A Note on k-decidibility

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§0 : Introduction

In [1] Barone-Netto has introduced the concept of *k-decidibility* for germs of $(R^n;O)$ in $(R;0)$. We say that a germ f has $k-jet$ if there is a polynomial P of degree m, $m \leq k$ such that $\lim_{x \to O} \frac{f(x)-P(x)}{\|x\|^k} = 0$, denote by $J(k;n)$ the set of germs who has *k-jet* (or, $J(k)$ when the context is clear enough) and by $J^k f$ the *k-jet* of f.

One can be interested in to study when *k-jet* of a germ f determine if it has a maximum, a minimum or neither (in this case we say that f has a saddle) in O, in this case we say that f is *k-decidible*.

In his paper Barone-Netto found a necessary and sufficient condition for f be *k-decidible*, unfortunately it's not an "algebraic condition" on $J^k f$. We think that the "operational complications" of the problem are motived by the "complication" of $J(k)$ and we show in this note that is very difficult to simplify the question, in sense of if we want know about k-decidibility of a germ we ought to look $J(k)$ in a whole, since we can't restrict our atention to a "conveniente subset" as, by example, the polynomials of $J(k)$ which are sufficient in the case $k = 2$ (see Remark 4 below).

Namely, we answer the next question: "There exists some polynomial p of degree k, *not* k-decidible but such that all $f \in J(k+1)$ with $J^k f = p$ has the same behavior that p?"(or: There are some polynomial p of degree k which is k-decidible in the world $J(k+1)$ but it *is not* k-decidible?). We give an example of such a polynomial P of 2 variables and degree 5 which has a saddle in O, every f wich has $6-jet$ and $J^5 f = P$ has saddle in O (in particular all germs C^6 with Taylor's polynomial of degree 5 equal to P has this propriety) and is not 5-decidible (of course, this is impossible in one variable).

We include two little remarks (3 and 4) which show that:

i- our example is the "simplest" we can find in R^2.

ii- in the case $k = 2$, the only we have an algebraic criteria to decide *k-decidibility*, it's enough to look by the polynomials of $J(2)$ to see if a germ is or not 2-decidible (see the third paragraph of this introduction).

§1 : An (counter-) Exemple

Let be $P(x;y) = y^4 - x^4 y$ and $f = y^4 - x^4 y + x^{5.2}$.

Remark 1 If $g \in J(6)$ and $J^5 g = P$ then g has saddle in O.

◁ :Take the germ of curve $y = x^{1.4}$ in O.

Write $g = J^6 g + R$, we have:

$$\begin{aligned} J^6 g(x;x^{1.4}) &= P(x;x^{1.4}) + Q(x;x^{1.4}) \\ &= x^{5.6} - x^{5.4} + Q(x;x^{1.4}) \end{aligned}$$

Q is an homogeneous polynomial of degree 6 since $J^5 g = P$, then $Q(x;y) = \sum_{j=0}^{6} a_j x^j y^{6-j}$ and $Q(x;x^{1.4}) = \sum_{j=0}^{6} a_j x^{j+1.4(6-j)}$.

Then, since in last line $j + 1.4(6-j) \geq 6$, we see that $J^6 g(x;x^{1.4}) = -x^{5.4} + R_1(x)$ where $\lim_{x\to 0} \frac{R_1(x)}{x^{5.4}} = 0$.

For this and because $\lim_{x\to 0} \frac{R(x;x^{1.4})}{x^{5.4}} = 0$ (since $J^6 R = 0$) we have proved that g has a maximum in O when restricted to $(x;x^{1.4})$.

Of course, this shows remark 1. :▷

Remark 2 f has a local minimum in O.

◁ :In fact, if $f(x_0;y_0) = 0$, with $(x_0;y_0) \neq O$, it's obvious that $y_0 > 0$ e $x_0 \neq 0$, then we have

$$P(x_0;y_0) = y_0(y_0^3 - x_0^4) = -x_0^{5.2} < 0\,.$$

So $y_0 < x_0^{\frac{4}{3}}$ (†).

Now choose $\epsilon > 0$ small enough to do $x^{5.2} > x^{\frac{16}{3}}$, if $0 < |x| < \epsilon$.

Finally we see that if $|x_0| < \epsilon$ by (†) we have $y_0 x_0^4 < x_0^{\frac{16}{3}} < x_0^{5.2}$, and this is a clear contradiction whith our assumption $f(x_0;y_0) = 0$, soon f has a local minimum in O. :▷

Remark 3 In R^2 if first non null jet is of order 2 this annomalous behaivor can't happen, in fact we have the

Lemma If $f \in J(k;2)$ $k \geq 2$ with $J^2 f \neq 0$ then if f is not k-decidible then there are two polynomials P_1 and P_2 such that $J^k f = J^k P_i$ $i \in \{1;2\}$, one of P_i has an extreme and other has saddle in O.

Notice that the lemma is false in R^3, to see this take $f(x;y;z) = z^2 + P(x;y)$ where P is the polynomial given in Remark 1.

◁ :Really this is a very simple application of spliting lemma.

We observe first that if a germ h has k-jet null the same is true for $h \circ \psi$, where ψ is a germ of C^∞ difeomorphism of (R^2, O) into itself. This shows that if h is k-decidible the same is true for $h \circ \psi$ (this is true if $\psi \in J(k)$).

Of course, we need to proof the lemma only by $J^k f$, then we can assume that f is a polynomial of degree k.

If f isn't 2-decidible and $J^2 f \neq 0$ we can suppose that $J^2 f$ has a non strict minimum at O, then by Milnor spliting lemma (see [2]) there is a change of coordinates C^∞ in R^2 such that in new coordinates $f(\xi;\eta) = \xi^2 + g(\eta)$ where g is a C^∞ function of one variable with $J^2 g = 0$.

Then $g \in J(k) \subset C^\infty$ and $J^k g = 0$.

In fact, this is obvious since f isn't k-decidible, and this doesn't depend on coordinates as we have already observed.

Now take $Q_1(\xi;\eta) = -J^{k+1}g + \eta^{k+1}$ and $Q_2(\xi;\eta) = -J^{k+2}g + \eta^{k+2}$ we see that $J^kQ_i = 0$ and one of the functions $f + Q_i$ has a minimum while the other has a saddle in O and both are (k+i)-decidible.

Finally return to the old variables $(x;y)$, it's clear now that J^kQ_i is null by the initial observation we have do, and the same observation shows that $f + Q_i$ is (k+i)-decidible, then $P_i = J^{k+i}(f + Q_i)$ are the desired polynomials. :▷

Remark 4 Note that if $f \in J(2)$ and f isn't 2-decidible than we have two polynomials P_1 and P_2 one of them has saddle in and other has extreme in O and $J^2P_i = J^2f$.

To see this is enough remember that then $J^1f = 0$ and J^2f is a homogeneous polynomial of degree 2 semi-defined (otherwise f would be 1 or 2-decidible).

So $P_i = J^2P + (-1)^i(x^4 + y^4)$ are the desird polynomials.

Remark 5 It's easy to see by similar arguments that $P(x;y) = (y^4 - x^3)^2 + x^2y^{12}$ and $f(x;y) = P(x;y) - x^{14.2}$ given another exemple of same kind that Remark 2, where P has a minimum 14-decidible in world $J(15)$ but it isn't 14-decidible.

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References :

[1] Barone-Netto, A. - Jet-Detectable Extrema - *Proc. of American Math. Soc.* - vol 92, n. 4 - pp 604-608;

[2] Milnor, J. - *Morse Theory* - Annals of Math. Studies, n. 51 - Priceton Univ. Press - 1973.