Proyecciones Journal of Mathematics
Vol. 44, No 2, pp. 281-293, April 2025.
Universidad Católica del Norte
Antofagasta - Chile

REVISTAS CIENTÍFICAS
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PROYECCIONES
Journal of Mathematics

$*$-Jordan-triple maps on alternative $*$-algebras

Tatiane Cardoso Batista Flores
Ruth Nascimento Ferreira
Bruno Leonardo Macedo Ferreira

April 2025

Recieved: July 2023. | Accepted: July 2024.

Abstract

Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two alternative $*$-algebras with identities $1_{\mathfrak{A}}$ and $1_{\mathfrak{A}'}$, respectively, and e_1 and $e_2 = 1_{\mathfrak{A}} - e_1$ nontrivial symmetric idempotents in $\mathfrak{A}$. In this paper we study the characterization of multiplicative $*$-Jordan-triple derivations on alternative $*$-algebras.

Keywords: *prime alternative $*$-algebra, alternative W^*-algebras*

MSC (2020): *17D05; 16W20*

Introduction and Preliminaries

Mathematicians have given the study of additivity of maps a good amount of attention. Martindale created a condition on a ring such that multiplicative bijective maps are all additive, which is where the first fairly surprise finding comes from [13]. Additionally, over the years, a number of studies examining various varieties of associative and non-associative algebras have been produced. Among these, we can note [2, 4, 6, 7, 8, 9].

Many researchers have focused their efforts on the investigation of two new products, which were provided by Brešar and Fošner in [1, 11] where the definition is as follows: For $a, b \in R$, where R is a $*$-ring, that is, a ring with involution $*$, we designate the $*$-Jordan product and the $*$-Lie product, respectively, by $\{a, b\}_* = ab + ba^*$ and $[a, b]_* = ab - ba^*$. The authors established in [3] that a map φ between two factor von Newmann algebras is a $*$-ring isomorphism if and only if $\varphi(\{a, b\}_*) = \{\varphi(a), \varphi(b)\}_*$.

In [5], Ferreira and Costa expanded these new products and created two further types of applications, referred to as multiplicative $*$-Jordan n-map and multiplicative $*$-Lie n-map, and utilized it to impose requirements such that a map between C^*-algebras is a $*$-ring isomorphism.

In [10], the authors described the Jordan triple derivation in terms of prime $*$-algebras and proved that such one mappings are $*$-derivations under specific circumstances. With this in mind, we will address a study of maps preserving a triple product for a large class of algebras, namely, alternative algebras. Our research was divided into two sections. In the first, we prove that our map is $*$-additive under specific circumstances, and in the second, we used the first result along with a different condition to proof that the map is a $*$-ring isomorphism.

Throughout the paper, the ground field is assumed to be of complex numbers and we consider an algebra $\mathfrak{A}$ endowed with a involution, called $*$-algebra. By involution, we mean a mapping $* : \mathfrak{A} \to \mathfrak{A}$ such that $(x + y)^* = x^* + y^*$, $(x^*)^* = x$, $(\lambda x)^* = \bar{\lambda} x^*$ and $(xy)^* = y^* x^*$ for all $x, y \in \mathfrak{A}$, $\lambda \in \mathbb{C}$. An element $s \in \mathfrak{A}$ satisfying $s^* = s$ is called *symmetric element* of $\mathfrak{A}$.

Consider the product $\{x, y\}_* = xy + yx^*$ and let us define the following sequence of polynomials:

$$q_{1*}(x) = x \; q_{2*}(x_1, x_2) = \{x_1, x_2\}_* \text{ and } q_{3*}(x_1, x_2, x_3) = \{\{x_1, x_2\}_*, x_3\}_*.$$

Note that q_{2*} is the product introduced by Brešar and Fošner [1, 11]. Then, we say that a map (not necessarily additive) $\varphi : \mathfrak{A} \longrightarrow \mathfrak{A}'$ is a *multiplicative $*$-Jordan triple derivation* if

$$\varphi(q_{3*}(x_1, x_2, x_3)) = q_{3*}(\varphi(x_1), \varphi(x_2), \varphi(x_3)).$$

Let be e_1 a nontrivial symmetric idempotent in $\mathfrak{A}$ and $e_2 = 1_{\mathfrak{A}} - e_1$ where $1_{\mathfrak{A}}$ is the identity of $\mathfrak{A}$. Let us consider an alternative $*$-algebra $\mathfrak{A}$ and fix a nontrivial symmetric idempotent $e_1 \in \mathfrak{A}$. It is easy to see that $(e_i a) e_j = e_i (a e_j)$ $(i, j = 1, 2)$

for all $a \in \mathfrak{A}$. Then $\mathfrak{A}$ has a Peirce decomposition $\mathfrak{A} = \mathfrak{A}_{11} \oplus \mathfrak{A}_{12} \oplus \mathfrak{A}_{21} \oplus \mathfrak{A}_{22}$, where $\mathfrak{A}_{ij} = e_i \mathfrak{A} e_j$ $(i, j = 1, 2)$ [12], satisfying the following multiplicative relations:

(i) $\mathfrak{A}_{ij}\mathfrak{A}_{jl} \subseteq \mathfrak{A}_{il}$ $(i, j, l = 1, 2)$;

(ii) $\mathfrak{A}_{ij}\mathfrak{A}_{ij} \subseteq \mathfrak{A}_{ji}$ $(i, j = 1, 2)$;

(iii) $\mathfrak{A}_{ij}\mathfrak{A}_{kl} = 0$, if $j \neq k$ and $(i, j) \neq (k, l)$, $(i, j, k, l = 1, 2)$;

(iv) $x_{ij}^2 = 0$, for all $x_{ij} \in \mathfrak{A}_{ij}$ $(i, j = 1, 2;\ i \neq j)$.

