ for all $a \in A$.*

Proof: By virtue of the previous lemma and our discussion so far, there is a group isomorphism $\mathbf{coalg}(B, H^\circ) \simeq \mathrm{Aut}_{k[[t]]}(A[[t]])$. The isomorphism sends $\alpha \in \mathbf{coalg}(B, H^\circ)$ to the automorphism $\sum c_n t^n \mapsto \sum_n \sum_{i+j=n}[\alpha(x^{(i)})(c_j)_{(0)}(c_j)_{(1)}]t^n$. In particular, this automorphism sends $c \in A$ to $\sum_n[\sum \alpha(x^{(n)})(c_{(0)})c_{(1)}]t^n$. Note that if $\alpha(x^{(0)}) = \epsilon_H$ then the associated automorphism preserves constants. Conversely, we argue that if $f = \alpha(x^{(0)}) \in H^\circ$ and $\sum f(c_{(0)})c_{(1)} = c$ for all $c \in A$ (i.e., the automorphism preserves constants) then $f = \epsilon_H$. But f is group-like, so $f \in \mathbf{alg}(H, k)$. The claim follows from our isomorphism $\mathbf{alg}(H, k) \simeq \mathrm{Aut}_k(A)$. The lemma is now a consequence of restricting the group isomorphism to coalgebra maps from B to H° which send $x^{(0)}$ to ϵ_H. ■

It is easy to see that the $D^{(1)}$-term of a higher derivation is always an ordinary derivation. We say that a derivation $D \in \mathrm{Der}_k(A)$ is *integrable* provided there exists a higher derivation $D^{(0)} = I, D^{(1)}, D^{(2)}, \ldots$ such that $D^{(1)} = D$.

Theorem 4. *Let A be a finite-dimensional k-algebra and assume that H represents the affine group scheme* $\mathbf{Aut}_A$. *Every ϵ-derivation of H has an infinite sequence of divided powers in* $\mathbf{Hyp}(H)$ *lying over it if and only if every derivation of A is integrable.*

Proof: It is easy to check directly that if d is an ϵ-derivation of H then the linear endomorphism D of A given by $D(a) = \sum d(a_{(0)})a_{(1)}$ is a derivation. It is well known that this map from ϵ-derivations to $\mathrm{Der}_k(A)$ is an isomorphism ([Wa]). (This can also be seen by replacing $k[[t]]$ with $k[t]/t^2$ in the arguments we have just presented.) Apply the lemma. ■

If the characteristic of k is zero then it is a well known consequence of the Leibniz rule (see [Hu], p.8) that any derivation D can be integrated to the higher derivation I, D, $D^2/2!$, ..., $D^n/n!$, ... Thus integrability of derivations is only an issue when $\mathrm{char}\, k > 0$, which brings us back to Sweedler's Theorem. We summarize our discussion for this section.

Corollary 5. *Assume that A is a finite-dimensional algebra over the perfect field k. The affine group scheme* $\mathbf{Aut}_A$ *is smooth if and only if every k-derivation of A is integrable.* ■

2 Integrable derivations

A derivation of a finite-dimensional algebra whose scalar field has characteristic zero always sends the radical into itself. We shall see that integrable derivations extend this behavior.

Lemma 6. *If S is a semiprime ring then so is $S[[t]]$.*

Proof: We must show that if $\alpha \in S[[t]]$ is nonzero then $\alpha S[[t]]\alpha \neq 0$. But $\alpha S \alpha \neq 0$, as can be seen by looking at the lowest term of α. ■
As consequence, if R is a ring and I is a nilpotent ideal of $R[[t]]$ then $I \subseteq (\text{prime rad}\,(R))[[t]]$.

