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# On Right-Nilalgebras of Index 4

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We first present a solution to a conjecture of [2] in the positive. We prove that if  $\mathfrak{A}$  is a commutative nonassociative algebra over a field of characteristic  $\neq 2, 3$ , satisfying the identity  $x(x(xx)) = 0$ , then  $L_{a^{t_1}}L_{a^{t_2}} \cdots L_{a^{t_s}} \equiv 0$  if  $t_1 + t_2 + \cdots + t_s \geq 10$ , where  $a \in \mathfrak{A}$ . For every  $b \in \mathfrak{A}$ , the linear operator  $L_b$  of  $\mathfrak{A}$  which is determined by  $b$  is defined by  $L_b: y \rightarrow by$  for all  $y \in \mathfrak{A}$ . In the last section we prove that every commutative non-associative nilalgebra of dimension  $\leq 7$ , over a field of characteristic zero or sufficiently large is solvable.

*Keywords:* solvable; commutative; nilalgebra;

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## 1 Introduction

Throughout this paper the term *algebra* is understood to be a commutative not necessarily associative algebra. We will use the notations and terminology of [7]. Let  $\mathfrak{A}$  be an (commutative nonassociative) algebra over a field  $F$ . We define inductively the following powers,  $\mathfrak{A}^1 = \mathfrak{A}$  and  $\mathfrak{A}^s = \sum_{i+j=s} \mathfrak{A}^i \mathfrak{A}^j$  for all positive integers  $s \geq 2$ . We shall say that  $\mathfrak{A}$  is *nilpotent* in case there is an integer  $s$  such that  $\mathfrak{A}^s = (0)$ . The algebra  $\mathfrak{A}$  is called *nilalgebra* in case the subalgebra  $\text{alg}(a)$  of  $\mathfrak{A}$  generated by  $a$  is nilpotent, for all  $a \in \mathfrak{A}$ . Therefore  $\mathfrak{A}$  is nilalgebra if and only if for every  $a \in \mathfrak{A}$  there exists an integer  $t$  such that every product of at least  $t$  factors each of them equal to  $a$ , in whatever association, vanishes. The (*principal*) *powers* of an element  $a$  in  $\mathfrak{A}$  are defined recursively by  $a^1 = a$  and  $a^{i+1} = aa^i$  for

all integers  $i \geq 1$ . The algebra  $\mathfrak{A}$  is called *right-nilalgebra* if for every  $a$  in  $\mathfrak{A}$  there exists an integer  $k = k(a)$  such that  $a^k = 0$ . The smallest positive integer  $k$  which this property is the *index*. Obviously, every nilalgebra is right-nilalgebra. For any element  $a$  in  $\mathfrak{A}$ , the linear mapping  $L_a$  of  $\mathfrak{A}$  defined by  $x \rightarrow ax$  is called *multiplication operator* of  $\mathfrak{A}$ . An *Engel algebra* is an algebra in which every multiplication operator is nilpotent in the sense that for every  $a \in \mathfrak{A}$  there exists a positive integer  $j$  such that  $L_a^j = 0$ .

An important question is that of the existence of simple nilalgebras in the class of finite-dimensional algebras. In [7] we proved that every nilalgebra  $\mathfrak{A}$  of dimension  $\leq 6$  over a field of characteristic  $\neq 2, 3, 5$  is solvable and hence  $\mathfrak{A}^2 \subsetneq \mathfrak{A}$ . For power-associative nilalgebras of dimension  $\leq 8$  over a field of characteristic  $\neq 2, 3, 5$ , we have shown in [4] that they are solvable, and hence there is no simple algebra (see also [5] and [7] for power-associative nilalgebras of dimension  $\leq 7$ ) in this subclass.

We show now the process of linearization of identities, which is an important tool in the theory of varieties of algebras. See [8], [11] and [12] for more information. Let  $P$  be the free commutative nonassociative polynomial ring in two generators  $x$  and  $y$  over a field  $F$ . For every  $\alpha_1, \dots, \alpha_r \in P$ , the *operator linearization*  $\delta[\alpha_1, \dots, \alpha_r]$  can be defined as follows: if  $p(x, y)$  is a monomial in  $P$ , then  $\delta[\alpha_1, \dots, \alpha_r]p(x, y)$  is obtained by making all the possible replacements of  $r$  of the  $k$  identical arguments  $x$  by  $\alpha_1, \dots, \alpha_r$  and summing the resulting terms if  $x$ -degree of  $p(x, y)$  is  $\geq r$ , and is equal to zero in other cases. Some examples of this operator are

$$\begin{aligned}\delta[y](x^2(xy)) &= 2(xy)^2 + x^2y^2 \\ \delta[x^2, y](x^2) &= 2x^2y, \quad \delta[y, xy^2, x](x^2) = 0.\end{aligned}$$

For simplicity,  $\delta[\alpha : r]$  will denote  $\delta[\alpha_1, \dots, \alpha_r]$ , where  $\alpha_1 = \dots = \alpha_r = \alpha$ . We observe that if  $p(x)$  is a polynomial in  $P$ , then  $p(x+y) = p(x) + \sum_{j=1}^{\infty} \delta[y : j]p(x)$ , where  $\delta[y : j]p(x)$  is the sum of all the terms of  $p(x+y)$  which have degree  $j$  with respect to  $y$ .

**Lemma 1** ([12]). *Let  $p(x, y)$  be a commutative nonassociative polynomial of  $x$ -degree  $\leq n$ . If  $F$  is a field of characteristic either zero or  $\geq n$ , and the  $F$ -algebra  $\mathfrak{A}$  satisfies the identity  $p(x, y)$ , then  $\mathfrak{A}$  satisfies all linearizations of  $p(x, y)$ .*

## 2 Right-nilalgebras of index 4

Throughout this section  $F$  is a field of characteristic different from 2 or 3 and all the algebras are over  $F$ . We will study right-nilalgebras of index  $\leq 4$ , that is the

variety  $\mathcal{V}$  of algebras over the field  $F$  satisfying the identity

$$x^4 = 0. \quad (1)$$

Let  $\mathfrak{A}$  be an algebra in  $\mathcal{V}$ . For simplicity, we will denote by  $L$  and  $U$  the multiplication operators,  $L_x$  and  $L_{x^2}$  respectively, where  $x$  is an element in  $\mathfrak{A}$ . The following known result is a basic tool in our investigation. See [2] and [3].

**Lemma 2.** *Let  $\mathfrak{A}$  be a commutative right-nilalgebra of index 4. Then  $\mathfrak{A}$  satisfies the identities*

$$x^2x^3 = -x(x^2x^2), \quad x^3x^3 = (x^2)^3 = x(x(x^2x^2)), \quad (2)$$

and  $p(x) = 0$ , for every monomial  $p(x)$  with  $x$ -degree  $\geq 7$ . Furthermore, we have

$$L_{x^3} = -LU - 2L^3, \quad (3)$$

$$L_{x^2x^2} = -U^2 - 2UL^2 - 2LUL + 4L^4, \quad (4)$$

$$L_{x(x^2x^2)} = -LU^2 - 2LUL^2 - 2L^2UL - 4L^3U - 12L^5, \quad (5)$$

$$L_{x(x(x^2x^2))} = 2L^2U^2 + 4L^2UL^2 + 4L^4U + 8L^6, \quad (6)$$

and also

Table i: Multiplication identities of degree 5

|        | $ULU$ | $LU^2$ | $UL^3$ | $LUL^2$ | $L^2UL$ | $L^3U$ | $L^5$ |
|--------|-------|--------|--------|---------|---------|--------|-------|
| $U^2L$ | 0     | -1     | 2      | 0       | 0       | -2     | -8    |

and two identities of  $x$ -degree 6 which may be written as

Table ii: Multiplication identities of degree 6

|          | $UL^2U$ | $(LU)^2$ | $L^2U^2$ | $UL^4$ | $LUL^3$ | $L^2UL^2$ | $L^3UL$ | $L^4U$ | $L^6$ |
|----------|---------|----------|----------|--------|---------|-----------|---------|--------|-------|
| $U^3$    | -2      | -2       | 2        | -8     | -8      | 0         | -4      | 8      | 40    |
| $(UL)^2$ | -1      | -1       | 1        | -4     | -2      | 2         | 0       | 4      | 24    |

We note that, for example, Table i means that  $U^2L = -LU^2 + 2UL^3 - 2L^3U - 8L^5$ . From the identities (3-6) we get that for any  $a \in \mathfrak{A}$  the associative algebra  $\mathfrak{A}_a$  generated by all  $L_c$  with  $c \in \text{alg}(a)$  is in fact generated by  $L_a$  and  $L_{a^2}$ . Furthermore, every algebra in  $\mathcal{V}$  is a nilalgebra of index  $\leq 7$ .

