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THE COMPLETE PRE-ORDER POLYTOPE: FACETS AND SEPARATION PROBLEM

Maria Angela M.C.Gurgel
Yoshiko Wakabayashi

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Maria Angela M.C. Gurgel
Yoshiko Wakabayashi

Instituto de Matemática e Estatística
Universidade de São Paulo
Caixa Postal 20570 - Ag. Jardim Paulistano
01452-990 - São Paulo - Brazil

Abstract

Let $D_n = (V_n, A_n)$ be the complete digraph on n nodes. A subset $A \subseteq A_n$ is called a *complete pre-order* of D_n if $H = (V_n, A)$ is a total transitive subdigraph of D_n (i.e., uv or $vu \in A$, for any two distinct vertices u, v in V_n; and if uv and $vw \in A$ then $uw \in A$ for any three distinct vertices u, v, w in V_n). Given weights $w_a \in \mathbb{R}$ for all $a \in A_n$, the *complete pre-order problem* is to find a complete pre-order $A \subseteq A_n$ such that $w(A) := \sum_{a \in A} w_a$ is as large as possible. This problem is known to be $\mathcal{NP}$-hard. Let $\mathcal{P}^n$ denote the *complete pre-order polytope*, i.e., $\mathcal{P}^n$ is the convex hull of the incidence vectors of all complete pre-orders of D_n. We show that various classes of inequalities like triangles, pentagons, heptagons and semi-k-fences are facet-defining for $\mathcal{P}^n$. We also show that the separation problem for the class of semi-k-fences is $\mathcal{NP}$-complete.

Keywords: Facets of polytopes, complete pre-order, transitive digraph, separation problem, $\mathcal{NP}$-complete.

1. Introduction and notation

We mention here a few concepts in graph theory with the purpose of establishing our notation. The other basic concepts not mentioned in this section can be found in Bondy and Murty (1976).

A *directed graph* or *digraph* $D = (V, A)$ consists of a finite nonempty set V of *nodes* and a set A of *arcs* which are ordered pairs of distinct elements of V. An arc $\alpha = (u, v)$ will also be denoted by uv or u, v.

If H is a subdigraph of a digraph $G = (V, A)$, we use the notation $H \subseteq G$. For $X \subseteq V$ ($X \subseteq A$), $G[X]$ denotes the subdigraph of G spanned by X. If $X \subseteq V$, $G - X$ denotes the subdigraph obtained from G, by deleting the set X. To simplify the notation, $D - v$ denotes the subdigraph $D - X$, if $X = \{v\}$, $v \in V$.

If H is a digraph (graph) sometimes we may denote its node set by VH and its arc set by AH.

A *triangle* in a digraph G is a set consisting of three arcs of the form uv, vw and uw.

The complete digraph on n nodes is denoted by $D_n = (V_n, A_n)$. If $\alpha = (u, v)$ is an arc of D_n, $\overline{\alpha}$ denotes the *reverse arc* of α, i.e., $\overline{\alpha} = (v, u)$. If $B \subseteq A_n$, $\overline{B}$ denotes the set $\{\overline{\alpha} \mid \alpha \in B\}$.

A nonempty sequence of arcs $P = \langle (v_1, v_2), \ldots, (v_{k-1}, v_k) \rangle$ in a digraph $D = (V, A)$ such that $v_i \neq v_j$ for $i \neq j$ is called a $(v_1 - v_k)$-*directed path*. If P is a $(v_1 - v_k)$-directed path and $(v_k, v_1) \in A$ then $C = \langle (v_1, v_2), \ldots, (v_{k-1}, v_k), (v_k, v_1) \rangle$ is called a *directed cycle*. The *length* of a path (cycle) is the number of its arcs. A *directed k-path* (*k-cycle*) is a path (cycle) of length k; it will be sometimes denoted by the sequence of its nodes, such as $\langle v_1, \ldots, v_k \rangle$. For convenience, if C is a cycle we also denote by C the set of arcs in C and by VC the set of nodes in C.

A binary relation on a set S, $R \subseteq S \times S$, is called

- *total* if $(a, b) \in R$ or $(b, a) \in R$ for all $a, b \in S$, $a \neq b$;
- *anti-symmetric* if $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in S$;
- *transitive* if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in S$, $a \neq b \neq c \neq a$.

Let $D = (V, A)$ be a digraph and $B \subseteq A$. We say that B is a *linear order* (*complete* *pre-order*) of D if B is an anti-symmetric, transitive and total (a transitive and total) binary relation on V.

If $D = (V, A)$ is a digraph with n nodes and $B \subseteq A$ is a linear order of D, then there is precisely one directed n-path in D whose arcs are in B (and whose transitive closure is B). The sequence of nodes of this path, say $O := \langle u_1, \ldots, u_n \rangle$, is another notation for this linear order. We also denote by $A(O)$ the set B. Thus, for the sequence O above, $A(O) = \{(u_i, u_j) \mid 1 \leq i < j \leq n\}$.

In this paper we will be concerned with the study of the facets of the polytope associated with the complete pre-order problem. For this purpose let us mention briefly the concepts of polyhedral theory we need in the sequel. Other related concepts can be found in Grünbaum (1967) and Schrijver (1986).

If $S \subseteq \mathbb{R}^m$ then *conv* S denotes the convex hull of S, that is, the set of elements in $\mathbb{R}^m$ obtained by a convex combination of finitely many elements in S.

A *polyhedron* $P \subseteq \mathbb{R}^m$ is the intersection of finitely many halfspaces of $\mathbb{R}^m$. A *polytope* is a bounded polyhedron, or equivalently, the convex hull of finitely many points. The *dimension* of a polyhedron P, denoted by *dim (P)*, is the maximum number of affinely independent points in P minus one. A polyhedron $P \subset \mathbb{R}^m$ is called *full-dimensional* if $\dim(P) = m$.

Let $P \subseteq \mathbb{R}^m$ be a polyhedron. An inequality $a^T x \leq a_0$ ($a \in \mathbb{R}^m$, $a_0 \in \mathbb{R}$) is *valid* for P if $P \subseteq \{x \in \mathbb{R}^m \mid a^T x \leq a_0\}$. If $a^T x \leq a_0$ is valid for P, then $F_a = \{x \in P \mid a^T x = a_0\}$ is the *face* (of P) *induced* or *defined* by $a^T x \leq a_0$. A valid inequality $a^T x \leq a_0$ defines a *facet* of P if $F_a \neq \emptyset$ and $\dim(F_a) = \dim(P) - 1$. If $a^T x \leq a_0$ is valid and F_a defines a *facet* of P, we say that $a^T x \leq a_0$ is *facet-defining* (for P).

For two sets A and B, $A \Delta B$ denotes their *symmetric difference*, that is, $A \Delta B := (A \backslash B) \cup (B \backslash A)$.

2. The complete pre-order polytope $\mathcal{P}^n$

Let $D_n = (V_n, A_n)$ be the complete digraph on n nodes, and set

$$\mathcal{A} := \{A \subseteq A_n \mid A \text{ is a complete pre-order of } D_n\} .$$

Given weights w_a for each arc $a \in A_n$, the *complete pre-order problem* (CPO, for short) is to solve $\max\{w(A) \mid A \in \mathcal{A}\}$, that is, to find a complete pre-order of maximum weight.

This problem - known to be $\mathcal{NP}$-hard (Wakabayashi (1992) - will be studied here from a polyhedral point of view. For a more detailed and complete study of this problem the reader should refer to Gurgel (1992).