Remark 0.1. *Observe that in the case associative $a_{ij}b_{ij} = 0$ for $i \neq j$ but in general in alternative algebras, due the property (ii), we do not have this relation.*

The following two claims play a very important role in the further development of the paper. By definition of involution clearly we get

Proposition 0.1. $(\mathfrak{A}_{ij})^* \subseteq \mathfrak{A}_{ji}$ *for $i, j \in \{1, 2\}$.*

Proof. If $a_{ij} \in \mathfrak{A}_{ij}$ then

$$a_{ij}^* = (e_i a_{ij} e_j)^* = (e_j)^* (a_{ij})^* (e_i)^* = e_j (a_{ij})^* e_i \in \mathfrak{A}_{ji}.$$

□

Proposition 0.2. *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two alternative $*$-algebras and $\varphi : \mathfrak{A} \to \mathfrak{A}'$ a bijective map which satisfies*

$$\varphi(q_{3*}(\xi, a, b)) = q_{3*}(\varphi(\xi), \varphi(a), \varphi(b))$$

for all $a, b \in \mathfrak{A}$ and $\xi \in \{1_{\mathfrak{A}}, e_1, e_2\}$. Let x, y and h be in $\mathfrak{A}$ such that $\varphi(h) = \varphi(x) + \varphi(y)$. Then, given $z \in \mathfrak{A}$,

$$\varphi(q_{3*}(t, h, z)) = \varphi(q_{3*}(t, x, z)) + \varphi(q_{3*}(t, y, z))$$

and

$$\varphi(q_{3*}(t, z, h)) = \varphi(q_{3*}(t, z, x)) + \varphi(q_{3*}(t, z, y))$$

for $t = 1_{\mathfrak{A}}$ or $t = e_i$, $i = 1, 2$.

Proof. Using the definition of φ and multilinearity of q_{n*} we obtain

$$\begin{aligned}
\varphi(q_{3*}(t, h, z)) &= q_{3*}(\varphi(t), \varphi(h), \varphi(z)) \\
&= q_{3*}(\varphi(t), \varphi(x) + \varphi(y), \varphi(z)) \\
&= q_{3*}(\varphi(t), \varphi(x), \varphi(z)) + q_{3*}(\varphi(t), \varphi(y), \varphi(z)) \\
&= \varphi(q_{3*}(t, x, z)) + \varphi(q_{3*}(t, y, z)).
\end{aligned}$$

In a similar way we have

$$\varphi(q_{3*}(t, z, h)) = \varphi(q_{3*}(t, z, x)) + \varphi(q_{3*}(t, z, y)).$$

□

1 Main theorem

We shall prove as follows a part of the the main result of this paper:

Theorem 1.1. *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two alternative $*$-algebras with identities $1_{\mathfrak{A}}$ and $1_{\mathfrak{A}'}$, respectively, and e_1 and $e_2 = 1_{\mathfrak{A}} - e_1$ nontrivial symmetric idempotents in $\mathfrak{A}$. Suppose that $\mathfrak{A}$ satisfies*

$$(\spadesuit) \qquad x(\mathfrak{A}\, e_i) = \{0\} \quad \textit{implies} \quad x = 0.$$

Even more, suppose that $\varphi : \mathfrak{A} \to \mathfrak{A}'$ is a bijective unital map which satisfies

$$\varphi(q_{3*}(\xi, a, b)) = q_{3*}(\varphi(\xi), \varphi(a), \varphi(b))$$

for all $a, b \in \mathfrak{A}$ and $\xi \in \{1_{\mathfrak{A}}, e_1, e_2\}$. Then φ is $*$-additive.

It is easy to see that any prime alternative $*$-algebra over ground field of characteristic different $2, 3$ satisfies $(\spadesuit)$ hence we have

Corollary 1.1. *Let $\mathfrak{A}$ be prime alternative $*$-algebra and $\mathfrak{A}'$ an alternative $*$-algebra with identities $1_{\mathfrak{A}}$ and $1_{\mathfrak{A}'}$, respectively, and e_1 and $e_2 = 1_{\mathfrak{A}} - e_1$ nontrivial symmetric idempotents in $\mathfrak{A}$. Suppose that $\varphi : \mathfrak{A} \to \mathfrak{A}'$ is a bijective unital map which satisfies*

$$\varphi(q_{3*}(\xi, a, b)) = q_{3*}(\varphi(\xi), \varphi(a), \varphi(b))$$

for all $a, b \in \mathfrak{A}$ and $\xi \in \{1_{\mathfrak{A}}, e_1, e_2\}$. Then φ is $*$-additive.

The following lemmas have the same hypotheses as the Theorem 1.1 and we need them to prove the $*$-additivity of φ.

Lemma 1.1. $\varphi(0) = 0$.