Theorem 7. *Let A be a finite-dimensional algebra. If the algebra map $\phi : A \to A[[t]]$ is a higher derivation then $\phi(\text{rad}\, A) \subseteq (\text{rad}\, A)[[t]]$.*

Proof: It suffices to prove that $\phi(\text{rad}\, A)$ lies in a nilpotent ideal of $A[[t]]$. Let $\widetilde{A}$ denote the algebra generated by $\phi(A)$ and t. (We do not ask that $\widetilde{A}$ be closed.) The condition that $\phi(a) = a+$"higher terms" implies that $\widetilde{A}$ is dense in $A[[t]]$. Let n be the index of nilpotence for rad A and choose $w_j \in \text{rad}\, A$. Choose $r_j, s_j \in \widetilde{A}$ for $j = 1, \ldots, n$. Because ϕ is an algebra map and t is central, $(r_1\phi(w_1)s_1)(r_2\phi(w_2)s_2)\cdots(r_n\phi(w_n)s_n) = 0$. By continuity, this identity extends to all $r_j, s_j \in A[[t]]$. Hence the ideal in $A[[t]]$ generated by $\phi(\text{rad}\, A)$ is nilpotent. ■

Corollary 8. *Let A be a finite-dimensional k-algebra. If $D \in \text{Der}_k(A)$ is integrable then $D(\text{rad}\, A) \subseteq \text{rad}\, A$.*

Proof: Let $I = D^{(0)}, D = D^{(1)}, D^{(2)}, \ldots$ be a higher derivation. According to the theorem, $D^{(m)}(\text{rad}\, A) \subseteq \text{rad}\, A$ for all m. ■

We shall see, when we examine monomial algebras, that it is possible for every derivation of a finite-dimensional algebra to leave the radical invariant even though its automorphism group scheme is not smooth. Nonetheless, the corollary does provide a useful test.

Theorem 9. *Assume that k is a perfect field of characteristic p and G is a non-trivial finite p-group. Then the group algebra $k[G]$ never has a smooth automorphism group scheme.*

Proof: Since G/G' is not trivial, there is a nonzero additive character $\lambda \in \text{Hom}(G, k^+)$. Define $D : k[G] \to k[G]$ by linearly extending the function $D(g) = \lambda(g)g$ with $g \in G$. It is easy to see that D is a derivation.

Choose $h \in G$ with $\lambda(h) \neq 0$. Then $h - 1$ lies in the augmentation ideal of $k[G]$, which coincides with the radical. But $D(h-1) = \lambda(h)h$ so $D(h-1)$ is not in the radical. ■

It is tempting to conjecture that $k[G]$ does not have a smooth automorphism group scheme whenever p divides the order of G. However, we will see in a few moments that inner derivations are always integrable. Thus a "prerequisite" to the conjecture is the knowledge that such group algebras possess outer derivations. The good news is that this weaker assertion is true ([FJL]). The bad new is that the only known proof requires the classification of finite simple groups.

We record some well known properties of integrable derivations for future use. (See, e.g., [Mat].) Let $\mathcal{Z}(\ )$ denote the center of a ring.

Proposition 10. *The integrable derivations of the k-algebra A comprise a $\mathcal{Z}(A)$-submodule of all derivations.*

Proof: If D and E are integrable derivations then there exist ϕ and ψ, constant preserving $k[[t]]$-automorphisms of $A[[t]]$, such that $\phi(a) = a + D(a)t + \cdots$ and $\psi(a) = a + E(a)t + \cdots$ for all $a \in A$. Then the composition $\psi\phi$ is an automorphism which preserves constants.

Explicitly, if $D^{(0)}, D^{(1)} = D, D^{(2)}, \ldots$ and $E^{(0)}, E^{(1)} = E, E^{(2)}, \ldots$ are the corresponding higher derivations then we have constructed a new higher derivation whose m^{th} term is $\sum_{i+j=m} E^{(i)}D^{(j)}$. In particular, the $m = 1$ term is $D + E$. Thus the collection of integrable derivations is closed under addition. For any central $\lambda \in A$, the sequence $\lambda^0 D^{(0)}$, $\lambda^1 D^{(1)}$, $\lambda^2 D^{(2)}$, ..., is also a higher derivation. ∎

Proposition 11. *Every inner derivation of the algebra A is integrable.*

Proof: Let $a \in A$. Conjugation by the unit $1 - at$ is an algebra automorphism of $A[[t]]$ and for any $r \in A$, $(1-at)^{-1}r(1-at) = r + (ar - ra)t +$ higher terms. Thus ad$a$ is integrable. ∎