Let $\mathbb{R}^m$, $m = |A_n| = n(n-1)$, denote the real vector space where every component of a vector $x \in \mathbb{R}^m$ is indexed by an arc $uv \in A_n$. For every arc set $A \subseteq A_n$, we denote by x^A the *incidence vector of* A (defined as $x_a^A = 1$ if $a \in A$ and $x_a^A = 0$ if $a \in A_n \backslash A$). The notation $x(A)$ stands for $\sum_{a \in A} x_a$.

The *complete pre-order polytope* $\mathcal{P}^n$ is the convex hull of the incidence vectors x^A of the complete pre-orders A of D_n, i.e,

$$\mathcal{P}^n = \text{conv}\,\{x^A \in \{0,1\}^{A_n} \mid A \text{ is a complete pre-order of } D_n\} .$$

Since every vertex of $\mathcal{P}^n$ corresponds to a complete pre-order and vice-versa, it is immediate that CPO can be viewed as the linear program

$$\begin{cases} \text{maximize} & w^T x \\ \text{subject to} & x \in \mathcal{P}^n . \end{cases}$$

In order to be able to apply linear programming techniques to solve the complete pre-order problem, we need a description of $\mathcal{P}^n$ by means of a system of linear inequalities. Since CPO is $\mathcal{NP}$-hard, it is unlikely (cf. Papadimitriou (1984)) that a complete and 'good' description of $\mathcal{P}^n$ will be found.

We present in this paper a partial characterization of $\mathcal{P}^n$, by exhibiting several classes of facet-defining inequalities for this polytope.

Initially, let us formulate CPO as an integer linear programming problem. Since $\mathcal{P}^n$ is contained in the unit hypercube, the trivial inequalities

$$0 \leq x_a \leq 1 \quad \text{for all } a \in A_n$$

are valid. Moreover, for all $u, v \in V_n$, the inequality below, called **total-inequality**,

$$x_{uv} + x_{vu} \geq 1$$

is clearly valid for $\mathcal{P}^n$. Also, if A is a complete pre-order of D_n, and uv and vw are two arcs of A such that $u \neq w$, then the arc uw must be in A. Thus, for every *triangle* $\{uv, vw, uw\}$ of D_n, the **triangle-inequality**

$$x_{uv} + x_{vw} - x_{uw} \leq 1$$

is satisfied by the incidence vector of any complete pre-order and hence is valid for $\mathcal{P}^n$.

Consider now the polytope

$$\begin{aligned} Q^n := \Big\{ x \in \mathbb{R}^{A_n} \mid\ & 0 \leq x_a \leq 1 && \text{for all } a \in A_n\ , \\ & x_{uv} + x_{vu} \geq 1 && \text{for all } u, v \text{ in } V_n\ , \\ & x_{uv} + x_{vw} - x_{uw} \leq 1 && \text{for all } u, v, w \text{ in } V_n \Big\}\ . \end{aligned}$$

It is easy to see that $\mathcal{P}^n \subseteq Q^n$ and that the integral points of Q^n coincide with the incidence vectors of the complete pre-orders of D_n. Therefore, $\mathcal{P}^n = \text{conv}\{x \in Q_n \mid x \text{ integral}\}$. Thus, we have the following integer linear programming formulation of CPO:

$$\begin{cases} \text{maximize} & w^T x \\ \text{subject to} & x \in Q^n \\ & x \text{ integral.} \end{cases}$$

The next result shows that $\mathcal{P}^n$ is full-dimensional, which implies that the facet-defining inequalities are unique up to a multiplication by a positive constant.

Theorem 1. For $n \geq 2$, $\dim \mathcal{P}^n = n(n-1)$.

Proof. Let $a^T x = b$ be a linear equation such that $\mathcal{P}^n \subseteq \{x \in \{0,1\}^{A_n} \mid a^T x = b\}$. Let $\alpha = (u,v)$ be an arc of A_n and $O = \langle v, u, w_1, \ldots, w_s \rangle$ be a linear order of D_n, where $\langle w_1, \ldots, w_s \rangle$ is any linear order of $D_n - \{u,v\}$. Since $A(O)$ and $A(O) \cup \{\alpha\}$ are complete pre-orders of D_n, it follows that

$$\sum_{\beta \in A(O)} a_\beta x_\beta = \sum_{\beta \in A(O)} a_\beta x_\beta + a_\alpha x_\alpha = b \ .$$

Thus, $a_\alpha = 0$ and therefore a is a zero vector. ■

3. Facets of $\mathcal{P}^n$

We present first a useful lifting theorem which shows that inequalities that are facet-defining for $\mathcal{P}^n$ and satisfy a certain condition can be 'lifted' to define a facet of $\mathcal{P}^m$, $m > n$.

In the next theorem we assume that the node set of D_n is $V_n = \{1, 2, \ldots, n\}$.

Theorem 2 (Lifting Theorem). Let $a^T x \leq a_0$ be a valid inequality for $\mathcal{P}^n$, $n \geq 2$. Define $\hat{a} \in \mathbb{R}^{n(n+1)}$ as

$$\hat{a}_\alpha = a_\alpha \ , \quad \text{if } \alpha \in A_n \ ;$$

$$\hat{a}_{i,n+1} = \hat{a}_{n+1,i} = 0 \ , \quad \text{for } i = 1, \ldots, n \ .$$

Then,

(i) $\hat{a}^T x \leq a_0$ is valid for $\mathcal{P}^{n+1}$;

(ii) If $a^T x \le a_0$ is facet-defining for $\mathcal{P}^n$ and there exists a linear order $A \subseteq A_n$ such that $a^T x^A = a_0$, then $\hat{a}^T x \le a_0$ is facet-defining for $\mathcal{P}^{n+1}$.

Proof. (i) It is not difficult to conclude that the inequality $\hat{a}^T x \le a_0$ is valid for $\mathcal{P}^{n+1}$.

(ii) Let $F_{\hat{a}}$ be the face of $\mathcal{P}^{n+1}$ defined as

$$F_{\hat{a}} = \{x \in \mathcal{P}^{n+1} \mid \hat{a}^T x = a_0\} .$$

Suppose that $\hat{b}^T x \le b_0$ defines a facet of $\mathcal{P}^{n+1}$ such that

$$F_{\hat{a}} \subseteq F_{\hat{b}} = \{x \in \mathcal{P}^{n+1} \mid \hat{b}^T x = b_0\} . \tag{1}$$

Let $O = \langle i_1, i_2, \ldots, i_n \rangle$ be a linear order of D_n such that $a^T x^{A(O)} = a_0$. For each $j \in \{1, \ldots, n\}$, let O_1^j and O_2^j be the following linear orders of D_{n+1}:

$$\begin{aligned} O_1^j &= \langle i_1, \ldots, i_j, n+1, i_{j+1}, \ldots, i_n \rangle \quad \text{and} \\ O_2^j &= \langle i_1, \ldots, i_{j-1}, n+1, i_j, \ldots, i_n \rangle . \end{aligned}$$

Consider now the arc sets $A_i^j = A(O_i^j)$, $i = 1, 2$. Since $a^T x^{A(O)} = a_0$ and A_i^j $(1 \le i \le 2, 1 \le j \le n)$ is obtained from $A(O)$ by adding arcs adjacent to $n+1$, it follows that

$$\hat{a}^T x^{A_1^j} = \hat{a}^T x^{A_2^j} = a_0 \quad \text{for all } j \in \{1, \ldots, n\} .$$

Since $F_{\hat{a}} \subseteq F_{\hat{b}}$, then

$$\hat{b}^T x^{A_1^j} = \hat{b}^T x^{A_2^j} = b_0 ,$$

and therefore $\hat{b}_{n+1,i_j} = \hat{b}_{i_j,n+1}$ for all $j \in \{1, \ldots, n\}$.