Proof. Since φ is surjective, there exists $x \in \mathfrak{A}$ such that $\varphi(x) = 0$. Then

$$0 = q_{3*}(\varphi(e_1), \varphi(e_2), \varphi(x)) = \varphi(q_{3*}(e_1, e_2, x)) = \varphi(0).$$

Therefore, $\varphi(0) = 0$. □

Lemma 1.2. *For any $a_{11} \in \mathfrak{A}_{11}$ and $b_{22} \in \mathfrak{A}_{22}$, we have*

$$\varphi(a_{11} + b_{22}) = \varphi(a_{11}) + \varphi(b_{22}).$$

Proof. Since φ is surjective, given $\varphi(a_{11}) + \varphi(b_{22}) \in \mathfrak{A}'$ there exists $t \in \mathfrak{A}$ such that $\varphi(t) = \varphi(a_{11}) + \varphi(b_{22})$, with $t = t_{11} + t_{12} + t_{21} + t_{22}$. Now, by Proposition 0.2

$$\varphi(q_{3*}(e_i, e_i, t)) = \varphi(q_{3*}(e_i, e_i, a_{11})) + \varphi(q_{3*}(e_i, e_i, b_{22})),$$

with $i = 1, 2$. It follows that

$$\varphi(2(e_i t + t e_i)) = \varphi(2(e_i a_{11} + a_{11} e_i)) + \varphi(2(e_i b_{22} + b_{22} e_i)).$$

Using the injectivity of φ we obtain $2(2t_{11} + t_{12} + t_{21}) = 2(2a_{11})$ and $2(2t_{22} + t_{12} + t_{21}) = 2(2b_{22})$. Then $t_{11} = a_{11}$, $t_{22} = b_{22}$ and $t_{12} = t_{21} = 0$. □

Lemma 1.3. *For any* $a_{12} \in \mathfrak{A}_{12}$ *and* $b_{21} \in \mathfrak{A}_{21}$*, we have* $\varphi(a_{12}+b_{21}) = \varphi(a_{12})+\varphi(b_{21})$*.*

Proof. Since φ is surjective, given $\varphi(a_{12}) + \varphi(b_{21}) \in \mathfrak{A}'$ there exists $t \in \mathfrak{A}$ such that $\varphi(t) = \varphi(a_{12}) + \varphi(b_{21})$, with $t = t_{11} + t_{12} + t_{21} + t_{22}$. Now, by Proposition 0.2

$$\begin{aligned}\varphi\left(q_{3*}\left(e_1, \frac{1}{2^2}e_1, t\right)\right) &= \varphi\left(q_{3*}\left(e_1, \frac{1}{2^2}e_1, a_{12}\right)\right) \\ &+ \varphi\left(q_{3*}\left(e_1, \frac{1}{2^2}e_1, b_{21}\right)\right) \\ &= \varphi(e_1a_{12} + a_{12}e_1) + \varphi(e_1b_{21} + b_{21}e_1) \\ &= \varphi(a_{12}) + \varphi(b_{21}) = \varphi(t).\end{aligned}$$

Since φ is injective, $e_1t + te_1 = t$, that is, $2t_{11} + t_{12} + t_{21} = t_{11} + t_{12} + t_{21} + t_{22}$. Then $t_{11} = t_{22} = 0$.

Now, observe that, for $c_{12} \in \mathfrak{A}_{12}$, $q_{3*}\left(\frac{1}{2}e_2, t, c_{12}\right) = t_{21}c_{12} + t_{12}c_{12} + c_{12}t_{21}^* + c_{12}t_{12}^*$. Thus, $q_{3*}\left(\frac{1}{2}e_2, a_{12}, c_{12}\right) = a_{12}c_{12}+c_{12}a_{12}^* \in \mathfrak{A}_{11}+\mathfrak{A}_{21}$ and $q_{3*}\left(\frac{1}{2}e_2, b_{21}, c_{12}\right) = b_{21}c_{12} + c_{12}b_{21}^* \in \mathfrak{A}_{21} + \mathfrak{A}_{22}$. Moreover,

$$q_{3*}\left(\frac{1}{2}e_2, q_{3*}\left(\frac{1}{2}e_2, t, c_{12}\right), e_2\right) = 2(t_{21}c_{12} + (t_{21}c_{12})^*),$$

thus

$$\begin{aligned}&\varphi\left(q_{3*}\left(\frac{1}{2}e_2, q_{3*}\left(\frac{1}{2}e_2, t, c_{12}\right), e_2\right)\right) \\ &= \varphi\left(q_{3*}\left(\frac{1}{2}e_2, q_{3*}\left(\frac{1}{2}e_2, a_{12}, c_{12}\right), e_2\right)\right) \\ &\quad + \varphi\left(q_{3*}\left(\frac{1}{2}e_2, q_{3*}\left(\frac{1}{2}e_2, b_{21}, c_{12}\right), e_2\right)\right) \\ &= \varphi(0) + \varphi(2(b_{21}c_{12} + (b_{21}c_{12})^*)) = \varphi(2(b_{21}c_{12} + (b_{21}c_{12})^*)).\end{aligned}$$