It is well known that a diagonalizable derivation of a k-algebra A is equivalent to a grading of A by the additive group k^+. Indeed, the eigenspaces of the derivation are the homogeneous components for the grading. Such gradings can be difficult to deal with when the characteristic of k is positive; it would be nice to lift $\mathbf{Z}/(p)$-gradings to $\mathbf{Z}$-gradings. This goal is encoded in the following definition. We say that a higher derivation $D^{(0)}, D^{(1)}, \ldots$ is diagonalizable when the $D^{(m)}$ are simultaneously diagonalizable k-endomorphisms of A. If $\phi : A \to A[[t]]$ is the algebra map version of the higher derivation then diagonalizability means that there is a basis $v_1, \ldots, v_n$ of A so that $\phi(v_i) = f_i v_i$ for some $f_i \in k[[t]]$. Moreover, the fact that ϕ preserves constants tells us that $f_i \in \mathcal{U}_1(k[[t]])$, the multiplicative group of units in $k[[t]]$ with constant term 1. With very little additional work, we have

Proposition 12. *There is a one-to-one correspondence between diagonalizable higher derivations of the finite-dimensional k-algebra A and $\mathcal{U}_1(k[[t]])$-gradings of A.* ∎

Observe that the group $\mathcal{U}_1(k[[t]]$ is always torsion free, no matter what the characteristic of k is. Thus if a diagonalizable derivation of A lifts to a diagonalizable higher derivation then a k^+-grading lifts to a grading by a torsion free abelian group. The converse is more valuable. Since $\mathcal{U}_1(k[[t]])$ is uncountable, it is abelian of infinite rank. As a consequence, every finitely generated torsion free abelian group embeds in $\mathcal{U}_1$. We conclude that if D is a diagonalizable derivation of A whose grading lifts to a second grading via a (finitely generated) torsion free abelian group then the D is integrable (and is the $D^{(1)}$-term of a diagonalizable higher derivation).

3 Commutative monomial algebras

We regard monomial algebras as a rich source of elementary examples. Our study of this family of rings begins with a more or less computable criterion for integrability in this case.

Recall (cf. [FGGM]) that if I is an ideal of the polynomial algebra $k[X_1, \ldots, X_n]$ then every derivation of $R = k[X_1, \ldots, X_n]/I$ lifts to a derivation of $k[X_1, \ldots, X_n]$ which stabilizes I. If I is a monomial ideal then every derivation of R is a linear combination of images of such derivations with the special form $m\frac{\partial}{\partial X_j}$ for some monomial m. In this section, we will always assume that I has finite codimension in the polynomial algebra.

Theorem 13. *Let I be a monomial ideal of $k[X, Y_1, \ldots Y_n]$ and set $R = k[X, Y_1, \ldots, Y_n]/I$. Assume that m is a monomial which does not involve X such that $m\frac{\partial}{\partial X}$ stabilizes I. Then the derivation D it induces on R is integrable if and only if for each monomial $X^e\nu \in I$, where ν does not involve X, $\binom{e}{j}X^{e-j}m^j\nu \in I$ for $j = 0, 1, \ldots, e$.*

Proof: Choose an automorphism ϕ of $R[[t]]$ such that $\phi\mid_R = I + Dt + \cdots$. Underline to denote the image of a polynomial in R. Suppose $d \leq e$. Since $\phi(\underline{X}) = \underline{X} + \underline{m}t + \cdots$, the coefficient of t^d in $\phi(\underline{X})^e$ has the form $\binom{e}{d}\underline{X}^{e-d}\underline{m}^d + \underline{X}^{e-d+1}s_d$ for some $s_d \in R$. (The pigeonhole principle is at work here: no more than d of the factors $\phi(\underline{X})$ in $\phi(\underline{X})^e$ can contribute a term rt^i for $i \geq 1$.) Similarly, $\phi(\underline{\nu}) = \sum_{h \geq 0} a_h t^h$ with $a_0 = \underline{\nu}$ Hence for $j \leq e$, the coefficient of t^j in $\phi(\underline{X})^e\phi(\underline{\nu})$ has the form

$$\sum_{d=0}^{j}(\binom{e}{d}\underline{X}^{e-d}\underline{m}^d + \underline{X}^{e-d+1}s_d)a_{j-d} = \binom{e}{j}\underline{X}^{e-j}\underline{m}^j\underline{\nu} + \underline{X}^{e-j+1}s$$

for some $s \in R$. On the other hand, $\phi(\underline{X})^e\phi(\underline{\nu}) = \phi(\underline{X}^e\underline{\nu}) = 0$. Since R is strongly graded by monomials, we see from the powers of $\underline{X}$ in our expression for 0 that $\binom{e}{j}\underline{X}^{e-j}\underline{m}^j\underline{\nu} = 0$. This proves one direction of the theorem.