In order to show that $\hat{b}_{n+1,i_j} = 0$, suppose that $j \in \{1, \ldots, n\}$ and consider $A_3^j = A_1^j \cup \{(n+1, i_j)\}$. Then A_1^j and A_3^j are complete pre-orders of D_{n+1} which satisfy $\hat{a}^T x^{A_1^j} = \hat{a}^T x^{A_3^j} = a_0$. Since $F_{\hat{a}} \subseteq F_{\hat{b}}$, it follows that $\hat{b}^T x^{A_1^j} = \hat{b}^T x^{A_3^j} = b_0$ and therefore $\hat{b}_{n+1,i_j} = 0$. Thus $\hat{b}_{i_j,n+1} = \hat{b}_{n+1,i_j} = 0$ for all $j \in \{1, \ldots, n\}$. This implies that $b^T x \le b_0$ is valid for $\mathcal{P}^n$, where b is the restriction of $\hat{b}$ to the arc set A_n.

Now using (1), we conclude that

$$\{x \in \mathcal{P}^n \mid a^T x = a_0\} \subseteq \{x \in \mathcal{P}^n \mid b^T x = b_0\} \ .$$

Since $a^T x \leq a_0$ induces a facet of $\mathcal{P}^n$ and this polytope is full-dimensional, it follows that $b^T x \leq b_0$ induces a facet of $\mathcal{P}^n$. Then, there exists a real non-negative μ such that $a^T = \mu b^T$. Thus $\hat{a}^T = \mu \hat{b}^T$, since $\hat{a}_{i,n+1} = \hat{a}_{n+1,i} = \hat{b}_{i,n+1} = \hat{b}_{n+1,i} = 0$ for $i \in \{1, \ldots, n\}$. Therefore $\hat{a}^T x \leq a_0$ induces a facet of $\mathcal{P}^{n+1}$. ■

3.1 Trivial facets

Using the lifting theorem, it is easy to prove that

$$x_a \leq 1 \ , \qquad \text{for any } a \in A_n \ ,$$

$$x_a + x_{\bar{a}} \geq 1 \ , \qquad \text{for any } a \in A_n$$

and $x_{uv} + x_{vw} - x_{uw} \leq 1$, for any triangle $\{uv, vw, uw\}$ of D_n are facet-defining inequalities for $\mathcal{P}^n$, $n \geq 3$. On the other hand, the trivial valid inequality for $\mathcal{P}^n$,

$$x_a \geq 0 \ , \quad \text{for any } a \in A_n \ ,$$

is not a facet-defining inequality for this polytope, since it is the sum of $-x_{\bar{a}} \geq -1$ and $x_a + x_{\bar{a}} \geq 1$.

3.2 Pentagon-inequalities

Let C be the arc set of a directed cycle of D_n, say $C = \langle a_1, \ldots, a_k \rangle$, where $a_i = v_i v_{i+1}$ for $i = 1, \ldots, k-1$ and $a_k = v_k v_1$. Then the set

$$CO := \{v_i v_{i+2} \in A_n \mid i = 1, \ldots, k-2\} \cup \{v_{k-1} v_1, v_k v_2\}$$

is called the set of *2-chords* of C. For any directed cycle $C \subset A_n$ of length at least 5,

$$x(C) - x(CO) \leq \left\lfloor \frac{1}{2} |C| \right\rfloor$$

is called the *2-chorded cycle inequality* (induced by C) and

$$x(C) - x(CO) + x(\overline{C}) - x(\overline{CO}) \leq \left\lfloor \frac{1}{2}|C| \right\rfloor$$

is called the *double 2-chorded cycle inequality* (induced by C).

The 2-chorded cycle inequality induced by a directed 5-cycle C of D_n,

$$3x(C) - 3x(CO) + x(\overline{C}) - x(\overline{CO}) \leq 4 ,$$

is called *pentagon-inequality.*

In order to show that the pentagon-inequalities are facet-defining for $\mathcal{P}^n$, $n \geq 5$, let us first prove the following simple lemma.

Lemma 3. Let C be a directed k-cycle of D_n, $5 \leq k \leq n$. Then the 2-chorded cycle inequality induced by C,

$$x(C) - x(CO) \leq \left\lfloor \frac{1}{2}|C| \right\rfloor$$

and the double 2-chorded cycle inequality induced by C,

$$x(C) - x(CO) + x(\overline{C}) - x(\overline{CO}) \leq \left\lfloor \frac{1}{2}|C| \right\rfloor$$

are valid for $\mathcal{P}^n$, $n \geq 5$.

Proof. From the lifting theorem, it suffices to prove for $k = n$. Suppose that $C = \langle(1,2),(2,3),\ldots,(n-1,n),(n,1)\rangle$. Then $CO = \{(1,3),(2,4),\ldots,(n-1,1),(n,2)\}$. For each $a \in CO$, let b and c be the two arcs in C such that $\{a,b,c\}$ is a triangle in D_n. Consider the associated triangle-inequality $x_b + x_c - x_a \leq 1$. Adding up, for all $a \in CO$, these triangle-inequalities we obtain

$$\sum_{a \in CO} (x_b + x_c - x_a) = 2x(C) - x(CO) \leq |CO| = |C| .$$

Since $-x(CO) \leq 0$ is valid for $\mathcal{P}^n$, adding this inequality to the latter one and then dividing by 2, we obtain that $x(C) - x(CO) \leq \frac{1}{2}|C|$ is valid for $\mathcal{P}^n$. Since for every vertex of $\mathcal{P}^n$ the left-hand side of this inequality is an integer, we can round the right-hand side down to the next integer and obtain the validity of the 2-chorded cycle inequality induced by C.

In the same way as we obtained the inequality $2x(C) - x(CO) \leq |C|$, adding up triangle-inequalities we obtain the the inequality $2x(\overline{C}) - x(\overline{CO}) \leq |C|$.

From the total-inequalities we obtain the validity of $-x(CO) - x(\overline{CO}) \leq -|C|$.

Adding up the three last inequalities, we have

$$2x(C) + 2x(\overline{C}) - 2x(CO) - 2x(\overline{CO}) \leq |C| \ .$$

Thus the validity of the double 2-chorded inequality induced by C can be obtained by dividing both sides of this last inequality by 2 and round the right-hand side down to the next integer. ■

Theorem 4. The pentagon-inequality

$$3x(C) - 3x(CO) + x(\overline{C}) - x(\overline{CO}) \leq 4 \ ,$$

induced by a directed 5-cycle C of D_n, as defined above, is facet-defining for $\mathcal{P}^n$, $n \geq 5$.

Proof. Let $a^T x \leq a_0$ denote the pentagon-inequality induced by C. We prove first the following assertions:

(i) $a^T x \leq a_0$ is valid for $\mathcal{P}^5$;

(ii) $a^T x \leq a_0$ is facet-defining for $\mathcal{P}^5$.