We now pass to study homogeneous identities in  $\mathfrak{A}$  with  $x$ -degree  $\geq 7$  and  $y$ -degree 1. From the relation  $0 = \delta[y, x, x^3, x^3](x^4) = 2y(x(x^3x^3)) + 4y(x^3(x^2x^2)) + 2x(y(x^3x^3)) + 4x^3(y(xx^3)) + 4x(x^3(yx^3)) + 4x^3(x(yx^3)) + 4x^3(x^3(xy)) = 2x((x^3)^2y) +$

$4x(x^3(x^3y)) + 4x^3(x(x^3y)) + 4x^3(x^3(xy)) = 2[LL_{x^3x^3} + 2LL_{x^3}L_{x^3} + 2L_{x^3}LL_{x^3} + 2L_{x^3}L_{x^3}L](y)$  we have

$$L^3U^2 = -2L^3UL^2 - L^4UL - 5L^5U - 20L^7, \quad (7)$$

since we can use the reductions (3-6) and replace the occurrences of  $(UL)^2$ . Multiplying the identity of Table i by  $U$  from the left and first replacing the occurrences of  $U^3$  we get

$$ULU^2 = 2(UL^2UL - UL^3U - LUL^2U - L^2ULU + 2UL^5 - 2LUL^4 - 2L^2UL^3 - 3L^4UL - L^5U - 16L^7). \quad (8)$$

Next, we can reduce the relation  $0 = \delta[y, x, x^2, x^2x^2]x^4$  using the above identities. This yields

$$UL^5 = -LUL^4 + \frac{1}{2}L^2UL^3 + \frac{3}{4}L^4UL + \frac{3}{4}L^5U + 8L^7. \quad (9)$$

Now combining (8) and (9) we obtain  $ULU^2 = 2UL^2UL - 2UL^3U - 2LUL^2U - 2L^2ULU - 8LUL^4 - 2L^2UL^3 - 3L^4UL + L^5U$ . Thus, we have three identities of  $x$ -degree 7 and  $y$ -degree 1 which may be written as multiplication identities:

Table iii: Multiplication identities of degree 7

|          | $UL^2UL$ | $UL^3U$ | $LUL^2U$ | $L(LU)^2$ | $LUL^4$ | $L^2UL^3$ | $L^3UL^2$ | $L^4UL$ | $L^5U$ | $L^7$ |
|----------|----------|---------|----------|-----------|---------|-----------|-----------|---------|--------|-------|
| $L^3U^2$ | 0        | 0       | 0        | 0         | 0       | 0         | -2        | -1      | -5     | -20   |
| $UL^5$   | 0        | 0       | 0        | 0         | -1      | 1/2       | 0         | 3/4     | 3/4    | 8     |
| $ULU^2$  | 2        | -2      | -2       | -2        | -8      | -2        | 0         | -3      | 1      | 0     |

In an analogous way, using successively the identities

$$0 = \delta[y, x, x, x(x(x^2x^2))]x^4, \quad 0 = \delta[y, x^2, x^2, x^2x^2]x^4, \quad 0 = \delta[y, x, x^2, x(x^2x^2)]x^4,$$

multiplying the second identity of Table ii with the operator  $U$  from the left and replacing the occurrences of  $UUL$ , and finally using  $0 = \delta[y, x, x^3, x^2x^2]x^4$ , we obtain the following 5 multiplication identities:

Table iv: Multiplication identities of degree 8

|            | $UL^4U$ | $LUL^2UL$ | $LUL^3U$ | $L^2UL^2U$ | $L^2UL^4$ | $L^4UL^2$ | $L^5UL$ | $L^6U$ | $L^8$ |
|------------|---------|-----------|----------|------------|-----------|-----------|---------|--------|-------|
| $L^3ULU$   | 0       | 0         | 0        | 0          | 0         | -1/2      | -2      | -11/2  | -20   |
| $UL^2U^2$  | -4      | -2        | -2       | 0          | 2         | -5/2      | 13      | 31/2   | 32    |
| $(UL^2)^2$ | 0       | 1         | 0        | -1         | -12       | -11/4     | -7/2    | 25/4   | 36    |
| $UL^3UL$   | -1      | -1        | -1       | 0          | -4        | -11/2     | -3      | 9/2    | 0     |
| $L^3UL^3$  | 0       | 0         | 0        | 0          | 0         | -3/4      | -3/2    | -3/4   | -8    |

Now, relations  $0 = \delta[y, x, x^3, x(x^2x^2)]x^4$ ,  $0 = \delta[y, x, x^2, x(x(x^2x^2))x^4$ ,  $0 = \delta[y, x^2, x^2, (x^2x^2)]x^4$ ,  $0 = \delta[y, x, x^2x^2, x^2x^2]x^4$ ,  $0 = \delta[y, x^2, x^3, x^2x^2]x^4$ , and multiplying the relation determined by the last row of Table iii with the operator  $U$  from the left and first replacing the occurrences of  $UUL$ , imply the following 6 multiplication identities:

Table v: Multiplication identities of degree 9

|             | $LUL^4U$ | $(L^2U)^2L$ | $L^7U$ | $L^9$   |
|-------------|----------|-------------|--------|---------|
| $L^6UL$     | 0        | 0           | -7     | -48     |
| $L(L^2U)^2$ | 0        | 0           | -217   | -4510/3 |
| $UL^4UL$    | 1        | 0           | -587/2 | -6155/3 |
| $L^2UL^3U$  | 0        | 0           | 29/3   | 422/9   |
| $UL^2ULU$   | 0        | 0           | 1318/3 | 27988/9 |
| $L^5UL^2$   | 0        | 0           | -23    | -496/3  |

The author used a MAPLE language program to discover these identities. We now present a solution of a Conjecture of [2] in the positive. We see that for every  $a \in \mathfrak{A}$ , the associative algebra  $\mathfrak{A}_a$ , generated by the multiplication operators  $L_a$  and  $L_{a^2}$ , is nilpotent of index  $\leq 10$ .

**Theorem 1.** *Let  $\mathfrak{A}$  be an algebra over a field  $F$  of characteristic  $\neq 2, 3$ , satisfying  $x^4 = 0$ . Then every monomial in  $P$  of  $x$ -degree  $\geq 10$  and  $y$ -degree 1 is an identity in  $\mathfrak{A}$ . In particular,  $L_a^{10} = 0$  for all  $a \in \mathfrak{A}$ .*

*Proof.* First we shall prove that every monomial of  $x$ -degree 10 and  $y$ -degree 1 is an identity in  $\mathfrak{A}$ . Multiplying the operators in the first line of Table v with  $L$  from the left and from the right, and the operators in the first line of Table iv with  $U$  from the left and from the right and next using reductions from Tables i-v we see that we only need to prove that  $L^2UL^4U = 0$ ,  $L^8U = 0$  and  $L^{10} = 0$  are multiplication identities in  $\mathfrak{A}$ . Now, for any  $x$  in  $\mathfrak{A}$  we have

$$\begin{aligned} L^7UL &= L(L^6UL) = -7L^8U - 48L^{10}, \\ L^6UL^2 &= (L^6UL)L = -7L^7UL - 48L^{10} = 49L^8U + 288L^{10}, \\ L^6UL^2 &= L(L^5UL^2) = -23L^8U - 496/3L^{10}. \end{aligned}$$

Therefore

$$27L^8U + 170L^{10} = 0. \tag{10}$$

Now,

$$\begin{aligned} L^5UL^3 &= (L^5UL^2)L = -23L^7UL - 496/3L^{10} = 161L^8U + 2816/3L^{10}, \\ L^5UL^3 &= L^2(L^3UL^3) = -3/4L^6UL^2 - 3/2L^7UL - 3/4L^8U - 8L^{10} \\ &= -27L^8U - 152L^{10}, \end{aligned}$$

and hence

$$141L^8U + 818L^{10} = 0. \quad (11)$$

Next

$$\begin{aligned} L^3UL^3U &= L(L^2UL^3U) = 29/3L^8U + 422/9L^{10}, \\ L^3UL^3U &= (L^3UL^3)U = -3/4L^4UL^2U - 3/2L^5ULU - 3/4L^6UU - 8L^8U \\ &= -3/4L(L^3UL^2U) - 3/2L^2(L^3ULU) - 3/4L^3(L^3U^2) - 8L^{10} \\ &= 9/4L^6UL^2 + 15/4L^7UL + 667/4L^8U + 2345/2L^{10} \\ &= 1003L^8U + 3281/2L^{10}, \end{aligned}$$

so that

$$17880L^8U + 28685L^{10} = 0. \quad (12)$$

Combining (10-12) we obtain that  $L^8U = 0$  and  $L^{10} = 0$ . Now, we have by Table v that  $0 = (L^2UL^3U)L = L^2(UL^3UL) = -L^2UL^4U - L^3UL^2UL - L^3UL^3U - 4L^4UL^4 - 11/2L^6UL^2 - 3L^7UL + 9/2L^8U = -L^2UL^4U - (L^3UL^2U)L - 4L(L^3UL^3)L = -L^2UL^4U$ . Therefore, we have  $L^2UL^4U = 0$ .

In an analogous way, we can see that every monomial of  $x$ -degree 11 and  $y$ -degree 1 is an identity in  $\mathfrak{A}$ . This proves the theorem.  $\square$

Now we shall investigate two subvarieties of  $\mathcal{V}$ . We start in Subsection 2.2 with the class of all nilalgebras in  $\mathcal{V}$  of index  $\leq 5$  and next in Subsection 2.3 we study the multiplication identities of the variety of all the nilalgebras in  $\mathcal{V}$  of index  $\leq 6$ . We shall use these results in Section 3 where we prove that every nilalgebra of dimension  $\leq 7$  over a field of characteristic zero is solvable.