As to the converse, assume that for a set of generating relations $X^e\nu \in I$ we have $\binom{e}{j}\underline{X}^{e-j}\underline{m}^j\underline{\nu} = 0$ in R for $j = 0, \ldots, e$. Consider the assignments $\psi(\underline{X}) = \underline{X} + \underline{m}t$ and $\psi(\underline{Y}_i) = \underline{Y}_i$ for $i = 1, \ldots, n$. Since $\psi(\underline{X})^e\psi(\underline{\nu}) = 0$, ψ extends to an algebra map from R to $R[[t]]$. It is easy to check that the coefficient of t in the expansion of ψ agrees with D on the generators $\underline{X}, \underline{Y}_1, \ldots, \underline{Y}_n$ of R. Hence D is integrable. ■

The previous theorem only handles images of $m\frac{\partial}{\partial X}$ when X does not appear in m. Fortunately, the remaining "monomial" derivations are always integrable.

Theorem 14. *Let I be a monomial ideal of $k[X, Y_1, \ldots Y_n]$ and set $R = k[X, Y_1, \ldots, Y_n]/I$. Assume that m is a monomial which involves X such that $m\frac{\partial}{\partial X}$ stabilizes I. Then the derivation D it induces on R is integrable.*

Proof: According to Proposition 2.1, it suffices to show that the image D of $X\frac{\partial}{\partial X}$ is integrable. Define $\phi(\underline{X}) = \underline{X} + \underline{X}t$ and $\phi(\underline{Y}_j) = \underline{Y}_j$ for $j = 1, \ldots n$. If $X^e\nu$ is a monomial in I such that X does not appear in ν then $(X + Xt)^e\nu = X^e\nu(1+t)^e \in I$. Thus ϕ extends to an algebra map from R to $R[[t]]$. Its t term agrees with D. ■

We use the previous two theorems to illustrate the metatheorem that an algebra whose relations do not interfere with the characteristic has a smooth automorphism group.

Theorem 15. *Assume that k is a field of characteristic $p > 0$. Let I be a monomial ideal of $k[X_1, \ldots X_n]$ and set $R = k[X_1, \ldots, X_n]/I$. If no minimal monomial in I has positive degree in any X_j which is divisible by p then the automorphism group scheme of R is smooth.*

Proof: By virtue of the previous theorem and Proposition 2.1, we need only prove that if m is a monomial which does not involve X_s and $m\frac{\partial}{\partial X_s}$ stabilizes I then its image derivation of R is integrable. We apply Theorem 3.1. It suffices to show that if $X_s^e\nu$ is a monomial in I such that X_s does not appear in ν then $X_s^{e-j}m^j\nu \in I$ for $j = 0, \ldots, e$.

By induction, we are reduced to the case $j = 1$. Since $m\frac{\partial}{\partial X_s}$ stabilizes I, $eX_s^{e-1}\nu m \in I$. We are done unless $p|e$. Suppose this is the case. Now $X_s^e\nu$ is divisible by some minimal monomial relation μ. But the X_s-degree of μ is either zero or a positive integer not divisible by p. In either event, we must have $X_s^{e-1}\nu \in I$. ■

The hope is to look at the minimal monomials generating an ideal and immediately tell whether the corresponding monomial algebra has a smooth automorphism group scheme. Since we do not yet know how to do this, we offer a more modest result.

Theorem 16. *Assume that k is a field of characteristic $p > 0$. Let I be a monomial ideal of $k[X_1, \ldots X_n]$ and set $R = k[X_1, \ldots, X_n]/I$. Every derivation of R stabilizes the radical if and only if for each j there exists a minimal monomial $\mu_j \in I$ such that the X_j-degree of μ_j is not divisible by p.*

Proof: First assume that every minimal monomial in I has the form $X_1^{ps}\alpha$ where α is a monomial not involving X_1. Then $\frac{\partial}{\partial X}(X_1^{ps}\alpha) = \frac{\partial}{\partial X}(X_1^{ps})\alpha + X_1^{ps}\frac{\partial}{\partial X}(\alpha) = 0$. Thus $\frac{\partial}{\partial X}$ stabilizes I. We see that $\frac{\partial}{\partial X}$ induces a derivation of R which sends the image of X_1, which is in the radical, to 1.