(i) This proof will be done by combining (and rounding down) valid inequalities for $\mathcal{P}^5$. From the total-inequalities, we obtain the validity of

$$- x(CO) - x(\overline{CO}) \leq -5 \ . \qquad (1)$$

From the triangle-inequalities, we obtain the validity of

$$2x(C) - x(CO) \leq 5 \qquad \text{and} \tag{2}$$

$$x(\overline{C}) + x(\overline{CO}) - x(CO) \leq 5 \ . \tag{3}$$

To simplify, denote by (4) and (5) the 2-chorded 5-cycle and the double 2-chorded 5-cycle inequalities induced by C, i.e.,

$$x(C) - x(CO) \leq 2 \qquad \text{and} \tag{4}$$

$$x(C) - x(CO) + x(\overline{C}) - x(\overline{CO}) \leq 2 \ . \tag{5}$$

Considering the following combination of these inequalities, 2.(1)+3.(2)+(3)+(4)+2.(5), we obtain the inequality $9x(C) - 9x(CO) + 3x(\overline{C}) - 3x(\overline{CO}) \leq 16$. By dividing this inequality by 3 and rounding down we obtain

$$3x(C) - 3x(CO) + x(\overline{C}) - x(\overline{CO}) \leq 5 \ . \tag{6}$$

By dividing (5) + (6) by 2 and rounding down we obtain

$$2x(C) - 2x(CO) + x(\overline{C}) - x(\overline{CO}) \leq 3 \ . \tag{7}$$

The sum 2.(1)+3.(2)+(3)+(4)+(6)+4.(7) results in the inequality $18x(C) - 18x(CO) + 6x(\overline{C}) - 6x(\overline{CO}) \leq 29$. The validity of the pentagon-inequality for $\mathcal{P}^5$ is obtained by dividing both sides of this last inequality by 6 and rounding down.

(ii) Suppose, w.l.o.g., that $C = \langle(1,2),(2,3),(3,4),(4,5),(5,1)\rangle$. Consider all the additions (for nodes) taken modulo 5. Denote by F_a the face defined by $a^T x \leq a_0$ and let F_b be a facet of $\mathcal{P}^5$ such that

$$F_a = \{x \in \mathcal{P}^5 \mid a^T x = a_0\} \subseteq F_b = \{x \in \mathcal{P}^5 \mid b^T x = b_0\} \ .$$

We will prove that there exists $\lambda \in \mathbb{R}^+$ such that $b = \lambda a$.

First, for each $i \in VC$, let O_i be the linear order of D_5 given by $\langle i, i+1, i+2, i+3, i+4 \rangle$. Consider the following sets of arcs:

$$\begin{aligned} M_i &= A(O_i) \cup \{(i+1, i), (i+3, i+2)\} \quad \text{and} \\ M_i' &= A(O_i) \cup \{(i+1, i), (i+4, i+3)\} \ . \end{aligned}$$

It is easy to see that

$$x^{M_i}, \ x^{M_i'} \in F_a \subseteq F_b \quad \text{for every } i \in VC \ .$$

Therefore

$$b^T x^{M_i} = b^T x^{M_i'} \quad \text{for every } i \in VC \ .$$

Since $M_i \Delta M_i' = \{(i+3, i+2), (i+4, i+3)\}$, we can conclude from the above results that $b_{i+3,i+2} = b_{i+4,i+3}$ for $i = 1, \ldots, 5$. Thus there exists $\lambda \in \mathbb{R}$ such that $b_\alpha = \lambda$ for every $\alpha \in \overline{C}$.

Consider now, for each $i \in VC$, the following arc set:

$$M_i'' = M_i \cup \{(i+4, i+3), (i+4, i+2)\} \ .$$

From $x^{M_i''} \in F_a \subseteq F_b$, we obtain $b^T x^{M_i} = b^T x^{M_i''}$. Thus $b_{i+4,i+2} = -b_{i+4,i+3} = -\lambda$. From this, we can easily conclude that $b_\alpha = -\lambda$ for all $\alpha \in \overline{CO}$.

Analogously, we can obtain $b_\beta = \mu$ for all $\beta \in C$ and $b_\beta = -\mu$ for all $\beta \in CO$, for some $\mu \in \mathbb{R}$.

The inequality $b^T x \leq b_0$ can be written as

$$\mu x(C) - \mu x(CO) + \lambda x(\overline{C}) - \lambda x(\overline{CO}) \leq b_0 = \lambda + \mu \ . \tag{8}$$

Consider now the linear order $O = \langle 2, 5, 3, 1, 4 \rangle$ and $A = A(O)$. It is easy to verify that $x^A \in F_a$. Since $F_a \subseteq F_b$, from (8) we can conclude that $2\mu - 2\lambda = \lambda + \mu$. Thus $\mu = 3\lambda$ and $b^T x \leq b_0$ can be written as $3\lambda x(C) - 3\lambda x(CO) + \lambda x(\overline{C}) - \lambda x(\overline{CO}) \leq 4\lambda$. Therefore $b = \lambda a$ for $\lambda \in \mathbb{R}^+$ and $a^T x \leq a_0$ is a facet of $\mathcal{P}^5$.

Now to conclude the proof it suffices to use the lifting theorem. Note that in the last paragraph we exhibited a linear order $A \subseteq A_5$ such that $a^T x^A = a_0$. ■

Corollary 5. The 2-chorded 5-cycle inequality $x(C) - x(CO) \leq 2$ is not facet-defining for $\mathcal{P}^n$, $n \geq 5$.

Proof. It suffices to note that $x(C) - x(CO) \leq 2$ can be obtained by dividing by 7 the sum of the following valid inequalities for $\mathcal{P}^n$, $n \geq 5$:

$$\begin{array}{rcrcrcrcr} 3x(C) & - & 3x(CO) & + & x(\overline{C}) & - & x(\overline{CO}) & \leq & 4 \\ -x(C) & & & & & + & 2x(\overline{CO}) & \leq & 5 \\ 6x(C) & - & 3x(CO) & & & & & \leq & 15 \\ -x(C) & & & - & x(\overline{C}) & & & \leq & -5 \\ & - & x(CO) & & & - & x(\overline{CO}) & \leq & -5 \end{array}$$

■

Corollary 6. The double 2-chorded 5-cycle inequality $x(C)-x(CO)+x(\overline{C})-x(\overline{CO}) \leq 2$ is not facet-defining for $\mathcal{P}^n$, $n \geq 5$.

Proof. It follows from the fact that this inequality can be obtained by dividing by 4 the sum of the following pentagon-inequalities:

$$3x(C) - 3x(CO) + x(\overline{C}) - x(\overline{CO}) \leq 4$$

$$x(C) - x(CO) + 3x(\overline{C}) - 3x(\overline{CO}) \leq 4 \,.$$

■

3.3 Heptagon-inequalities

We could not extend the pentagon-inequality to any directed cycle with odd length, but the heptagon-inequality to be presented in this section can be seen as an extension of this inequality.

In what follows the symbol $\oplus$ indicates that the addition is taken modulo 7.

Definition 7. Let G be a digraph with $VG = \{1, 2, \ldots, 7\}$ and $AG = C \cup CO \cup \overline{C} \cup \overline{CO} \cup D \cup \overline{D}$, where $C = \langle (1,2), (2,3), \ldots, (6,7), (7,1) \rangle$, $CO = \{(i, i \oplus 2) \mid i = 1, \ldots, 7\}$

and $D = \{(i, i \oplus 3) : i = 1, \ldots, 7\}$. Then the inequality

$$4x(C) - 4x(CO) + 2x(\overline{C}) - 2x(\overline{CO}) + x(D) - x(\overline{D}) \leq 9$$

induced by G is called ***heptagon-inequality.***

Due to its length the proof of the next theorem will be only outlined here, but the interested reader can find the details in Gurgel (1992).