Therefore, $(t_{21}-b_{21})c_{12}+c_{12}^*(t_{21}^*-b_{21}^*) = 0$, for all $c_{12} \in \mathfrak{A}_{12}$. Consider $ic_{12} \in \mathfrak{A}_{12}$, we have $(t_{21} - b_{21})c_{12} - c_{12}^*(t_{21}^* - b_{21}^*) = 0$. Thus, $(t_{21} - b_{21})c_{12} = 0$ implies $(t_{21} - b_{21})(\mathfrak{A} e_2) = 0$. Using (♠), we obtain $t_{21} - b_{21} = 0$, that is, $t_{21} = b_{21}$. Note that, with similar calculations, if we replace c_{12} by c_{21} and e_1 by e_2, obtain that $t_{12} = a_{12}$. □

Lemma 1.4. *For any* $a_{11} \in \mathfrak{A}_{11}$*,* $b_{12} \in \mathfrak{A}_{12}$*,* $c_{21} \in \mathfrak{A}_{21}$ *and* $d_{22} \in \mathfrak{A}_{22}$ *we have*

$$\varphi(a_{11} + b_{12} + c_{21}) = \varphi(a_{11}) + \varphi(b_{12}) + \varphi(c_{21})$$

and

$$\varphi(b_{12} + c_{21} + d_{22}) = \varphi(b_{12}) + \varphi(c_{21}) + \varphi(d_{22}).$$

Proof. Since φ is surjective, given $\varphi(a_{11})+\varphi(b_{12})+\varphi(c_{21}) \in \mathfrak{A}'$ there exists $T \in \mathfrak{A}$ such that $\varphi(t) = \varphi(a_{11}) + \varphi(b_{12}) + \varphi(c_{21})$, with $t = t_{11} + t_{12} + t_{21} + t_{22}$. Now, observing that $q_{3*}(e_2, e_2, a_{11}) = 0$ and using Proposition 0.2 and Lemma 1.3, we obtain

$$\begin{aligned}\varphi(q_{3*}(e_2, e_2, t)) &= \varphi(q_{3*}(e_2, e_2, a_{11})) + \varphi(q_{3*}(e_2, e_2, b_{12})) \\ &\quad + \varphi(q_{3*}(e_2, e_2, c_{21})) \\ &= \varphi(q_{3*}(e_2, e_2, b_{12}) + q_{3*}(e_2, e_2, c_{21})).\end{aligned}$$

By injectivity of φ we have $q_{3*}(e_2, e_2, t) = q_{3*}(e_2, e_2, b_{12}) + q_{3*}(e_2, e_2, c_{21})$, that is, $2t_{22} + t_{12} + t_{21} = b_{12} + c_{21}$. Therefore, $t_{22} = 0$, $t_{12} = b_{12}$ and $t_{21} = c_{21}$. Again, observing that

$$q_{3*}(1_{\mathfrak{A}}, e_1 - e_2, b_{12}) = q_{3*}(1_{\mathfrak{A}}, e_1 - e_2, c_{21}) = 0$$

and using Proposition 0.2, we obtain

$$\begin{aligned}\varphi(q_{3*}(1_{\mathfrak{A}}, e_1 - e_2, t)) &= \varphi(q_{3*}(1_{\mathfrak{A}}, e_1 - e_2, a_{11})) \\ &\quad + \varphi(q_{3*}(1_{\mathfrak{A}}, e_1 - e_2, b_{12})) \\ &\quad + \varphi(q_{3*}(1_{\mathfrak{A}}, e_1 - e_2, c_{21})) \\ &= \varphi(q_{3*}(1_{\mathfrak{A}}, e_1 - e_2, a_{11})).\end{aligned}$$

By injectivity of φ we have $q_{3*}(1_{\mathfrak{A}}, e_1 - e_2, t) = q_{3*}(1_{\mathfrak{A}}, e_1 - e_2, a_{11})$, that is, $2t_{11} - 2t_{22} = 2a_{11}$. Therefore, $t_{11} = a_{11}$.

The other identity we obtain in a similar way. □

Lemma 1.5. *For any $a_{11} \in \mathfrak{A}_{11}$, $b_{12} \in \mathfrak{A}_{12}$, $c_{21} \in \mathfrak{A}_{21}$ and $d_{22} \in \mathfrak{A}_{22}$ we have*

$$\varphi(a_{11} + b_{12} + c_{21} + d_{22}) = \varphi(a_{11}) + \varphi(b_{12}) + \varphi(c_{21}) + \varphi(d_{22}).$$

Proof. Since φ is surjective, given $\varphi(a_{11}) + \varphi(b_{12}) + \varphi(c_{21}) + \varphi(d_{22}) \in \mathfrak{A}'$ there exists $T \in \mathfrak{A}$ such that $\varphi(t) = \varphi(a_{11}) + \varphi(b_{12}) + \varphi(c_{21}) + \varphi(d_{22})$, with $t = t_{11}+t_{12}+t_{21}+t_{22}$. Now, observing that $q_{3*}(e_1, e_1, e_{22}) = 0$ and using Proposition 0.2 and Lemma 1.4, we obtain

$$\begin{aligned}\varphi(q_{3*}(e_1, e_1, t)) &= \varphi(q_{3*}(e_1, e_1, a_{11})) + \varphi(q_{3*}(e_1, e_1, b_{12})) \\ &\quad + \varphi(q_{3*}(e_1, e_1, c_{21})) + \varphi(q_{3*}(e_1, e_1, d_{22})) \\ &= \varphi(q_{3*}(e_1, e_1, a_{11})) + \varphi(q_{3*}(e_1, e_1, b_{12})) \\ &\quad + \varphi(q_{3*}(e_1, e_1, c_{21})) \\ &= \varphi(q_{3*}(e_1, e_1, a_{11}) + q_{3*}(e_1, e_1, b_{12}) + q_{3*}(e_1, e_1, c_{21})).\end{aligned}$$