## 2.1 The identity $x((xx)(xx))=0$

We will now consider the class of all algebras in  $\mathcal{V}$  satisfying the identity  $x(x^2x^2) = 0$ . First, linearization  $\delta[y]\{x(x^2)^2\}$  implies

$$L_{x^2x^2} = -ALUL, \quad (13)$$

and identity  $\delta[y]\{x^2x^3\} = 0$  forces

$$UU = -2ULL + 2LUL + 4L^4. \quad (14)$$

Next, using above identity and  $\delta[y, x^2]\{x(x^2)^2\} = 0$  we get that  $0 = 4UUL + 4LUU + 8LL_xL = 4(UUL + LUU - 2LLUL - 4L^5) = 8(-UL^3 + LULL + 2L^5 - LULL + LLUL + 2L^5 - LLUL - 2L^5) = 8(-UL^3 + 2L^5)$ . Hence  $UL^3 = 2L^5$ . Now identity  $L_{x(x^2x^2)} = 0$  and relations (5) and (14) imply  $L^2UL = -L^3U - 4L^5$ . Thus, we have the following multiplication identities

Table vi: Multiplication identities of degree 5

|         | $ULU$ | $LUL^2$ | $L^3U$ | $L^5$ |
|---------|-------|---------|--------|-------|
| $UUL$   | 0     | 2       | 0      | 0     |
| $LUU$   | 0     | -2      | -2     | -4    |
| $L^2UL$ | 0     | 0       | -1     | -4    |
| $UL^3$  | 0     | 0       | 0      | 2     |

From Table ii, we can prove that

$$(UL)^2 = -UL^2U - (LU)^2 + 2L^3UL + 4L^4U + 16L^6, \quad (15)$$

and  $\delta[x^2]\{x^2(x(xy)) - 2x(x(x(xy)))\} = 0$  forces

$$(UL)^2 + UL^2U + 2L^3UL + 4L^6 = 0. \quad (16)$$

Combining (15) and (16), we have  $(LU)^2 = 4L^6$  and  $(UL)^2 = -UL^2U + 2L^4U + 4L^6$ .

Now, we can check easily the following multiplication identities

Table vii: Multiplication identities of degree 6

|        | $ULLU$ | $L^4U$ | $L^6$ |           | $ULLU$ | $L^4U$ | $L^6$ |
|--------|--------|--------|-------|-----------|--------|--------|-------|
| $UUU$  | -2     | 4      | 8     | $LLUU$    | 0      | -4     | -4    |
| $UULL$ | 0      | 0      | 4     | $UL^4$    | 0      | 0      | 2     |
| $ULUL$ | -1     | 2      | 4     | $LUL^3$   | 0      | 0      | 2     |
| $LUUL$ | 0      | 2      | 0     | $L^2UL^2$ | 0      | 1      | 0     |
| $LULU$ | 0      | 0      | 4     | $L^3UL$   | 0      | -1     | -4    |

**Theorem 2.** *Let  $\mathfrak{A}$  be an algebra over a field  $F$  of characteristic  $\neq 2$  or  $3$ , satisfying the identities  $x^4 = 0$  and  $x(x^2x^2) = 0$ . Then every monomial in  $P$  of  $x$ -degree  $\geq 7$  and  $y$ -degree  $1$  is an identity in  $\mathfrak{A}$ . In particular,  $L_a^7 = 0$  for all  $a \in \mathfrak{A}$ . Furthermore, the algebra generated by  $L_x$  and  $L_{x^2}$  is spanned, as vector space, by  $L, U, L^2, UL, LU, L^3, UL^2, LUL, L^2U, L^4, ULU, LUL^2, L^3U, L^5, UL^2U, L^4U, L^6$ .*

*Proof.* We shall prove that every monomial of  $x$ -degree  $\geq 7$  and  $y$ -degree 1 is an identity in  $\mathfrak{A}$ . Multiplying the operators in the first line of Table vii with  $L$  and  $U$  from the left and from the right, and the operators in the first line of Table vi with  $U$  from the left and from the right, and next using reductions from Tables i-vii we see that we only need to prove that  $LUL^2U = 0$ ,  $L^5U = 0$  and  $L^7 = 0$  are multiplication identities in  $\mathfrak{A}$ . Now, we have  $0 = \delta[y, x^2x^2]\{x(x^2)^2\} = 4L_{x^2x^2}UL + 4LUL_{x^2x^2} = -16LULUL - 16LULUL = -32LULUL = -32(LU)^2L = -2^7L^7$ , so that  $L^7 = 0$ . Also  $0 = LULUL = L(UL)^2 = -LUL^2U + 2L^5U$ . Therefore,  $LUL^2U = 2L^5U$ . Finally, from Table vi we have that  $0 = (L^2UL + L^3U + 4L^5)L^2 = L^2UL^3 + L^3UL^2 = L^3UL^2 = L(L^2UL^2) = L^5U$ . This proves the theorem.  $\square$

## 2.2 The identity $x(x((xx)(xx)))=0$

In this subsection we consider the class of all algebras in  $\mathcal{V}$  satisfying the identity  $x(x(x^2x^2)) = 0$ . Because we use linearization process of identities and  $x(x(x^2x^2))$  has degree 6, we need consider the field  $F$  of characteristic not 5 (2 or 3).

From linearization  $\delta[y]\{x(x(x^2x^2))\}$ , we get the multiplication identity  $L_{x(x^2x^2)} + LL_{x^2x^2} + 4L^2UL = 0$  and Lemma 2 forces

$$LUU = -2LUL^2 - 2L^3U - 4L^5, \quad (17)$$

The relation  $0 = \delta[y, x^2]\{x(x(x^2x^2))\} = UL_{x^2x^2} + 4LL^{x^2x^2}L + 4ULUL + 4LUUL + 8L^2L_{x^3}L + 4L^2UU$  implies

$$LUL^3 = -\frac{1}{2}(L^2UL^2 + L^3UL), \quad (18)$$

since we can use identities from Tables i-v. Next, by  $0 = \delta[y, x^3]\{x(x(x^2x^2))\}$  and  $0 = \delta[y, x^2, x^2]\{x(x(x^2x^2))\}$  we get

$$L^4UL = -3L^5U - 16L^7, \quad (19)$$

$$L^2ULU = -L^3UL^2 + 5L^5U + 28L^7, \quad (20)$$

and identities  $0 = \delta[y, x^2, x^2, x^2]\{x(x(x^2x^2))\}$  and  $0 = \delta[y, x^2, x^3]\{x(x(x^2x^2))\}$  imply

$$UL^4U = -\frac{1}{2}L^2UL^2U + 24L^6U + 62L^8, \quad (21)$$

$$L^2UL^2U = 48L^6U + 156L^8. \quad (22)$$

Now, identity  $0 = \delta[y, x^2x^2]\{x^3x^3\}$  forces

$$L^6U = -2L^8. \quad (23)$$

**Theorem 3.** *Let  $\mathfrak{A}$  be a commutative algebra over a field  $F$  of characteristic not 2, 3 or 5, satisfying the identities  $x^4 = 0$  and  $x(x(x^2x^2)) = 0$ . Then every monomial in  $P$  of  $x$ -degree  $\geq 9$  and  $y$ -degree 1 is an identity in  $\mathfrak{A}$ . In particular,  $L_a^9 = 0$  for all  $a \in \mathfrak{A}$ .*

*Proof.* By Tables i-v, we only need to prove that  $LUL^4U = 0$ ,  $L^2UL^2UL = 0$ ,  $L^7U = 0$  and  $L^9 = 0$  are multiplication identities in  $\mathfrak{A}$ . From (19-23) may be deduced immediately  $L^7U = -2L^9$  and  $2L^9 = 2L^8L = -L^6UL = -L^2(L^4UL) = 3L^7U + 16L^9 = -6L^9 + 16L^9 = 10L^9$ . Therefore  $L^9 = 0$  and  $L^7U = 0$  are identities in  $\mathfrak{A}$ . Now  $L^2UL^2UL = (L^2UL^2U)L = 48L^6UL + 156L^9 = 0$  and  $LUL^4U = L(UL^4U) = -(1/2)L^3UL^2U + 24L^7U + 62L^9 = -(1/2)L(L^2UL^2U) = -24L^7U - 78L^9 = 0$ . This proves the theorem.  $\square$

### 3 Nilalgebras of dimension $\leq 7$

In this section we study (commutative nonassociative) nilalgebras of dimension  $\leq 7$ , over a field  $F$  of characteristic zero or sufficiently large. We show that nilalgebras over  $F$  with dimension  $\leq 7$ , are solvable. An algebra  $\mathfrak{A}$  is called *solvable* if there exists a positive integer  $t$  such that  $\mathfrak{A}^{[t]} = (0)$ , where we define inductively  $\mathfrak{A}^{[1]} = \mathfrak{A}$  and  $\mathfrak{A}^{[j+1]} = \mathfrak{A}^{[j]}\mathfrak{A}^{[j]}$  for all positive integers  $j$ .

Let  $\mathfrak{A}$  be a finite-dimensional nilalgebra over  $F$ . We will denote by  $\deg(\mathfrak{A})$ , the *degree* of  $\mathfrak{A}$ , the smallest number  $m$  such that for every  $a \in \mathfrak{A}$ , the subalgebra  $\text{alg}(a)$  of  $\mathfrak{A}$  generated by  $a$  has  $\dim(\text{alg}(a)) \leq m$ . If  $\deg(\mathfrak{A}) \leq 2$ , then  $\mathfrak{A}$  satisfies the identity  $x^3 = 0$  and hence this algebra is Jordan. It is well-known that any finite-dimensional Jordan nilalgebra is nilpotent. Therefore  $\mathfrak{A}$  is nilpotent if  $\deg(\mathfrak{A}) \leq 2$ . Because any nilpotent algebra is solvable, we have that  $\mathfrak{A}$  is solvable if  $\deg(\mathfrak{A}) \leq 2$ .