Theorem 8. The heptagon-inequality induced by the digraph G, as in Definition 7,

$$4x(C) - 4x(CO) + 2x(\overline{C}) - 2x(\overline{CO}) + x(D) - x(\overline{D}) \leq 9$$

is facet-defining for $\mathcal{P}^n$, $n \geq 7$.

Proof. Let $a^T x \leq 9$ be the heptagon-inequality induced by G. To prove the theorem we first show the following assertions:

(i) $a^T x \leq 9$ is valid for $\mathcal{P}^7$;

(ii) $a^T x \leq 9$ is facet-defining for $\mathcal{P}^7$.

(i) We were not able to prove the validity of $a^T x \leq 9$ for $\mathcal{P}^7$ by means of combinations of valid inequalities, as we did for the pentagon-inequality. We prove that if A is a complete pre-order of D_7, then $a^T x^A \leq 9$. To this end we write $a^T x \leq 9$ as $2a_1^T x + a_2^T x \leq 9$, where

$$\begin{aligned} a_1^T x &= 2x(C) - 2x(CO) + x(\overline{C}) - x(\overline{CO}) \qquad \text{and} \\ a_2^T x &= x(D) - x(\overline{D}) \, , \end{aligned}$$

and prove the following assertions:

Claim 1: $a_1^T x \leq 6$ and $a_2^T x \leq 7$ are valid for $\mathcal{P}^7$.

Claim 2: If $a_2^T x^A \leq -3$ or $a_2^T x^A \geq 6$ then $a^T x^A \leq 9$.

Claim 3: If $a_1^T x^A = 6$ then $a^T x^A \leq 9$.

Claim 4: If $4 \leq a_2^T x^A \leq 5$ then $a_1^T x^A < 3$.

Claim 5: If $2 \leq a_2^T x^A \leq 3$ then $a_1^T x^A < 4$.

Claim 6: If $0 \leq a_2^T x^A \leq 1$ then $a_1^T x^A < 5$.

Note that once Claims 1, 2 and 3 are proved, we may restrict our attention to the cases in which

$$3 \leq a_1^T x^A \leq 5 \quad \text{and} \quad 0 \leq a_2^T x^A \leq 5 \ .$$

Then, using claims 4, 5 and 6 we obtain the validity of $a^T x \leq 9$ for $\mathcal{P}^7$.

(ii) To prove that $a^T x \leq 9$ is facet-defining for $\mathcal{P}^7$ we use arguments and techniques similar to the ones used for the pentagon-inequality. We leave this proof to the reader.

To conclude the proof of the theorem we use the lifting theorem.

3.4 Semi-k-fence inequalities

Definition 9. A digraph D is a *k-fence* if $|VD| = 2k$, $k \geq 3$, and there is a partition of VD into two subsets $S = \{s_1, \ldots, s_k\}$ and $I = \{i_1, \ldots, i_k\}$ such that $AD = \{(s_j, i_j) \mid j = 1, \ldots, k\} \cup \{(i_j, s_l) \mid 1 \leq j \neq l \leq k\}$. The inequality $x(AD) \leq k^2 - k + 1$ induced by D is called a *simple k-fence inequality.* The arcs (s_j, i_j) are called *pales.*

Grötschel, Jünger and Reinelt (1985) proved that the simple k-fence inequalities are facet-defining for the polytope associated with the linear order problem. It is easy to see that they are not valid for $\mathcal{P}^n$. The semi-k-fence inequalities to be defined in the sequel were obtained by adding up simple k-fence inequalities with some facet-defining inequalities for $\mathcal{P}^n$. As it will be shown, this new class is facet-defining for $\mathcal{P}^n$.

Definition 10. A *semi-k-fence* is a digraph that can be obtained from a k-fence by changing the direction of precisely $k - 1$ pales. In other words, it is a digraph G such that:

(i) $|VG| = 2k$, $k \geq 2$;

(ii) VG can be partitioned into two subsets, say $S = \{s_1, \ldots, s_k\}$ and $I = \{i_1, \ldots, i_k\}$, and there is an index t, $1 \leq t \leq k$, such that

$$AG = \{(s_t, i_t)\} \cup \{(i_l, s_j) \mid 1 \leq j, l \leq k\} \setminus \{(i_t, s_t)\} .$$

The set S is called the set of *superior nodes* of G, I is the set of *inferior nodes* and (s_t, i_t) is the *pivot* of the semi-k-fence. In Figure 1 it is shown a semi-3-fence with pivot (s_3, i_3).

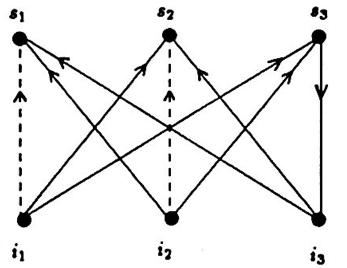

Figure 1. A semi-3-fence with pivot (s_3, i_3)

The arcs (i_j, s_j) such that $j \neq t$ are refered as *negative arcs* of G; the set of these arcs is denoted by AG_-. The complementary set $AG_+ = AG \setminus AG_-$ is the set of the *positive arcs* of G. The inequality

$$x(AG_+) - x(AG_-) \leq k^2 - 2k + 2$$

induced by G is called *semi-k-fence inequality.*

Theorem 11. The semi-k-fence inequality (induced by a semi-k-fence G),

$$x(AG_+) - x(AG_-) \leq k^2 - 2k + 2 ,$$

is facet-defining for $\mathcal{P}^n$, $n \geq 2k$, $k \geq 3$.

Proof. Let $a^T x \leq a_0$ denote the semi-k-fence inequality. We first show that $a^T x \leq a_0$ is valid for $\mathcal{P}^n$, $n \geq 2k$. Suppose that the pivot of G is (s_k, i_k). Thus $AG_- = \{(i_l, s_l) \mid 1 \leq l \leq k-1\}$. Let H be a subdigraph of D_n such that x^{AH} is a vertex of $\mathcal{P}^n$ and suppose that $A^+ = AG_+ \cap AH$ and $A^- = AG_- \cap AH$. Therefore $|A^+| \leq k^2 - k + 1$ and $|A^-| \leq k - 1$. To prove that $x(A^+) - x(A^-) \leq k^2 - 2k + 2$ we analyse three cases:

Case 1: $|A^-| \geq k - 2$

If $|A^+| = k^2 - k + 1$, that is, H has all the positive arcs of G, it follows from the transitivity of AH that $|A^-| = k - 1$ and $x(A^+) - x(A^-) \leq k^2 - 2k + 2$. If $|A^+| \leq k^2 - k$ the last inequality is immediate.

Case 2: $|A^-| = k - 3$

Let $\alpha = (i_l, s_l)$ and $\beta = (i_p, s_p)$ be two arcs in $AG_- \backslash AH$. Since $\overline{\alpha}, \overline{\beta} \in AH$, then $(i_l, s_p) \notin A^+$ or $(i_p, s_l) \notin A^+$. If $(s_k, i_k) \notin A^+$ then $|A^+| \leq k^2 - k - 1$. Otherwise, either $(i_l, s_k) \notin A^+$ or $(i_k, s_l) \notin A^+$. Thus $|A^+| \leq k^2 - k - 1$ and $|A^+| - |A^-| \leq k^2 - 2k + 2$.