Conversely, assume that I has minimal generators as described in the theorem. We must show that if D is derivation of $k[X_1, \ldots, X_n]$ and $D(I) \subseteq I$ then $D(X_j) \in (X_1, \ldots, X_n)$ for $j = 1, \ldots, n$. Choose a minimal monomial $X_j^f\beta \in I$ such that β does not involve X_j and p does not divide f, $fX_j^{f-1}D(X_j)\beta + X_j^f D(\beta) = D(X_j^f\beta) \in I$. Write $D(X_j) = c + H$ where $c \in k$ and $H \in (X_1, \ldots, X_n)$. Then $cfX_j^{f-1}\beta + fX_j^{f-1}\beta H + X_1^f D(\beta) \in I$. If we write the second term as a nonredundant linear combination of monomials then each monomial which appears has length greater than the length of $X_j^{f-1}\beta$. Each monomial in the support of the third term has X_j-degree at least f. Thus the first monomial term cannot be in the support of $fX_j^{f-1}\beta H + X_1^f D(\beta)$. The fact that I is a monomial ideal implies now that $cfX_j^{f-1}\beta \in I$. But $X_j^{f-1}\beta \notin I$ by minimality and f is not zero in k. We conclude that $c = 0$, i.e., $D(X_j) \in (X_1, \ldots, X_n)$. ■

As promised, we can now construct many examples of finite-dimensional algebras all of whose derivations stabilize the radical, but which do not have smooth automorphism group schemes. For example, suppose that k is a perfect field of characteristic p, that $n \geq 2$, and $p < e(1) \leq e(2) \leq \cdots \leq e(n)$. Let I be the ideal of $k[X_1, \ldots X_n]$ generated by $X_1^{e(1)}, \ldots, X_n^{e(n)}$, and $X_1^p X_2 \cdots X_n$. We first observe that if p is relatively prime to $e(1)$ then $R = k[X_1, \ldots, X_n]/I$ does not have a smooth automorphism group scheme. Set $w = X_2 \cdots X_n$. It is easy to check that $w\frac{\partial}{\partial X_1}$ induces a derivation of R. We argue that this derivation is not integrable. Otherwise, we may apply theorem 3.1 to conclude that

$$X_1 X_2^{e(1)-1} \cdots X_n^{e(1)-1} = \binom{e(1)}{e(1)-1} w^{e(1)-1} \in I\ .$$

However, $e(1) \leq e(j)$ for all j, so $e(1) - 1 < e(j)$ for all $j \geq 2$. We have created an element of I which is not divisible by any of the defining generators for I.

If we assume, in addition, that all $e(j)$ are relatively prime to p then Theorem 3.4 tells us that every derivation of R sends the radical into itself. The curious aspect of this example is that when $e(1) \geq 2p$ we can add the relation $(X_2 \cdots X_n)^2$ and the new algebra has a smooth automorphism group scheme.

4 Morita invariance

We prove that having a smooth automorphism group scheme is a Morita invariant for finite-dimensional algebras over perfect fields. In fact, we establish a stronger result with no restriction on the scalar field. It is well known that Hochschild cohomology is a Morita invariant.

According to Proposition 2.2, for any finite-dimensional algebra A it makes sense to define the subspace $\int \mathrm{HH}^1(A)$ = integrable derivations of A / inner derivations of A in $\mathrm{HH}^1(A)$. We show that $\int \mathrm{HH}^1$ is a Morita invariant in the sense that if A and B are Morita equivalent finite-dimensional algebras then there is an isomorphism from $\mathrm{HH}^1(A)$ to $\mathrm{HH}^1(B)$ which carries $\int \mathrm{HH}^1(A)$ isomorphically to $\int \mathrm{HH}^1(B)$. Although this subspace is the correct one for our strategy of proof, there are good reasons to "cointerpret" it. As a consequence of our theorem, the obstruction to integrability $\mathrm{Der}_k(A)$ / integrable derivations of A is a Morita invariant.

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