The following lemma, proved in [7], is an immediate consequence of a result of M. Gerstenhaber [9] and B. Mathes et al. [10] for linear spaces of nilpotent matrices.

**Lemma 3.** *Let  $\mathfrak{A}$  be a nilalgebra over the field  $F$ . Then  $\mathfrak{A}^2\mathfrak{A}^2 \subset \mathfrak{B}$  for every subalgebra  $\mathfrak{B}$  of codimension  $\leq 2$ .*

By above lemma, if  $\deg(\mathfrak{A}) \geq \dim(\mathfrak{A}) - 2$ , then  $\mathfrak{A}^2\mathfrak{A}^2$  is nilpotent and hence  $\mathfrak{A}$  is solvable. Summarizing,  $\mathfrak{A}$  is solvable in the following cases: (i)  $\dim(\mathfrak{A}) \leq 5$ ; (ii)  $\dim(\mathfrak{A}) = 6$  and  $\deg(\mathfrak{A}) \neq 3$ ; (iii)  $\dim(\mathfrak{A}) = 7$  and  $\deg(\mathfrak{A}) \neq 3$  or 4. Thus, for  $\dim(\mathfrak{A}) \leq 7$ , it remains to be shows that  $\mathfrak{A}$  is solvable if  $\deg(\mathfrak{A}) = 3$  or 4.

The following lemma is clear from Lemma 2. For any subset  $\{a_1, \dots, a_t\}$  of  $\mathfrak{A}$  we denote by  $\langle a_1, \dots, a_t \rangle$  the vector space spanned by the elements  $a_1, \dots, a_t$ .

**Lemma 4.** *Let  $\mathfrak{A}$  be an algebra over  $F$  satisfying the identity  $x^4 = 0$ . Consider an element  $a$  in  $\mathfrak{A}$ . (i) If  $a(a(a^2a^2)) \neq 0$ , then  $\dim(\text{alg}(a)) = 6$ ; (ii) If  $a(a(a^2a^2)) = 0$  and  $a(a^2a^2) \neq 0$ , then  $\dim(\text{alg}(a)) = 5$ .*

*Proof.* We first observe that  $\text{alg}(a) = \langle a, a^2, a^3, a^2a^2, a(a^2a^2), a(a(a^2a^2)) \rangle$ . Assume  $a(a(a^2a^2)) \neq 0$ . We will prove that  $a, a^2, a^3, a^2a^2, a(a^2a^2), a(a(a^2a^2))$  are linearly independent. Let  $\lambda_1 a + \lambda_2 a^2 + \lambda_3 a^3 + \lambda_4 a^2 a^2 + \lambda_5 a(a^2 a^2) + \lambda_6 a(a(a^2 a^2)) = 0$ . Then  $0 = L_a^2 L_{a^2} L_a(0) = L_a^2 L_{a^2} L_a(\lambda_1 a + \lambda_2 a^2 + \lambda_3 a^3 + \lambda_4 a^2 a^2 + \lambda_5 a(a^2 a^2) + \lambda_6 a(a(a^2 a^2))) = \lambda_1 a(a(a^2 a^2)) = \lambda_1 a(a(a^2 a^2))$  and hence  $\lambda_1 = 0$ . Analogously,  $0 = L_a^2 L_{a^2}(0) = L_a^2 L_{a^2}(\lambda_2 a^2 + \lambda_3 a^3 + \lambda_4 a^2 a^2 + \lambda_5 a(a^2 a^2) + \lambda_6 a(a(a^2 a^2))) = \lambda_2 a(a(a^2 a^2))$  so that  $\lambda_2 = 0$ . Next,  $0 = L_a L_{a^2}(0) = L_a L_{a^2}(\lambda_3 a^3 + \lambda_4 a^2 a^2 + \lambda_5 a(a^2 a^2) + \lambda_6 a(a(a^2 a^2))) = -\lambda_3 a(a(a^2 a^2))$  so that  $\lambda_3 = 0$ . And analogously we can prove that  $\lambda_4 = \lambda_5 = \lambda_6 = 0$ . The case (ii) is similar.  $\square$

**Corollary 1.** *Let  $\mathfrak{A}$  be an algebra over  $F$  satisfying the identity  $x^4 = 0$ . Assume  $\deg(\mathfrak{A}) = 3$  or 4 and let  $a$  be an element in  $\mathfrak{A}$ . Then  $\text{alg}(a) = \langle a, a^2, a^3, a^2a^2 \rangle$  and  $\langle a^3, a^2a^2 \rangle \cdot \text{alg}(a) = 0$ .*

### 3.1 The case $\text{degree}(\mathfrak{A})=3$

Now we will study nilalgebras of degree 3. In this subsection  $\mathfrak{A}$  will be a nilalgebra of degree 3 and dimension  $\leq 7$  over the field  $F$ . Consider  $a$  an element in  $\mathfrak{A}$ . Because  $\mathfrak{A}$  is nilalgebra, there exists a positive integer  $t$  such that  $a^t = 0$ . We can assume that  $a^t = 0$  and  $a^{t-1} \neq 0$ . Clearly, the elements  $a, a^2, \dots, a^{t-1}$  are linearly independent, and hence  $t \leq 4$ , since  $\deg(\mathfrak{A}) = 3$ . Consequently, the algebra  $\mathfrak{A}$  satisfies the identity  $x^4 = 0$ . By Corollary 1, the sequence  $a^3, a^2a^2$  is linearly dependent and  $\mathfrak{A}$  satisfies the identities  $x(x^2x^2) = 0$ ,  $x^2x^3 = 0$ . Consequently,  $\mathfrak{A}$  satisfies multiplication identities (13), (14), Tables vi and vii and we can use Theorem 2.

**Lemma 5.** *Let  $\mathfrak{A}$  be a nilalgebra over the field  $F$  with dimension  $\leq 7$  and degree 3. Then  $L_a^6 = 0$  is a multiplication identity in  $\mathfrak{A}$ .*

*Proof.* Assume that there exist  $a, b \in \mathfrak{A}$  such that  $L_a^6(b) \neq 0$ . Then the sequence  $\Psi = \{L_a^i(b) : i = 0, 1, \dots, 6\}$  is a basis of  $\mathfrak{A}$ . Note that from Table vii and (13) we have

$$L_a^6(b) = \frac{1}{2}a(a^2(a(a(ab)))) = -\frac{1}{8}(a^2a^2)(a(ab)),$$

so that  $a^2a^2 \neq 0$ . Because  $\Psi$  is a basis and  $a(a^2a^2) = 0$ , we get that

$$a^2a^2 = \lambda L_a^6(b),$$

for any  $0 \neq \lambda \in F$ . Combining above relations we have  $a^2a^2 = (a^2a^2)[(-\lambda/8)a(ab)]$ . But this is impossible because  $\mathfrak{A}$  is an Engel algebra. This forces  $L_a^6 = 0$ .  $\square$

We may use (3) combined with (13) to yield

$$L_{x^2x^2}L - 4L_{x^3}L^2 = 8L^5. \quad (24)$$

We shall use this formula now.

**Lemma 6.** *Let  $\mathfrak{A}$  be a nilalgebra over the field  $F$  with dimension  $\leq 7$  and degree 3. Then  $L^5 = 0$  is a multiplication identity in  $\mathfrak{A}$ .*

*Proof.* Assume that there exist  $a, b \in \mathfrak{A}$  such that  $L_a^5(b) \neq 0$ . By identity (24) we have that either  $a^3 \neq 0$  or  $a^2a^2 \neq 0$ . The proof now splits into two cases.

*Case 1.* If  $a^3 = \beta a^2a^2$  with  $\beta \in F$ , then using (24) we obtain  $8L_a^5 = L_{a^2a^2}L_a - 4\beta L_{a^2a^2}L_a^2$  and multiplying this relation from the right side with  $L_a$  yields  $L_{a^2a^2}L_a^2 = 0$ , so that  $L_{a^3}L_a^2 = 0$  and  $L_{a^2a^2}L_a = 8L_a^5$ . Now, it is easy to prove that  $\Psi = \{a^2a^2, L_a^i(b) : i = 0, 1, \dots, 5\}$  is linearly independent and hence a basis of  $\mathfrak{A}$ . Let  $a = \lambda a^2a^2 + \sum_{i=0}^5 \mu_i L_a^i(b)$ . Multiplying by  $a$ , 3 times, we get  $0 = \mu_0 L_a^3(b) + \mu_1 L_a^4(b) + \mu_2 L_a^5(b)$ , so that  $\mu_0 = \mu_1 = \mu_2 = 0$ . Next, multiplying by  $a$  two time, we have  $a^3 = \mu_3 L_a^5(b) = (\mu_3/8)(a^2a^2)(ab)$  and hence  $\mu_3 = 0$  since  $a^3 \in \langle a^2a^2 \rangle$  and  $\mathfrak{A}$  is an Engel algebra. Now, multiplying by  $a$ , it follows  $a^2 = \mu_4 L_a^5(b) = (\mu_4/2)a^2(a(ab))$  hence  $\mu_4 = 0$  since  $\mathfrak{A}$  is an Engel algebra. Thus, we have  $a = \lambda a^2a^2 + \mu_5 L_a^5(b)$ , but multiplying this relation from the left side, first with  $a$  and next with  $a^2$ , we get that  $a^2a^2 = \lambda a^2(a(a^2a^2)) + \mu_5 L_{a^2}L_a^6(b) = 0$  and this is impossible.