Case 3: $|A^-| < k - 3$.

Let α and β be two arcs in $AG_- \backslash AH$, $\alpha = (i_l, s_l)$ and $\beta = (i_p, s_p)$, $l \neq p$. Since $x^{AH} \in \mathcal{P}^n$, then $\overline{\alpha}$ and $\overline{\beta}$ are in H. In this case, either $(i_l, s_p) \notin AH$ or $(i_p, s_l) \notin AH$, otherwise α and β would be in H, a contradiction. We can conclude here that for each two negative arcs of G that do not belong to H, at least one positive arc of G is not in H. Clearly, all these positive arcs are distinct. Since $(k - 1 - |A^-|)$ arcs are in $AG^- \backslash AH$, at least $\binom{k-1-|A^-|}{2}$ positive arcs are not in H. Thus

$$|A^+| \leq k^2 - k + 1 - \binom{k - 1 - |A^-|}{2} .$$

Since $|A^-| < k - 3$, then $k - 1 - |A^-| > 2$ and $\binom{k-1-|A^-|}{2} \geq k - 1 - |A^-|$. Therefore $|A^+| \leq k^2 - k + 1 - (k - 1 - |A^-|)$ and $|A^+| - |A^-| \leq k^2 - 2k + 2$.

Let us prove now that the semi-k-fence inequality is facet-defining for $\mathcal{P}^{2k}$. Assume that $F_b = \{x \in \mathcal{P}^{2k} \mid b^T x = b_0\}$ is a facet of $\mathcal{P}^{2k}$ such that $F_b \supseteq F_a = \{x \in \mathcal{P}^{2k} \mid a^T x = a_0\}$.

Let us prove that exists $\lambda \in \mathbb{R}^+$ such that $b = \lambda a$, in the following order:

(a) $b_\alpha = 0$ for all $\alpha \in \overline{AG}$;

(b) $b_\alpha = 0$ for all $\alpha \in AD_{2k} \backslash (AG \cup \overline{AG})$;

(c) $b_\alpha = \lambda a_\alpha$ for all $\alpha \in AG$.

(a) Let O be the following linear order of D_{2k}:

$$O = \langle i_1, i_2, \ldots, i_{k-1}, s_k, i_k, s_1, s_2, \ldots, s_{k-1} \rangle \ .$$

Then $A(O)$ is a complete pre-order of D_{2k} such that $x^{A(O)} \in F_a \subseteq F_b$. We leave to the reader the proof that $b_\alpha = 0$ for all $\alpha \in \overline{A(O)}$. Since $AG \subseteq A(O)$, it follows that $b_\alpha = 0$ for all $\alpha \in \overline{AG}$.

(b) Consider $S' = S \backslash \{s_k\}$ and $I' = I \backslash \{i_k\}$. Let us show first that $b_{u,v} = 0$ if $u, v \in S'$.

Suppose w.l.o.g. that $u = s_i$ and $v = s_j$, $1 \leq i \neq j \leq k-1$. Consider the following linear order of D_{2k}:

$$\hat{O} = \langle i_1, i_2, \ldots, i_{k-1}, s_k, i_k, s_j, s_i, O' \rangle \ ,$$

where O' is any sequence of nodes in $S' \backslash \{s_j, s_i\}$. Then $A(\hat{O})$ and $A(\hat{O}) \cup \{(s_i, s_j)\}$ are complete pre-orders whose incidence vectors are in F_a. Since $F_a \subseteq F_b$, these vectors are in F_b and

$$0 = b^T x^{A(\hat{O})} - b^T x^{A(\hat{O}) \cup \{(s_i, s_j)\}} \ .$$

Thus $b_{s_i, s_j} = 0$.

With similar arguments one can verify that $b_{u,v} = 0$ if $u, v \in I'$. Thus to complete the proof of (b), it remains to verify the following assertions, for all $r \neq k$ and $l \neq k$:

$$(\mathrm{b}_1) \ \ b_{s_r, s_k} = 0 \ , \qquad (\mathrm{b}_2) \ \ b_{s_k, s_r} = 0 \ ,$$

$$(\mathrm{b}_3) \ \ b_{i_k, i_l} = 0 \ , \qquad (\mathrm{b}_4) \ \ b_{i_l, i_k} = 0 \ .$$

Since (s_k, s_r) and (i_l, i_k) are arcs of $A(O)$, where O is the linear order defined in (a), it follows that $b_{s_r, s_k} = b_{i_k, i_l} = 0$. This verifies ($\mathrm{b}_1$) and ($\mathrm{b}_3$). To verify ($\mathrm{b}_2$), let us construct the digraph H from G, by changing the orientation of the arcs $\alpha = (s_k, i_k)$ and $\beta = (i_j, s_j)$, where $j \neq r$. Note that such an arc β exists, since $k \geq 3$. Thus,

$$|AH \cap AG_+| = k^2 - k \quad \text{and} \quad |AH \cap AG_-| = k - 2 ,$$

and therefore,

$$x(AH \cap AG_+) - x(AH \cap AG_-) = k^2 - 2k + 2 .$$

The set AH can be extended to the following linear order:

$$\hat{O} = \langle O', s_j, i_j, O'', s_r, s_k \rangle ,$$

where O' is any sequence of nodes in $I \backslash \{i_j\}$ and O'' is any sequence of nodes in $S \backslash \{s_j, s_r, s_k\}$. Note that $A(\hat{O})$ and $A(\hat{O}) \cup \{s_k, s_r\}$ are complete pre-orders whose incidence vectors are in F_a and therefore in F_b. Thus,

$$0 = b^T x^{A(\hat{O})} - b^T x^{A(\hat{O}) \cup \{(s_k, s_r)\}} ,$$

which implies that $b_{s_k, s_r} = 0$. With similar arguments one can verify (b$_4$) completing the proof of (b). This last proof is left to the reader.

(c) Suppose that $b_{s_k, i_k} = \lambda$, $\lambda \in \mathbb{R}$ and IND $=: \{1, 2, \ldots, k-1\}$. Let us prove that:

(c$_1$) $b_{i_j, s_j} = -\lambda , \quad \forall j \in \text{IND}$; (c$_2$) $b_{i_j, s_k} = \lambda , \quad \forall j \in \text{IND}$;

(c$_3$) $b_{i_k, s_r} = \lambda , \quad \forall r \in \text{IND}$ and (c$_4$) $b_{i_j, s_r} = \lambda , \quad \forall r, j \in \text{IND}, \ r \neq j$.

(c$_1$) Consider the digraph H obtained from G by changing the orientation of (s_k, i_k) and (i_j, s_j). Then AG and AH can be extended to the linear orderings O_G and O_H, respectively.

$$\begin{aligned} O_G &= \langle i_1, \ldots, i_{k-1}, s_k, i_k, s_1, \ldots, s_{k-1} \rangle \\ O_H &= \langle i_1, \ldots, i_{j-1}, i_k, i_{j+1}, \ldots, i_{k-1}, s_j, i_j, s_1, \ldots, s_{j-1}, s_k, s_{j+1}, \ldots, s_{k-1} \rangle . \end{aligned}$$

Then $A(O_G)$ and $A(O_H)$ are complete pre-orders and $x^{A(O_H)}$, $x^{A(O_G)} \in F_a \subseteq F_b$. From (a) and (b), we can conclude that $b_{i_j,s_j} + b_{s_k,i_k} = O$. Thus $b_{i_j,s_j} = -\lambda$.