By injectivity of φ we have $q_{3*}(e_1, e_1, t) = q_{3*}(e_1, e_1, a_{11}) + q_{3*}(e_1, e_1, b_{12}) + q_{3*}(e_1, e_1, c_{21})$, that is, $2t_{11} + t_{12} + t_{21} = 2a_{11} + b_{12} + c_{21}$. Therefore, $t_{11} = a_{11}$, $t_{12} = b_{12}$ and $t_{21} = c_{21}$.

In a similar way, using $q_{3*}(e_2, e_2, t)$, we obtain $2t_{22}+t_{12}+t_{21} = 2d_{22}+b_{12}+c_{21}$ and then $t_{22} = d_{22}$. □

Lemma 1.6. *For every* $a_{12}, b_{12} \in \mathfrak{A}_{12}$ *and* $c_{21}, d_{21} \in \mathfrak{A}_{21}$ *we have*

$$\varphi(a_{12}b_{12} + a_{12}^*) = \varphi(a_{12}b_{12}) + \varphi(a_{12}^*)$$

and

$$\varphi(c_{21}d_{21} + c_{21}^*) = \varphi(c_{21}d_{21}) + \varphi(c_{21}^*).$$

Proof. Since $q_{3*}\left(1_{\mathfrak{A}}, a_{12}, \frac{1}{2}(e_2 + b_{12})\right) = a_{12} + a_{12}b_{12} + a_{12}^* + b_{12}a_{12}^*$ we get from Lemma 1.5 that

$$\begin{aligned}
\varphi(a_{12}) + &\varphi(a_{12}b_{12} + a_{12}^*) + \varphi(b_{12}a_{12}^*) \\
&= \varphi(a_{12} + a_{12}b_{12} + a_{12}^* + b_{12}a_{12}^*) \\
&= \varphi\left(q_{3*}\left(1_{\mathfrak{A}}, a_{12}, \frac{1}{2}(e_2 + b_{12})\right)\right) \\
&= \left(q_{3*}\left(\varphi(1_{\mathfrak{A}}), \varphi(a_{12}), \varphi(e_2) + \varphi\left(\frac{1}{2}b_{12}\right)\right)\right) \\
&= \varphi(a_{12}) + \varphi(a_{12}^*) + \varphi(a_{12}b_{12}) + \varphi(b_{12}a_{12}^*),
\end{aligned}$$

which implies $\varphi(a_{12}b_{12} + a_{12}^*) = \varphi(a_{12}b_{12}) + \varphi(a_{12}^*)$. Similarly, we prove the other case using the identity $q_{3*}\left(1_{\mathfrak{A}}, c_{21}, \frac{1}{2}(e_1 + d_{21})\right) = c_{21} + c_{21}d_{21} + c_{21}^* + d_{21}c_{21}^*$ □

Lemma 1.7. *For all* $a_{ij}, b_{ij} \in \mathfrak{A}_{ij}$, $1 \leq i \neq j \leq 2$, *we have*

$$\varphi(a_{ij} + b_{ij}) = \varphi(a_{ij}) + \varphi(b_{ij}).$$

Proof. Since $q_{3*}\left(1_{\mathfrak{A}}, \frac{1}{2}(e_i + a_{ij}), e_j + b_{ij}\right) = a_{ij} + b_{ij} + a_{ij}^* + a_{ij}b_{ji} + b_{ij}a_{ij}^*$ we get from Lemma 1.6 that

$$\begin{aligned}
&\varphi(a_{ij} + b_{ij}) + \varphi(a_{ij}^* + a_{ij}b_{ij}) + \varphi(b_{ij}a_{ij}^*) \\
&= \varphi\left(q_{3*}(1_{\mathfrak{A}}, \frac{1}{2}(e_i + a_{ij}), e_j + b_{ij})\right) \\
&= q_{3*}\left(\varphi(1_{\mathfrak{A}}), \varphi\left(\frac{1}{2}(e_i + a_{ij})\right), \varphi(e_j + b_{ij})\right) \\
&= q_{3*}\left(\varphi(1_{\mathfrak{A}}), \varphi(e_i)\varphi\left(\frac{1}{2}a_{ij}\right), \varphi(e_j) + \varphi(b_{ij})\right) \\
&= \varphi(b_{ij}) + \varphi(a_{ij} + a_{ij}^*) + \varphi(a_{ij}b_{ij} + b_{ij}a_{ij}^*) \\
&= \varphi(b_{ij}) + \varphi(a_{ij}) + \varphi(a_{ij}^*) + \varphi(a_{ij}b_{ij}) + \varphi(b_{ij}a_{ij}^*),
\end{aligned}$$

which implies $\varphi(a_{ij} + b_{ij}) = \varphi(b_{ij}) + \varphi(a_{ij})$. □

Lemma 1.8. *For all* $a_{ii}, b_{ii} \in \mathfrak{A}_{ii}$, *we have* $\varphi(a_{ii} + b_{ii}) = \varphi(a_{ii}) + \varphi(b_{ii})$ *for* $i \in \{1, 2\}$.