*Case 2.* If  $a^2a^2 = 0$ , then  $L_{a^3}L_a^2 = -2L_a^5$ . Now, it is easy to prove that  $\Phi = \{a^3, L_a^i(b) : i = 0, 1, \dots, 5\}$  is linearly independent and hence a basis of  $\mathfrak{A}$ . Let  $a = \lambda a^3 + \sum_{i=0}^5 \mu_i L_a^i(b)$ . Multiplying by  $a$ , 3 times, we get  $0 = \mu_0 L_a^3(b) + \mu_1 L_a^4(b) + \mu_2 L_a^5(b)$ , so that  $\mu_0 = \mu_1 = \mu_2 = 0$ . Next, multiplying by  $a$  two time, we have  $a^3 = \mu_3 L_a^5(b)$ , but this is impossible because  $\Phi$  is a basis.  $\square$

**Lemma 7.** *Let  $\mathfrak{A}$  be a nilalgebra over the field  $F$  with dimension  $\leq 7$  and degree 3. Then every monomial in  $P$  of  $x$ -degree  $\geq 6$  and  $y$ -degree 1 is an identity in  $\mathfrak{A}$ .*

*Proof.* By Theorem 2 and Lemma 5 we only need to prove that  $L^4U = 0$  and  $UL^2U = 0$  are multiplication identities in  $\mathfrak{A}$ . Using identity (13), Table vii and

relation  $0 = \delta[x^2]\{x(x(x(xy))))\}$  we have that  $0 = UL^4 + LUL^3 + L^2UL^2 + L^3UL + L^4U = L^2UL^2 + L^3UL + L^4U = L^4U$  and hence from Table vii we see that

$$ULLU = -ULUL = \frac{1}{4}UL_{x^2x^2} = UL_{x^3}L,$$

since  $LU = -L_{x^3} - 2L^3$  and  $UL^4 = L^5 = 0$ . Let  $a \in \mathfrak{A}$ . If  $a^2a^2 = 0$ , it follows immediately that  $L_{a^2}L_a^2L_{a^2} = 0$ . If  $a^2a^2 \neq 0$ , then there exists  $\lambda \in F$  such that  $a^3 = \lambda a^2a^2$ . Therefore,  $L_{a^2}L_a^2L_{a^2} = L_{a^2}L_{a^3}L_a = \lambda L_{a^2}L_{a^2a^2}L_a = 0$ . This proves the lemma.  $\square$

Using Lemma 2, Lemma 6 and identity (13) and Table vi, we can prove easily the following multiplication identities

$$\begin{aligned} L^3U &= -L^2UL = -L^2L_{x^3} = LL_{x^3}L = \frac{1}{4}LL_{x^2x^2}, \\ LUL^2 &= -L_{x^3}L^2 = -\frac{1}{4}L_{x^2x^2}L, \end{aligned}$$

for nilalgebra of dimension  $\leq 7$  and degree 3 over the field  $F$ . We shall use these formulas now.

**Lemma 8.** *Let  $\mathfrak{A}$  be a nilalgebra over the field  $F$  with dimension  $\leq 7$  and degree 3. Then  $L^3U = 0$  and  $LUL^2 = 0$  are multiplications identities in  $\mathfrak{A}$ .*

*Proof.* Let  $a$  be an element in  $\mathfrak{A}$ . If  $a^2a^2 = 0$  then, from above identities we obtain immediately that  $L_a^3L_{a^2} = (1/4)L_aL_{a^2a^2} = 0$  and  $L_aL_{a^2}L_a^2 = -(1/4)L_{a^2a^2}L_a = 0$ . If  $a^2a^2 \neq 0$  then there exists  $\lambda \in F$  such that  $a^3 = \lambda a^2a^2$ . This means that  $L_{a^3} = \lambda L_{a^2a^2}$ . By Lemma 7 we have  $L_a^3L_{a^2} = L_a^2L_{a^3} = \lambda L_a^2L_{a^2a^2} = 0$  and  $L_aL_{a^2}L_a^2 = -L_{a^3}L_a^2 = \lambda L_{a^2a^2}L_a^2 = 0$ . This proves the lemma.  $\square$

**Lemma 9.** *Let  $\mathfrak{A}$  be a nilalgebra over the field  $F$  with dimension  $\leq 7$  and degree 3. Then  $LUL = 0$  is a multiplication identity in  $\mathfrak{A}$ .*

*Proof.* We will assume the contrary, there exist two elements  $a, b \in \mathfrak{A}$  such that  $a(a^2(ab)) \neq 0$ . We know by (13) that  $a(a^2(ab)) = -(1/4)(a^2a^2)b$ . Therefore,  $a^2a^2 \neq 0$  and also the sequence  $\{a^2a^2, a(a^2(ab))\}$  is linearly independent, because  $L_b$  is nilpotent. For any  $\lambda \in F$  we have that  $a^3 = \lambda a^2a^2$ . Obviously, this forces  $L_{a^3} = \lambda L_{a^2a^2}$ . From identity (3) we have immediately,  $L_aL_{a^2}L_a = -L_{a^3}L_a - 2L_a^4 = -\lambda L_{a^2a^2}L_a - 2L_a^4 = 4\lambda L_aL_{a^2}L_a^2 - 2L_a^4 = -2L_a^4$ , that is

$$L_aL_{a^2}L_a = -2L_a^4.$$

We will now prove that  $\Psi = \{b, ab, a^2(ab), a(a^2(ab)), a, a^2, a^2a^2\}$  is a basis of  $\mathfrak{A}$ . Let  $\lambda_1 b + \lambda_2 ab + \lambda_3 a^2(ab) + \lambda_4 a(a^2(ab)) + \mu_1 a + \mu_2 a^2 + \mu_3 a^2a^2 = 0$ , with  $\lambda_i, \mu_j \in F$ . Multiplying with  $a, a^2$  and  $a$  successively, we get  $\lambda_1 = 0$ . Multiplying with  $a^2$  and  $a$  successively, we have  $\lambda_2 = 0$ . Multiplying with  $a$  and  $a^2$  successively, we obtain  $\mu_1 = 0$ , so that

$$\lambda_3 a^2(ab) + \lambda_4 a(a^2(ab)) + \mu_2 a^2 + \mu_3 a^2a^2 = 0.$$

Multiplying with  $a$  it follows that  $\lambda_3 = 0$  since  $a^2a^2, a(a^2(ab))$  are linearly independent and  $a^3 \in \langle a^2a^2 \rangle$ . Multiplying with  $a^2$  we have  $\mu_2 = 0$ . Now, relation  $\lambda_4 a(a^2(ab)) + \mu_3 a^2a^2 = 0$  forces  $\lambda_4 = \mu_3 = 0$ . Therefore, we have proved that the sequence  $\Psi$  is linearly independent. Since  $\dim(\mathfrak{A}) \leq 7$ , it follows that  $\Psi$  is a basis of  $\mathfrak{A}$ .

On the other hand, because  $\Psi$  is a basis of  $\mathfrak{A}$ , we have a representation  $a(ab) = \alpha_1 b + \alpha_2 ab + \alpha_3 a^2(ab) + \alpha_4 a(a^2(ab)) + \alpha_5 a + \alpha_6 a^2 + \alpha_7 a^2a^2$ , with  $\alpha_i \in F$ . Using the operators  $L_a L_{a^2} L_a, L_a L_{a^2}, L_{a^2} L_a$  and  $L_a L_a$ , we prove that  $\alpha_1 = 0, \alpha_2 = 0, \alpha_5 = 0$  and  $a(a(a(ab))) = 0$  respectively, but this is impossible. This proves the lemma.  $\square$

It was proved in [4] the following result for power-associative nilalgebras.

**Lemma 10.** *Every commutative power-associative nilalgebra of dimension  $\leq 8$  over a field of characteristic  $\neq 2, 3$  or  $5$  is solvable.*

**Theorem 4.** *Let  $\mathfrak{A}$  be a nilalgebra over the field  $F$  with dimension  $\leq 7$  and degree 3. The algebra  $\mathfrak{A}$  is solvable.*

*Proof.* By Lemma 9 we have that  $a^2a^2$  belong to the annihilator of the algebra  $\mathfrak{A}$ , for every  $a \in \mathfrak{A}$ . This means that the linear subspace  $J = \langle a^2a^2 : a \in \mathfrak{A} \rangle$  is an ideal of  $\mathfrak{A}$  and  $\mathfrak{A}J = 0$ . Thus,  $\mathfrak{A}/J$  is a commutative power-associative nilalgebra of dimension  $\leq 7$ , and hence solvable. This implies that  $\mathfrak{A}$  is solvable.  $\square$

### 3.2 The case $\text{degree}(\mathfrak{A}) = 4$ and $x(x(xx))=0$

For any subalgebra  $\mathfrak{B}$  of an algebra  $\mathfrak{A}$ , the set  $\text{st}(\mathfrak{B}) = \{x \in \mathfrak{A} : x\mathfrak{B} \subset \mathfrak{B}\}$  is called *stabilizer* of  $\mathfrak{B}$  in  $\mathfrak{A}$ . For every element  $a \in \text{st}(\mathfrak{B})$ , we can define a linear transformation  $\overline{L}_a$  on the quotient vector space  $\overline{\mathfrak{A}} = \mathfrak{A}/\mathfrak{B}$  as follows,

$$\overline{L}_a(x + \mathfrak{B}) = ax + \mathfrak{B},$$

for all  $x \in \mathfrak{A}$ . We will now denote by  $M_{\mathfrak{B}}$  the linear space  $\{\overline{L}_a : a \in \text{st}(\mathfrak{B})\}$  and by  $N_{\mathfrak{B}}$  the linear subspace  $\{\overline{L}_b : b \in \mathfrak{B}\}$ . Evidently, we have that  $N_{\mathfrak{B}} \subset M_{\mathfrak{B}}$ .