(c_2) Consider the digraph H obtained from G by changing the orientations of (i_j, s_k) and (i_j, s_j). Like in case (c_1), consider the linear order O_H obtained by extending the set AH:

$$O_H = \langle i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_{k-1}, s_k, i_k, s_j, i_j, s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_{k-1} \rangle \ .$$

It is not difficult to see that $x^{A(O)} \in F_a \subseteq F_b$. Thus $b^T x^{A(O_H)} - b^T x^{A(O_G)} = 0$, where O_G is the linear order defined in (c_1). Using previous results we can conclude that $b_{i_j,s_k} + b_{i_j,s_j} = 0$. From ($c_1$), it follows that $b_{i_j,s_k} = \lambda$.

(c_3) Consider now the digraph H obtained from G by changing the orientations of (i_k, s_r) and (i_r, s_r). As in (c_2), we can have the linear order:

$$O_H = \langle i_1, \dots, i_{r-1}, i_{r+1}, \dots, i_{k-1}, s_r, i_r, s_k, i_k, s_1, \dots, s_{r-1}, s_{r+1}, \dots, s_{k-1} \rangle \ .$$

Analogously to (c_2), we obtain $b_{i_r,s_r} + b_{i_k,s_r} = 0$. From ($c_1$), it follows that $b_{i_k,s_r} = \lambda$.

(c_4) Suppose w.l.o.g that $r < j$ and consider the digraph H obtained from G changing the orientations of the arcs (s_k, i_k), (i_j, s_r), (i_j, s_j) and (i_r, s_r). The set AH can be extended to the following linear order:

$$\begin{aligned} O_H = \ & \langle i_1, \dots, i_{r-1}, i_{r+1}, \dots, i_{j-1}, i_{j+1}, \dots, i_k, s_r, i_r, s_j, i_j, \\ & s_1, \dots, s_{r-1}, s_{r+1}, \dots, s_{j-1}, s_{j+1}, \dots, s_k \rangle \ . \end{aligned}$$

Note that $A(O_H)$ is a complete pre-order and that all the positive arcs of G, with exception of the arcs (i_j, s_r) and (s_k, i_k), are in $A(O_H)$. Since only the negative arcs (i_j, s_j) and (i_r, s_r) of G are not in $A(O_H)$, it follows that

$$x(A(O_H) \cap AG_+) + x(A(O_H) \cap AG_-) = k^2 - 2k + 2 \ .$$

Thus $x^{A(O_H)} \in F_A$. Since $F_a \subseteq F_b$, it follows that $b^T x^{A(O_G)} - b^T x^{A(O_H)} = 0$, where O_G is the linear order defined in (c$_1$). From (a) and (b), we can conclude that $b_{s_k,i_k} + b_{i_r,s_r} + b_{i_j,s_j} + b_{i_j,s_r} = 0$. Since $b_{i_r,s_r} = b_{i_j,s_j} = -\lambda$ by (c$_1$) and $b_{s_k,i_k} = \lambda$ by hypothesis, it follows that $b_{i_j,s_r} = \lambda$, completing the proof of (c).

From (c$_1$), (c$_2$), (c$_3$) and (c$_4$), we conclude that $b_\alpha = \lambda a_\alpha$, if $\alpha \in AG$, $\lambda \in \mathbb{R}$. From (a), (b) and (c), we conclude that $b = \lambda a$, for $\lambda \in \mathbb{R}^+$. Therefore $a^T x \leq a_0$ is facet-defining for $\mathcal{P}^{2k}$. Since the hypotheses of the lifting theorem are satisfied by this inequality, it follows that $x(AG_+) - x(AG_-) \leq k^2 - 2k + 2$ is a facet-defining for $\mathcal{P}^n$, $n \geq 2k$. ∎

Note that for $n \geq 6$ the class of semi-k-fence inequalities, where $3 \leq k \leq \lfloor \frac{n}{2} \rfloor$, is very large. Its cardinality is $\sum_{k=3}^{\lfloor \frac{n}{2} \rfloor} \left[\binom{n}{2k} \binom{2k}{k} k \right]$.

4. Complexity of the separation problem for the facets of $\mathcal{P}^n$

It is easy to see that the separation problem for the triangle-, the pentagon- and the heptagon-inequalities is polynomial. We are refering here to the straightforward algorithm where one has to check all subsets of 3, resp. 5 and 7 nodes of D_n (and all possible different inequalities induced by them). We shall prove now that the separation problem for the semi-k-fence inequalities is $\mathcal{NP}$-complete. To this end consider the following problem, which we call SFENCE.

Instance: A complete digraph $D_n = (V_n, A_n)$, a weight function $x : A_n \to [0, 1]$ and an integer k, $3 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

Question: Does D_n contain a semi-k-fence D' such that $x(AD'_+) - x(AD'_-) > k^2 - 2k + 2$?

Inspired by a result obtained recently by Müller (1992) — who has shown the $\mathcal{NP}$-

completeness of the separation problem for the simple t-reinforced k-fence inequalities for the acyclic subgraph polytope — we could prove that the problem CLIQUE can be reduced to SFENCE.

The problem CLIQUE, defined below, is well-known to be $\mathcal{NP}$-complete (see Garey and Johnson (1979)).

Instance: A simple connected graph $G = (V, E)$ and an integer k, $k \geq 3$.

Question: Does G contain a clique with at least k nodes?

Theorem 12. The problem SFENCE is $\mathcal{NP}$-complete.

Proof. It is immediate that SFENCE is in $\mathcal{NP}$. We show that CLIQUE can be reduced to SFENCE.

Let $G = (V, E)$ and k be given as an instance of CLIQUE. Suppose that $VG = \{v_1, \ldots, v_{n-1}\}$, $n \geq 4$, and $k \geq 3$. The corresponding instance of SFENCE is constructed as follows: a complete digraph D_p of order $p = 2n$ with node set $VD_p = \{i, \hat{i} : v_i \in VG\} \cup \{n, \hat{n}\}$; a weight function $x : AD_p \to [0, 1]$ defined as

$$x(a) = \begin{cases} \alpha & \text{if } a = (\hat{i}, j) \text{ and } v_i v_j \in AG \ , \\ \alpha & \text{if } a \in \{(\hat{i}, n), (\hat{n}, i) : i = 1, \ldots, n-1\} \ , \\ 1 & \text{if } a = (n, \hat{n}) \ , \\ 0 & \text{otherwise} \ , \end{cases}$$

where $\alpha = (k^2 - k + 1)/k^2$.

Note that $0 < \alpha < 1$ and if $v_i v_j \in AG$ then $x(\hat{i}, j) = x(\hat{j}, i) = \alpha$.