Proof. Since φ is surjective, given $\varphi(a_{ii})+\varphi(b_{ii}) \in \mathfrak{A}'$, $i=1,2$, there exists $t \in \mathfrak{A}$ such that $\varphi(t) = \varphi(a_{ii}) + \varphi(b_{ii})$, with $t = t_{11} + t_{12} + t_{21} + t_{22}$. By Proposition 0.2, for $j \neq i$,

$$\varphi(q_{3*}(e_j, e_j, t)) = \varphi(q_{3*}(e_j, e_j, a_{ii})) + \varphi(q_{3*}(e_j, e_j, b_{ii})) = 0.$$

Then, $t_{ij} = t_{ji} = t_{jj} = 0$. We just have to show that $t_{ii} = a_{ii} + b_{ii}$. Given $c_{ij} \in \mathfrak{A}_{ij}$, using Lemma 1.7 and Proposition 0.2 we have

$$\begin{aligned}\varphi(q_{3*}(e_i, t, c_{ij})) &= \varphi(q_{3*}(e_i, a_{ii}, c_{ij})) + \varphi(q_{3*}(e_i, b_{ii}, c_{ij}))\\ &= \varphi(q_{3*}(e_i, a_{ii}, c_{ij}) + q_{3*}(e_i, b_{ii}, c_{ij})).\end{aligned}$$

By injectivity of φ we obtain

$$q_{3*}(e_i, t, c_{ij}) = q_{3*}(e_i, a_{ii}, c_{ij}) + q_{3*}(e_i, b_{ii}, c_{ij}),$$

that is,

$$(t_{ii} - a_{ii} - b_{ii})c_{ij} = 0.$$

Finally, by (♠) we conclude that $t_{ii} = a_{ii} + b_{ii}$. □

Now we are able to show that φ preserves $*$-addition.

Using Lemmas 1.5, 1.7, 1.8 it is easy to see that φ is additive. Besides, on the one hand, since φ is additive it follows that

$$\varphi(a + a^*) = \varphi(a) + \varphi(a^*).$$

On the other hand, by additivity of φ,

$$\begin{aligned}2\varphi(a + a^*) &= \varphi(2(a + a^*)) = \varphi(q_{3*}(1_{\mathfrak{A}}, a, 1_{\mathfrak{A}}))\\ &= q_{3*}(1_{\mathfrak{A}'}, \varphi(a), 1_{\mathfrak{A}'}) = 2(\varphi(a) + \varphi(a)^*).\end{aligned}$$

Therefore $\varphi(a^*) = \varphi(a)^*$ and Theorem 1.1 is proved.

Now we focus our attention on investigate the problem of when φ is a $*$-ring isomorphism. We prove the following result:

Theorem 1.2. *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two alternative $*$-algebras with identities $1_{\mathfrak{A}}$ and $1_{\mathfrak{A}'}$, respectively, and e_1 and $e_2 = 1_{\mathfrak{A}} - e_1$ nontrivial symmetric idempotents in $\mathfrak{A}$. Suppose that $\mathfrak{A}$ and $\mathfrak{A}'$ satisfy:*

$$(\spadesuit) \qquad x(\mathfrak{A} e_i) = \{0\} \quad \textit{implies} \quad x = 0$$

and

$$(\clubsuit) \qquad y(\mathfrak{A}'\varphi(e_i)) = \{0\} \quad \textit{implies} \quad y = 0.$$

If $\varphi : \mathfrak{A} \to \mathfrak{A}'$ is a bijective unital map which satisfies

$$\varphi(q_{3*}(\xi, a, b)) = q_{3*}(\varphi(\xi), \varphi(a), \varphi(b)),$$

for all $a, b \in \mathfrak{A}$ and $e \in \{1_{\mathfrak{A}}, e_1, e_2\}$ then φ is $*$-ring isomorphism.

Again it is easy to see that prime alternative algebras over ground field of characteristic different $2,3$ satisfy (♠) and (♣) hence we get

Corollary 1.2. *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two prime alternative $*$-algebras with identities $1_{\mathfrak{A}}$ and $1_{\mathfrak{A}'}$, respectively, and e_1 and $e_2 = 1_{\mathfrak{A}} - e_1$ nontrivial symmetric idempotents in $\mathfrak{A}$. If $\varphi : \mathfrak{A} \to \mathfrak{A}'$ is a bijective unital map which satisfies*

$$\varphi(q_{3*}(\xi, a, b)) = q_{3*}(\varphi(\xi), \varphi(a), \varphi(b))$$

for all $a, b \in \mathfrak{A}$ and $\xi \in \{1_{\mathfrak{A}}, e_1, e_2\}$, then φ is $*$-ring isomorphism.