The following result will be useful. Items (iv), (v), (vi) and (vii) follow immediately from (i)-(iii) proved in [7].

**Lemma 11.** *Let  $V$  be a vector space of dimension 3 over a field  $F$  of characteristic  $\neq 2$  and let  $\mathfrak{M}$  be a vector space of nilpotent endomorphisms in  $\text{End}(V)$ . Then  $\dim \mathfrak{M} \leq 3$  and either  $\mathfrak{M}^3 = 0$  or: (i)  $\dim \mathfrak{M} = 2$ ; (ii) for every nonzero  $f \in \mathfrak{M}$  we have that  $\text{rank}(f) = 2$ ; (iii) if  $\mathfrak{M} = \langle f_1, f_2 \rangle$ , then there exists a basis  $\phi$  of  $V$  and  $0 \neq \lambda \in F$  such that the matrices (using columns) of  $f_1$  and  $\lambda f_2$  with respect to  $\phi$  are*

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

*respectively; (iv) if  $f, g$  and  $fg$  are all in  $\mathfrak{M}$ , then  $f = 0$  or  $g = 0$ ; (v) if  $f, g, h$  and  $f(g + fh)$  are all in  $\mathfrak{M}$  and  $f \neq 0$ , then  $g = 0$  and  $h \in \langle f \rangle$ ; (vi) if  $f, g, h \in \mathfrak{M}$  and  $f(f + gh) = 0$ , then  $f = 0$ ; (vii) if  $f, g \in \mathfrak{M}$  and  $f^2g^2 = 0$ , then the sequence  $\{f, g\}$  is linearly dependent.*

Let  $\mathfrak{A}$  be a nilalgebra over the field  $F$  with degree 4 and dimension  $\leq 7$  satisfying the identity  $x^4 = 0$ . From Lemma 3,  $\mathfrak{A}$  is solvable if  $\dim(\mathfrak{A}) \leq 6$ , so that throughout this subsection we will assume that  $\mathfrak{A}$  has dimension 7. By Corollary 1, the algebra  $\mathfrak{A}$  satisfies the identities  $x(x^2x^2) = 0$  and  $x^2x^3 = 0$ . Now we may take an element  $b$  in  $\mathfrak{A}$  such that  $\mathfrak{B}$ , the subalgebra of  $\mathfrak{A}$  generated by  $b$ , has dimension 4. By Corollary 1, we have

$$\mathfrak{B} = \langle b, b^2, b^3, b^2b^2 \rangle,$$

and

$$\langle b^3, b^2b^2 \rangle \mathfrak{B} = (0). \quad (25)$$

If  $\dim N_{\mathfrak{B}} = 0$ , then  $\mathfrak{B}$  is an ideal of  $\mathfrak{A}$  and hence  $\mathfrak{A}$  is solvable because  $\mathfrak{A}/\mathfrak{B}$  is solvable. If  $M_{\mathfrak{B}}$  is nilpotent, then there exists  $a \in \mathfrak{A}$  but not in  $\mathfrak{B}$  such that

$$f(a + \mathfrak{B}) = 0 + \mathfrak{B}, \quad (26)$$

for all  $f \in M_{\mathfrak{B}}$ . There exists a smallest integer  $m$ ,  $1 \leq m \leq 3$ , such that  $M_{\mathfrak{B}}^m = (0)$ . If  $m = 1$  take  $a \in \mathfrak{A}$  but not in  $\mathfrak{B}$ ; if  $m > 1$ , take  $0 \neq g \in M_{\mathfrak{B}}^{m-1}$  and  $a + \mathfrak{B}$  in  $g(\mathfrak{A}/\mathfrak{B})$  with  $a + \mathfrak{B} \neq 0 + \mathfrak{B}$ . Then (26) is satisfied. Since  $a \in \text{st}(\mathfrak{B})$  we have that  $\overline{L}_a \in M_{\mathfrak{B}}$ . Then relation (26) implies that  $0 + \mathfrak{B} = \overline{L}_a(a + \mathfrak{B})$  and hence  $a^2 \in \mathfrak{B}$ .

Let  $\mathfrak{B}' = \langle b, b^2, b^3, b^2b^2, a \rangle$ . We have that  $\mathfrak{B}'$  is a subalgebra of  $\mathfrak{A}$  with codimension 2. Using Lemma 3 we get that  $\mathfrak{A}^2\mathfrak{A}^2 \subset \mathfrak{B}'$  so that  $\mathfrak{A}$  is solvable.

We now consider the case  $N_{\mathfrak{B}} \neq (0)$  and  $M_{\mathfrak{B}}^3 \neq (0)$ . Then  $M_{\mathfrak{B}}$  satisfies properties (i)-(v) of Lemma 11. By Theorem 1,  $N_{\mathfrak{B}}$  is nilpotent, so Lemma 11 implies  $\dim(N_{\mathfrak{B}}) = 1$  since if  $\dim(N_{\mathfrak{B}}) > 1$  then either  $N_{\mathfrak{B}} = M_{\mathfrak{B}}$  or  $\dim(M_{\mathfrak{B}}) = 3$  and hence  $M_{\mathfrak{B}}$  is nilpotent. Let  $0 \neq h \in N_{\mathfrak{B}}$ . Then  $\overline{L_{b^i}} = \alpha_i h$  for  $i = 1, 2, 3$ . From identities (13) and (3) we have  $\overline{L_{b^2b^2}} = -4\alpha_1^2\alpha_2h^3 = 0$ ,  $\alpha_3h = \overline{L_{b^3}} = -\alpha_1\alpha_2h^2 - 2\alpha_1^3h^3 = -\alpha_1\alpha_2h^2$  so that  $\overline{L_{b^3}} = 0$  since  $h^3 = 0$ . Next (14) forces  $\overline{L_{b^2}{}^2} = -2\alpha_1^2\alpha_2h^3 + 2\alpha_1^2\alpha_2h^3 + 4\alpha_1^4h^4 = 0$ . Therefore  $\overline{L_{b^2}} = 0$  since  $\overline{L_{b^2}} \in M_{\mathfrak{B}}$  and from Lemma 11 every nonzero element in  $M_{\mathfrak{B}}$  is nilpotent of index 3. Thus, we have proved that

$$\mathfrak{B}^2\mathfrak{A} = \langle b^2, b^3, b^2b^2 \rangle \mathfrak{A} \subset \mathfrak{B}.$$

This yields  $N_{\mathfrak{B}} = \langle \overline{L_b} \rangle$ . By Lemma 11 we can take  $a \in \text{st}(\mathfrak{B})$ , but not in  $\mathfrak{B}$  such that  $M_{\mathfrak{B}} = \langle \overline{L_b}, \overline{L_a} \rangle$  and there exists a basis  $\Phi = \{v_1 + \mathfrak{B}, v_2 + \mathfrak{B}, v_3 + \mathfrak{B}\}$  of  $\mathfrak{A}/\mathfrak{B}$  such that the matrices of  $\overline{L_b}$  and  $\overline{L_a}$  with respect to  $\Phi$  are respectively

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This means that  $v_3 + \mathfrak{B} = \alpha a + \mathfrak{B}$ ,  $v_2 + \mathfrak{B} = \alpha a^2 + \mathfrak{B}$  and  $v_1 + \mathfrak{B} = \alpha a^3 + \mathfrak{B}$  for any  $\alpha \in F$ ,  $\alpha \neq 0$ . We can assume, without loss of generality, that  $\alpha = 1$ . By equation (13) and (25) we have  $(b^2b^2)a = -4b(b^2ba) \subset b(b^2\mathfrak{B}) = (0)$ , so that

$$(b^2b^2)a = 0.$$

On the other hand,  $ab$  can be expressed as a linear combination of  $b, b^2, b^3, b^2b^2$ . Let  $ab = \mu_1b + \mu_2b^2 + \mu_3b^3 + \mu_4b^4b^2$ . Then  $cb = \mu_1b + \mu_4b^2b^2$ , where  $c = a - \mu_2b - \mu_3b^2$ . Therefore  $c(cb) = \mu_1cb + \mu_4c(b^2b^2) = \mu_1cb$ . Since  $\mathfrak{A}$  is an Engel algebra,  $L_c$  is nilpotent and hence either  $\mu_1 = 0$  or  $cb = 0$ . This implies

$$ab \in \mathfrak{B}^2.$$

Using relation (14) we have that  $b^2(b^2a) = -2b^2(b(ba)) + 2b(b^2(ba)) + 4b(b(b(ba))) = 0$ . This forces

$$b^2a \in \mathfrak{B}^3.$$

Finally, using (3) and (13) we have

$$\begin{aligned} b^3a &= -b(b^2a) - 2b(b(ba)) = 0, \\ (b^2b^2)a^2 &= -4b(b^2(ba^2)) \in \langle b(b^2(-a + \mathfrak{B})) \rangle = \langle b(b^2a) \rangle = 0, \\ (b^2b^2)a^3 &= -a(a^2(b^2b^2)) - 2a(a(b^2b^2)) = 0, \end{aligned}$$

and hence  $b^2b^2 \in \text{ann}(\mathfrak{A})$ . Let  $J = \langle b^2b^2 : b \in \mathfrak{A}, \dim(\text{alg}(b)) = 4 \rangle$ . Then  $\mathfrak{A}/J$  is a commutative nilalgebra of dimension  $\leq 6$  and degree  $\leq 3$ , so that solvable. This implies that  $\mathfrak{A}$  is solvable.