We claim that G contains a clique with at least $k - 1$ nodes if and only if D_p contains a semi-k-fence D' such that

$$x(AD'_+) - x(AD'_-) > k^2 - 2k + 2 \ .$$

Suppose first that G contains a clique with $k - 1$ nodes $v_{i_1}, v_{i_2}, \ldots, v_{i_{k-1}}$. Then the node sets $\{i_1, \ldots, i_{k-1}, n\}$, $\{\hat{i}_1, \ldots, \hat{i}_{k-1}, \hat{n}\}$ determine a semi-k-fence D' of D_p (with pivot

$n\hat{n}$) such that

$$x(AD'_+) - x(AD'_-) = \alpha(k^2 - k) + 1 = k^2 - 2k + 3 - 1/k > k^2 - 2k + 2 .$$

Suppose now that D_p contains a semi-k-fence D' such that $x(AD'_+) - x(AD'_-) > k^2 - 2k + 2$. Let $s_1, \ldots, s_k$ be the superior nodes, and $l_1, \ldots, l_k$ be the inferior nodes and (s_k, l_k) the pivot of D'. Note that the arc $(n, \hat{n}) \in AD'_+$, otherwise, noting that D' has no directed 2-path with weight 2α, we can conclude that

$$x(AD'_+) - x(AD'_-) \leq \alpha(k^2 - k) = k^2 - 2k + 2 - 1/k ,$$

a contradiction. Also $x(s_k, l_k) > 0$ and $x(l_j, s_i) > 0$ for $i \neq j$, otherwise we have

$$x(AD'_+) - x(AD'_-) \leq k^2 - 2k + 2 - 1/k^2 ,$$

a contradiction.

Now it is easy to conclude that $(s_k, l_k) = (n, \hat{n})$, otherwise D' would have $k-1$ arcs in AD'_+, with one endnode l_k or s_k, with weight zero, a contradiction. Thus $\{s_1, \ldots, s_{k-1}\} \subseteq \{1, \ldots, n-1\}$, $\{l_1, \ldots, l_{k-1}\} \subseteq \{\hat{1}, \ldots, \widehat{n-1}\}$ and each pair of arcs (l_j, s_i), (l_i, s_j) for $1 \leq i \neq j \leq k-1$ in D' corresponds to an arc in G. Therefore G has a clique with at least $k-1$ nodes and the proof is complete. ∎

Since we have a partial characterization of $\mathcal{P}^n$ we can use this to design a linear programming based cutting plane algorithm to solve CPO. In such an algorithm it would be natural to consider first the triangle-inequalities, since they are facet-defining and can be tested in polynomial time. Thus, the following question arises naturally: can we solve the separation problem for the semi-k-fence inequalities knowing that the triangle-inequalities are satisfied?

The next theorem shows that the separation problem for the semi-k-fences is $\mathcal{NP}$-complete even in this case.

Theorem 13. The problem SFENCE is $\mathcal{NP}$-complete even if the weight function satisfies the triangle-inequalities.

Proof. Consider the same reduction as described in the proof of Theorem 12, except that we define another weight function, given below:

$$x(a) = \begin{cases} \beta & \text{if } a = (\hat{i}, j) \text{ and } v_i v_j \in AG \ , \\ 1 & \text{if } a \in \{(\hat{n}, i), (\hat{i}, n) : i = 1, \ldots, n-1\} \ , \\ 0 & \text{otherwise,} \end{cases}$$

where $\beta = (k^2 - 4k + 4.1)/(k^2 - 3k + 2)$.

Note that $0.1 < \beta < 1$ and the weight function x satisfies the triangle-inequalities.

For any semi-k-fence $\widehat{D}$ of D_{2n} let us call *k-fence-weight of $\widehat{D}$*, and denote by $x(\widehat{D})$ the difference $x(A\widehat{D}_+) - x(\widehat{AD})$. That is,

$$x(\widehat{D}) := x(\widehat{AD}_+) - x(\widehat{AD}_-) \ .$$

As in the proof of Theorem 12, suppose that G has a clique with $k-1$ nodes $v_{i_1}, \cdots, v_{i_{k-1}}$. Then the node sets $\{i_1, \ldots, i_{k-1}, n\}$, $\{\hat{i}_1, \ldots, \hat{i}_{k-1}, \hat{n}\}$, determine a semi-$k$-fence D' in D_{2n} (with pivot $n\hat{n}$) such that

$$x(D') = \beta[(k-1)^2 - (k-1)] + 2(k-1) = k^2 - 2k + 2.1 > k^2 - 2k + 2 \ .$$

Suppose now that D_{2n} contains a semi-k-fence D' such that $x(D') > k^2 - 2k + 2$. Let $L = \{l_1, \ldots, l_k\}$ be the set of inferior nodes, $S = \{s_1, \ldots, s_k\}$ the set of superior nodes and (s_k, l_k) the pivot of D'.

Note that the subdigraph of D_{2n} spanned by the arcs with positive weights has a bipartition, say (V^-, V^+), such that all its arcs are oriented from V^- to V^+.

Considering the weights defined on the arcs of D_{2n}, it is easy to see that if $\widehat{D}$ is a semi-k-fence of D_{2n} with fence-weight greater than $k^2 - 2k + 2$ then $\widehat{D}$ is of maximum k-fence-weight, $\widehat{D}$ has precisely $2(k-1)$ positive arcs with weight 1, precisely $k^2 - 3k + 2$ positive arcs with weight β and $x(\widehat{D}) = k^2 - 2k + 2.1$.

Suppose by contradiction that $x(s_k, l_k) > 0$. Thus $s_k \in V^-$ and $l_k \in V^+$. Therefore $x(l_i, s_k) = x(l_k, s_i) = 0$ for $1 \leq i \leq k-1$. We have two cases:

Case 1: $s_k = \hat{n}$

In this case D' can have at most k positive arcs with weight 1, since only one arc of AD_+ incident to s_k may have positive weight. As D' has $k^2 - 3k + 2$ positive arcs not incident to the pivot, we have

$$x(D') \leq \beta(k^2 - 3k + 2) + k < k^2 - 2k + 2 \ .$$

Case 2: $s_k \neq \hat{n}$

If $l_k = n$, we can have at most k positive arcs with weight 1, since only one arc of AD'_+ incident to n may have positive weight. The conclusion follows as in Case 1.

Suppose now that $l_k \neq n$. Therefore $x(s_k, l_k) = \beta$. Observe that D' can have at most $2(k-2)$ positive arcs with weight 1 and this can occur if $n \in S$ and $\hat{n} \in L$. Since D' has $k^2 - 3k + 2$ positive arcs not incident to the pivot, we have

$$x(D') \leq \beta(k^2 - 5k + 6) + 2(k-2) + \beta < k^2 - 2k + 2 \ .$$

In both cases we have a contradiction, thus $x(s_k, l_k) = 0$. Since $k^2 - 2k + 2 < x(D') \leq k^2 - 2k + 2.1$ and $\beta > 0.1$, it follows that D' is a semi-k-fence of maximum weight. Therefore $S \subseteq V^+$, $L \subseteq V^-$, $n \in S$, $\hat{n} \in L$ and $x(l_i, s_j) > 0$ for all $1 \leq i \neq j \leq k$. We can conclude that each pair of arcs (l_j, s_i), (l_i, s_j) in D' ($l_i, l_j \in L \backslash \{\hat{n}\}$ and $s_i, s_j \in S \backslash \{n\}$) corresponds an arc in G. Thus G has a clique with at least $k-1$ nodes and the proof is complete. ■

Remark 1. The weight function considered in the last theorem does not satisfy the total-inequalities. It would be interesting to know the complexity status of the problem SFENCE when the weight function satisfies both the triangle- and the total-inequalities.

Remark 2. The construction and the weights defined by Müller (1992) for the *partial fences* can be used to show — for the *linear order polytope* — that the (general) separation problem for the simple k-fence inequalities is $\mathcal{NP}$-complete. One has to consider the complete digraph obtained from the completion of the *partial fence*, and give weights zero to the newly added arcs.

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