Now since φ is $*$-additive, by Theorem 1.1, it is enough to verify that $\varphi(ab) = \varphi(a)\varphi(b)$. Firstly, let us prove the following lemmas:

Lemma 1.9. *$f_i = \varphi(e_i)$ is an idempotent in $\mathfrak{A}'$, with $i \in \{1, 2\}$.*

Proof. By additivity of φ we have

$$\begin{aligned} 2^2 f_i = 2^2\varphi(e_i) = \varphi(2^2 e_i) &= \varphi(q_{3*}(1_{\mathfrak{A}}, e_i, e_i)) \\ &= q_{3*}(1_{\mathfrak{A}'}, \varphi(e_i), \varphi(e_i)) = 2^2\varphi(e_i)\varphi(e_i) = 2^2 f_i f_i. \end{aligned}$$

Therefore, $f_i f_i = f_i$. □

Lemma 1.10. *If $x \in \mathfrak{A}_{ij}$ then $\varphi(x) \in \mathfrak{A}'_{ij}$.*

Proof. Firstly, given $x \in \mathfrak{A}_{ij}$, with $i \neq j$, we observe that

$$\begin{aligned} 2\varphi(x) = \varphi(2x) &= \varphi(q_{3*}(e_j, e_j, x)) = q_{3*}(\varphi(e_j), \varphi(e_j), \varphi(x)) \\ &= 2(f_j\varphi(x) + \varphi(x)f_j), \end{aligned}$$

that is, since φ is unital we have $f_i\varphi(x)f_i = f_j\varphi(x)f_j = 0$. Even more,

$$\begin{aligned} 0 &= \varphi(q_{3*}(e_i, x, e_i)) = q_{3*}(f_i, \varphi(x), f_i) \\ &= f_i\varphi(x)f_i + \varphi(x)f_i + f_i\varphi(x)^* f_i + f_i\varphi(x)^*. \end{aligned}$$

Multiplying left side by f_j we obtain $f_j\varphi(x)f_i = 0$. Therefore, $\varphi(x) \in \mathfrak{A}'_{ij}$. In a similar way, if $x \in \mathfrak{A}_{ii}$ we conclude that $\varphi(x) \in \mathfrak{A}'_{ii}$. □

Lemma 1.11. *If $a_{ii} \in \mathfrak{A}_{ii}$ and $b_{ij} \in \mathfrak{A}_{ij}$, with $i \neq j$, then $\varphi(a_{ii}b_{ij}) = \varphi(a_{ii})\varphi(b_{ij})$.*

Proof. Let $a_{ii} \in \mathfrak{A}_{ii}$ and $b_{ij} \in \mathfrak{A}_{ij}$, with $i \neq j$. Then, by Lemma 1.10 and additivity of φ,

$$\begin{aligned} 2\varphi(a_{ii}b_{ij}) &= \varphi(2a_{ii}b_{ij}) = \varphi(q_{3*}(e_i, a_{ii}, b_{ij})) \\ &= q_{3*}(\varphi(e_i), \varphi(a_{ii}), \varphi(b_{ij})) = 2\varphi(a_{ii})\varphi(b_{ij}). \end{aligned}$$

Therefore, $\varphi(a_{ii}b_{ij}) = \varphi(a_{ii})\varphi(b_{ij})$. □

Lemma 1.12. *If* $a_{ii}, b_{ii} \in \mathfrak{A}_{ii}$ *then* $\varphi(a_{ii}b_{ii}) = \varphi(a_{ii})\varphi(b_{ii})$.

Proof. Let x be an element of $\mathfrak{A}_{ij}$, with $i \neq j$. Using Lemmas 1.10, 1.11 and the flexibility of alternative algebras we obtain

$$\begin{aligned}\varphi(a_{ii}b_{ii})\varphi(x) = &\quad \varphi((a_{ii}b_{ii})x) = \varphi(a_{ii}(b_{ii}x)) = \varphi(a_{ii})\varphi(b_{ii}x) \\ = &\quad \varphi(a_{ii})(\varphi(b_{ii})\varphi(x)) = (\varphi(a_{ii})\varphi(b_{ii}))\varphi(x)\end{aligned}$$

that is,

$$(\varphi(a_{ii}b_{ii}) - \varphi(a_{ii})\varphi(b_{ii}))\varphi(x) = 0.$$

Now, by Lemma 1.10, since $\varphi(x) \in \mathfrak{A}'_{ij}$ and $\varphi(a_{ii}b_{ii}) - \varphi(a_{ii})\varphi(b_{ii}) \in \mathfrak{A}'_{ii}$, we have

$$(\varphi(a_{ii}b_{ii}) - \varphi(a_{ii})\varphi(b_{ii}))(\mathfrak{A}'\varphi(e_j)) = \{0\}.$$

Finally, (♣) ensures that $\varphi(a_{ii}b_{ii}) = \varphi(a_{ii})\varphi(b_{ii})$. □

Lemma 1.13. *If* $a_{ij} \in \mathfrak{A}_{ij}$ *and* $b_{ji} \in \mathfrak{A}_{ji}$, *with* $i \neq j$, *then* $\varphi(a_{ij}b_{ji}) = \varphi(a_{ij})\varphi(b_{ji})$.

Proof. Let $a_{ij} \in \mathfrak{A}_{ij}$ and $b_{ji} \in \mathfrak{A}_{ji}$, with $i \neq j$. Then, by Lemma 1.10 and additivity of φ,

$$\begin{aligned}\varphi(a_{ij}b_{ji}) &= \varphi(a_{ij}b_{ji}) = \varphi(q_{3*}(e_i, a_{ij}, b_{ji})) \\ &= q_{3*}(\varphi(e_i), \varphi(a_{ij}), \varphi(b_{ji})) = \varphi(a_{ij})\varphi(b_{ji}).\end{aligned}$$

Therefore, $\varphi(a_{ij}b_{ji}) = \varphi(a_{ij})\varphi(b_{ji})$.