### 3.3 The case $\text{degree}(\mathfrak{A}) = 4$

Let  $\mathfrak{A}$  be a nilalgebra with degree 4. If  $a \in \mathfrak{A}$ , then there exists a smallest integer  $t$  such that  $a^t \neq 0$  and  $a^{t+1} = 0$  so that the elements  $a, a^2, \dots, a^t$  are linearly independent. Since  $\text{deg}(\mathfrak{A}) = 4$ , we have that  $t \leq 4$  and hence  $\mathfrak{A}$  satisfies the identity

$$x^5 = 0.$$

Now we will prove that  $\mathfrak{A}$  is a nilalgebra of index  $\leq 9$ . Let  $\mathfrak{B}$  be a subalgebra of  $\mathfrak{A}$  generated by one element and let  $k_1$  be the index of  $\mathfrak{B}$  as right-nilalgebra, that is  $k_1$  is the smallest integer such that  $x^{k_1} = 0$  for all  $x \in \mathfrak{B}$ . Evidently,  $\dim \mathfrak{B} \leq 4$  and  $k_1 \leq 5$ . If  $k_1 \leq 3$ , then  $\mathfrak{B}$  is a Jordan algebra and hence nilpotent with  $\mathfrak{B}^{k_1} = 0$ . If  $k_1 = 4$ , then by Lemma 2 and Lemma 4, we have that  $p(x) = 0$  is an identity in  $\mathfrak{B}$  for every monomial  $p(x)$  of degree  $\geq 5$ . Finally, if  $k_1 = 5$ , then there exists  $b \in \mathfrak{B}$  such that  $\mathfrak{B} = \langle b, b^2, b^3, b^4 \rangle$ . Now, because  $\mathfrak{B}$  is nilpotent, we have that

$$b^2b^2 \in \langle b^3, b^4 \rangle, \quad b^2b^3, b^3b^3 \in \langle b^4 \rangle, \quad b^4\mathfrak{B} = (0).$$

Thus,  $\mathfrak{B}^2 = \langle b^2, b^3, b^4 \rangle$ ,  $\mathfrak{B}^3 = \mathfrak{B}\mathfrak{B}^2 = \langle b^3, b^4 \rangle$ ,  $\mathfrak{B}^4 = \mathfrak{B}\mathfrak{B}^3 + \mathfrak{B}^2\mathfrak{B}^2 \subset \langle b^3, b^4 \rangle$ ,  $\mathfrak{B}^5 = \mathfrak{B}\mathfrak{B}^4 + \mathfrak{B}^2\mathfrak{B}^3 \subset \langle b^4 \rangle$ ,  $\mathfrak{B}^6 = \mathfrak{B}\mathfrak{B}^5 + \mathfrak{B}^2\mathfrak{B}^4 + \mathfrak{B}^3\mathfrak{B}^3 = \langle b^4 \rangle$ ,  $\mathfrak{B}^7 \subset \langle (b^2b^2)^2, (b^3)^3 \rangle \subset \langle b^4 \rangle$ ,  $\mathfrak{B}^8 \subset \langle (b^2b^2)^2 \rangle \subset \langle b^4 \rangle$ , and  $\mathfrak{B}^t = 0$  for all  $t \geq 9$ . It has the following consequences.

**Lemma 12.** *The algebra  $\mathfrak{A}$  satisfies the identities*

$$x^i(x^j(x^t x^2)) = 0, \quad i, j, t \geq 1,$$

and  $p(x) = 0$  for every monomial  $p$  of degree  $\geq 9$ .

Linearizing the above identities we have (in order to simplify, we will write  $L_3$  instead of  $L_{x^3}$  and  $L_4$  instead of  $L_{x^4}$ ):

$$L_4 + LL_3 + L^2U + 2L^4 = 0, \quad (27)$$

$$L_{x^2x^3} + 2LL_3L + LUU + 2LUL^2 = 0, \quad (28)$$

$$2L_4L^2 + L_4U + L_3L_3 + L_3LU + 2L_3L^3 = 0, \quad (29)$$

$$L_{x^3x^3} + 2LL_3U + 4LL_3L^2 = 0, \quad (30)$$

$$U^3 + 2UUL^2 + 2UL_3L + 2L_{x^2x^3}L = 0. \quad (31)$$

**Lemma 13.** ([8]) *Every nilalgebra of bounded index over  $F$  is an Engel algebra.*

We have proved that the index of a nilalgebra of degree 4 is  $\leq 9$ . We then apply Lemma 13 to obtain

**Corollary 2.** *Every nilalgebra of degree 4 over  $F$  is an Engel algebra.*

**Theorem 5.** *Let  $\mathfrak{A}$  be a nilalgebra over a field  $F$  of characteristic zero or sufficiently large. If  $\dim(\mathfrak{A}) \leq 7$ , then  $\mathfrak{A}$  is solvable.*

*Proof.* We already prove that  $\mathfrak{A}$  is solvable if either  $\deg(\mathfrak{A}) \neq 4$  or  $x^4 = 0$  is an identity. Thus, it remains to prove that  $\mathfrak{A}$  is solvable if  $\dim(\mathfrak{A}) = 7$ ,  $\deg(\mathfrak{A}) = 4$  and  $x^4 = 0$  is not an identity in  $\mathfrak{A}$ .

Let  $\mathfrak{A}$  be a nilalgebra of dimension 7 and degree 4 such that there exists  $b \in \mathfrak{A}$  with  $b^4 \neq 0$ . Let  $\mathfrak{B}$  be the subalgebra of  $\mathfrak{A}$  generated by  $b$ . Because  $\mathfrak{A}$  has degree 4, we have  $\mathfrak{B} = \langle b, b^2, b^3, b^4 \rangle$  and  $b^5 = 0$ . As in Subsection 3.2, if  $M_{\mathfrak{B}}$  is nilpotent, then the algebra  $\mathfrak{A}$  is solvable. Also, the algebra is solvable if  $N_{\mathfrak{B}} = 0$ . Thus, we can assume that  $M_{\mathfrak{B}}$  is not nilpotent and  $\dim N_{\mathfrak{B}} \geq 1$ . By Lemma 11, we have that  $\dim(M_{\mathfrak{B}}) = 2$ . From (27) we have

$$\overline{L_{b^4}} = -\overline{L_b L_{b^3}} - \overline{L_b^2 L_{b^2}} = -\overline{L_b(L_{b^3} + L_b L_{b^2})} \in N_{\mathfrak{B}}.$$

Combining above relation and (v) of Lemma 11 we get that  $\overline{L_{b^4}} = 0$ . Now (29) implies  $\overline{L_{b^3}(L_{b^3} + L_b L_{b^2})} = 0$  and by (vi) of Lemma 11 we get that  $\overline{L_{b^3}} = 0$ . This means that

$$\mathfrak{B}^3 \mathfrak{A} \subset \mathfrak{B}, \quad (32)$$

and  $N_{\mathfrak{B}} = \langle \overline{L_b}, \overline{L_{b^2}} \rangle$ . Now relation (31) for  $x = b$  forces  $0 = \overline{L_{b^2}^3} + 2\overline{L_{b^2}^2 L_b^2} + 2\overline{L_{b^2} L_{b^3} L_b} + 2\overline{L_{b^2}^2 L_b^2} = 2\overline{L_{b^2}^2 L_b^2}$  and hence using (vii) of Lemma 11 we have that

$$\dim(N_B) = 1. \quad (33)$$

We can assume, without loss of generality, that

$$\overline{L}_b \neq 0, \quad (34)$$

since if  $\overline{L}_b = 0$ , then  $\overline{L}_{b^2} \neq 0$  and we can take  $0 \neq \lambda \in F$  such that  $(b + \lambda b^2)^4 = b^4 + \lambda[b(b^2b^2) + b^2b^3] + \lambda^2(b^2)^3 \neq 0$ . Because  $\dim(N_{\mathfrak{B}}) = 1$ , there exists  $\alpha \in F$  such that  $\overline{L}_{b^2} = \alpha \overline{L}_b$ . As in Subsection 3.2 there exists  $a \in \mathfrak{A}$  such that  $M_{\mathfrak{B}} = \langle f, \overline{L}_a \rangle$  and the matrices of  $f$  and  $\overline{L}_a$  with respect to the basis  $\Phi = \{a^3 + \mathfrak{B}, a^2 + \mathfrak{B}, a + \mathfrak{B}\}$  of  $\mathfrak{A}/\mathfrak{B}$  are respectively

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This means that

$$ba^3 - a^2, b^2a^3 - \alpha a^2, ba^2 + a, b^2a^2 + \alpha a, ba, b^2a, a^4 \in \mathfrak{B}. \quad (35)$$

By (35) we have that  $ba \in \mathfrak{B}$  so that  $ba = \lambda_1 b + \lambda_2 b^2 + \lambda_3 b^3 + \lambda_4 b^4$ , with  $\lambda_i \in F$ . Therefore  $[a - \lambda_2 b - \lambda_3 b^3 - \lambda_4 b^3]b = \lambda_1 b$ . This implies that  $\lambda_1 = 0$  and hence

$$ba \in \mathfrak{B}^2, \quad (36)$$

since every multiplication operator on  $\mathfrak{A}$  is nilpotent. Let  $j$  be a positive integer. From (27) and (32) we get

$$b^4 a^j = -b(b^3 a^j) - b(b(b^2 a^j)) - 2b(b(b(b a^j))) \in \mathfrak{B}^2, \quad (37)$$

so that

$$b^4 \mathfrak{A} \subset \mathfrak{B}^2. \quad (38)$$

By (29) we see that  $2b^4(b(b a^j)) + b^4(b^2 a^j) + b^3(b^3 a^j) + b^3(b(b^2 a^j)) + 2b^3(b(b(b a^j))) = 0$ . This identity for  $j = 2$  forces  $\alpha b^4 a \in \langle b^4 \rangle$ , and now for  $j = 3$  implies  $-2b^4 a + \alpha(b^4 a^2 - b^3 a) \in \langle b^4 \rangle$ , so that  $b^4 a \in \langle b^4 \rangle$ . Therefore

$$b^4 a = 0, \quad \alpha(b^4 a^2 - b^3 a) \in \langle b^4 \rangle, \quad (39)$$

since  $\mathfrak{A}$  is an Engel algebra.

The proof now splits into two cases:

*Case 1.* The relation  $x^3 x^3 = 0$  is not an identity in  $\mathfrak{A}$ . In this case, we can assume without loss of generality that

$$b^4 \neq 0, \quad \text{and} \quad b^3 b^3 \neq 0. \quad (40)$$

Let  $b_1$  be an element in  $\mathfrak{A}$  such that  $b_1^3 b_1^3 \neq 0$  and  $\mathfrak{B}$  the subalgebra of  $\mathfrak{A}$  generated by  $b_1$ . Then  $\{b_1, b_1^2, b_1^3, b_1^3 b_1^3\}$  is a basis of  $\mathfrak{B}$  and products satisfy the following properties,  $b_1^2 b_1^2 \in \langle b_1^3, b_1^3 b_1^3 \rangle$ ,  $b_1^3 \mathfrak{B} \subset \langle b_1^3 b_1^3 \rangle$  and  $(b_1^3 b_1^3) \mathfrak{B} = (0)$ , because  $\mathfrak{B}$  is nilpotent. By Corollary 1 we have that  $x^4 = 0$  is not an identity in  $\mathfrak{B}$ . Thus, there exists an element  $b$  in  $\mathfrak{B}$  of the form  $\lambda_1 b_1 + \lambda_2 b_1^2 + \lambda_3 b_1^3 + \lambda_4 b_1^3 b_1^3$  such that  $b^4 \neq 0$  and also we can assume that  $\overline{L}_b \neq 0$ . Evidently, we have  $\lambda_1 \neq 0$ . Now  $b^2 \in \lambda_1^2 b_1^2 + \langle b_1^3, b_1^3 b_1^3 \rangle$ ,  $b^3 \in \lambda_1^3 b_1^3 + \langle b_1^3 b_1^3 \rangle$  and  $b^3 b^3 = \lambda_1^6 b_1^3 b_1^3 \neq 0$ . Then (40) is satisfied. Evidently, we have  $b^3 b^3 = \gamma b^4$  for any  $0 \neq \gamma \in F$ . Combining (30) and (39) it follows  $0 = (b^3 b^3) a^2 + 2b(b^3(b^2 a^2)) + 4b(b^3(b(ba^2))) = (b^3 b^3) a^2 - 2\alpha b(b^3 a) = \gamma b^4 a^2 - 2\alpha b(b^4 a^2)$ . Thus  $b^4 a^2 = (2\alpha/\gamma)b(b^4 a^2)$ . Since  $\mathfrak{A}$  is an Engel algebra it follows that

$$b^4 a^2 = 0.$$

Combining this identity with (39) we have that

$$\alpha b^3 a \in \langle b^4 \rangle. \quad (41)$$

Now, relation (28) with  $x = b$  for the element  $a^2$  implies  $2b(b^3 a) + \alpha b(b^2 a) \in \langle b^4 \rangle$ . Combining this relation with (41) we see that  $b(b^3 a) \in \langle b^4 \rangle$  so that  $b^3 a \in \langle b^3, b^4 \rangle$ . Therefore

$$b^3 a \in \langle b^4 \rangle,$$

since  $\mathfrak{A}$  is an Engel algebra. Next, we put  $x = b$  in (27) to obtain  $0 = b^4 a + bb^3 a + b(b(b^2 a)) + 2b(b(b(ba))) = b(b(b^2 a))$ , and hence

$$b^2 a \in \langle b^3, b^4 \rangle.$$

Now, by Lemma 12 we know that  $x(x^2 x^3) = 0$  is an identity in  $\mathfrak{A}$  and hence  $0 = (1/48)\delta[b : 4, a : 2]\{x(x^2 x^3)\} = b(b^2(ba^2)) + 2b(b^2(a(ba))) + 4b((ba)(b(ba))) + 2a(b^2(b(ba))) + 2b((ba)(b^2 a)) + a(b^2(b^2 a)) + b(a^2 b^3) + 2a((ba)b^3)$ , forces  $b(b^3 a^2) \in \langle b^4 \rangle$  so that

$$b^3 a^2 \in \langle b^4 \rangle.$$

Finally, (30) implies  $0 = (b^3 b^3) a^3 + 2b(b^3(b^2 a^3)) + 4b(b^3(b(ba^3))) = (b^3 b^3) a^3 + 2b(b^3(\alpha a^2 - 2a)) = (b^3 b^3) a^3$ . Therefore we must have  $b^4 a^3 = 0$ . Consequently, we have proved, in this case, that  $b^4 \in \text{ann}(\mathfrak{A})$ . Let  $J = \langle c^4 : c^3 c^3 \neq 0, c \in \mathfrak{A} \rangle$ . Then  $\overline{\mathfrak{A}} = \mathfrak{A}/J$  is a nilalgebra of dimension  $\leq 6$  and hence solvable. This forces the solvability of  $\mathfrak{A}$ .

*Case 2.* The relation  $x^3 x^3 = 0$  is an identity in  $\mathfrak{A}$ . Linearizing this identity we have that  $\mathfrak{A}$  satisfies the identity

$$x^3(x^2 y) + 2x^3(xy) = 0.$$

Taking  $x = b$  and  $y = a^2$  it follows immediately that  $\alpha b^3 a \in \langle b^4 \rangle$  and for  $x = b$  and  $y = a^3$  this identity forces  $\alpha b^3 a^2 - 2b^3 a \in \langle b^4 \rangle$ . Therefore

$$b^3 a \in \langle b^4 \rangle.$$

Next, (27) forces  $0 = b^4 a + b(b^3 a) + b(b(b^2 a)) + 2b(b(b(ba))) = b(b(b^2 a))$ , so that

$$b^2 a \in \langle b^3, b^4 \rangle.$$

Now, taking the identity  $\delta[b : 4, a : 3]\{x^3 x^4\} = 0$  we have

$$\begin{aligned} -b^4 a^3 &= [b^3 a + b(b^2 a) + 2b(b(ba))] \cdot [ba^2 + 2a(ab)] + \\ &\quad [b(ba^2) + 2b(a(ba)) + 2a(b(ba)) + a(ab^2)] \cdot [b^2 a + 2b(ba)] + \\ &\quad [ba^3 + a(a^2 b) + 2a(a(ab))] \cdot [b^3] \\ &\in \langle b^4 \rangle \cdot \langle a, b, b^2, b^3, b^4 \rangle + \mathfrak{B} \cdot \langle b^3, b^4 \rangle + \mathfrak{B} \cdot \langle b^3 \rangle \subset \langle b^4 \rangle \end{aligned}$$

since by (35) we have that  $ba^3 + a(a^2 b) \in \mathfrak{B}$ . This means that

$$b^4 a^3 = 0,$$

because  $L_{a^3}$  is nilpotent. From  $\delta[b : 3, a : 2]x^5 = 0$  we get that  $-b(b(ba^2)) = a(ab^3) + a(b(ab^2)) + 2a(b(b(ba))) + b(a(ab^2)) + 2b(a(b(ba))) + 2b(b(a(ba))) = 0$ . This means that

$$b(ba^2) \in \langle b^4 \rangle.$$

Finally, from  $\delta[b : 4, a : 2]\{x^4 x^2\} = 0$  it follows that

$$\begin{aligned} -b^4 a^2 &= [b^3 a + b(b^2 a) + 2b(b(ba))] \cdot [2ba] + \\ &\quad [b(ba^2) + 2b(a(ba)) + 2a(b(ba)) + a(ab^2)] \cdot [b^2] \\ &\in \langle b^4 \rangle \cdot \langle b^2, b^3, b^4 \rangle + \langle b^4 \rangle \cdot \langle b^2 \rangle = (0). \end{aligned}$$

Therefore  $b^4 \in \text{ann}(\mathfrak{A})$  and as in the case 1, this implies the solvability of  $\mathfrak{A}$ .  $\square$

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