□

Lemma 1.14. *If* $a_{ij} \in \mathfrak{A}_{ij}$ *and* $b_{jj} \in \mathfrak{A}_{jj}$, *with* $i \neq j$, *then* $\varphi(a_{ij}b_{jj}) = \varphi(a_{ij})\varphi(b_{jj})$

Proof. Let x be an element of $\mathfrak{A}_{ji}$, with $i \neq j$. Using Lemmas 1.11 and 1.13 we obtain

$$\varphi(a_{ij}b_{jj})\varphi(x) = \varphi(a_{ij}b_{jj}x) = \varphi(a_{ij})\varphi(b_{jj}x) = \varphi(a_{ij})\varphi(b_{jj})\varphi(x),$$

that is,

$$(\varphi(a_{ij}b_{jj}) - \varphi(a_{ij})\varphi(b_{jj}))\varphi(x) = 0.$$

Now, by Lemma 1.10, since $\varphi(x) \in \mathfrak{A}'_{ji}$ and $\varphi(a_{ij}b_{jj}) - \varphi(a_{ij})\varphi(b_{jj}) \in \mathfrak{A}'_{ij}$, we have

$$(\varphi(a_{ij}b_{jj}) - \varphi(a_{ij})\varphi(b_{jj})(\mathfrak{A}'\varphi(e_i)) = \{0\}.$$

Finally, (♣) ensures that $\varphi(a_{ij}b_{jj}) = \varphi(a_{ij})\varphi(b_{jj})$. □

As in general $a_{ij}b_{ij} \neq 0$ for $i \neq j$ in alternative $*$-algebras, we need to prove.

Lemma 1.15. *If* $a_{ij} \in \mathfrak{A}_{ij}$ *and* $b_{ij} \in \mathfrak{A}_{ij}$, *with* $i \neq j$, *then* $\varphi(a_{ij}b_{ij}) = \varphi(a_{ij})\varphi(b_{ij})$

Proof. Let $a_{ij}, b_{ij} \in \mathfrak{A}_{ij}$, with $i \neq j$. Then, by Lemma 1.10 and additivity of φ,

$$\begin{aligned} 2\varphi(a_{ij}b_{ij}) &= \varphi(2a_{ij}b_{ij}) = \varphi(q_{3*}(e_i, a_{ij}, b_{ij})) \\ &= q_{3*}(\varphi(e_i), \varphi(a_{ij}), \varphi(b_{ij})) = 2\varphi(a_{ij})\varphi(b_{ij}). \end{aligned}$$

Therefore, $\varphi(a_{ij}b_{ij}) = \varphi(a_{ij})\varphi(b_{ij})$. □

Thus, by additivity of φ, proved in the Theorem 1.1, and the lemmas above we conclude that $\varphi(ab) = \varphi(a)\varphi(b)$. Therefore φ is a $*$-ring isomorphism.

2 Application

A complete normed alternative complex $*$-algebra $\mathfrak{A}$ is called of alternative C^*-algebra if it satisfies the condition: $\|a^*a\| = \|a\|^2$, for all elements $a \in \mathfrak{A}$. Alternative C^*-algebras are non-associative generalizations of C^*-algebras and appear in various areas in Mathematics (see more details in the references [14] and [15]). An alternative C^*-algebra A is called of alternative W^*-algebra if it is a dual Banach space and a prime alternative W^*-algebra is called alternative W^*-factor. It is well known that non-zero alternative W^*-algebras are unital.

Theorem 2.1. *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two alternative W^*-factors. If a map $\varphi : \mathfrak{A} \to \mathfrak{A}'$ satisfies*

$$\varphi(q_{3*}(\xi, a, b)) = q_{3*}(\varphi(\xi), \varphi(a), \varphi(b)),$$

for all $a, b \in \mathfrak{A}$ and $\xi \in \{1_{\mathfrak{A}}, e_1, e_2\}$, then φ is an additive $*$-ring isomorphism.

Corollary 2.1. *Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two alternative W^*-factors. In this case, φ is a nonlinear $*$-Jordan triple map if and only if φ is an additive $*$-ring isomorphism.*

3 Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Addresses of Tatiane Cardoso Batista Flores
Federal University of Technology,
Avenida Brasil, 4232,
85884-000, Medianeira, Brazil
E-mail address: tatianebatista@utfpr.edu.br

Addresses of Ruth Nascimento Ferreira
Federal University of Technology,
Avenida Professora Laura Pacheco Bastos, 800,
85053-510, Guarapuava, Brazil
E-mail address: ruth.nas@gmail.com

Addresses of Bruno Leonardo Macedo Ferreira
Federal University of Technology-Paraná-Brazil,
Avenida Professora Laura Pacheco Bastos, 800,
85053-510, Guarapuava, Brazil and
University of São Paulo,
Rua do Matão, 1010 - 05508-090 - São Paulo, Brazil and
University of ABC,
Av. dos Estados, 5001 - 09280-560 - Santo André, Brazil
E-mail address: brunoferreira@utfpr.edu.br or brunolmfalg@gmail.com