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Equivalence between geometric and optimization approaches on computing the power system closest bifurcation

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Abstract. This work aims to demonstrate the equivalence between the geometric approach and the optimization approach when computing the closest bifurcation in electric power systems. By proving this equivalence, we can explore the algorithmic aspects of both approaches to elaborate fast performance methods to fast compute the closest bifurcation in real time.

Keywords. Closest bifurcation, Bifurcation surface, Optimization

1 Introduction

With modern electric power systems operating closer to their limits, voltage stability assessment became an essential security analysis to ensure the system can withstand different operating conditions. These different conditions are defined by parameter variations, such as loading variations. If the system cannot endure such variation, the system may be prone to a voltage collapse.

To assess voltage stability, it is of vital importance to determine how much the system parameters can deviate from its original value without the system collapses. Therefore, one main safety indicator in the study of voltage stability is the computation of the closest bifurcation point, i.e. the closest operating point for which the system loses its stable equilibrium point and collapses.

The closest bifurcation point can be computed by the methods described in [3]. These methods are based on a geometric approach of the problem, dealing specifically with the vector normal to the bifurcation surface in the parameter space. The closest bifurcation point also can be computed by solving an optimization method.

In this paper, we prove the equivalence between the two aforementioned approaches on computing the closest bifurcation point. By proving this equivalence, we can merge the

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two methods to provide high performance algorithms to compute the closest bifurcation in a power system.

2 Power system modelling

On voltage stability studies, we generally model a power system as a system of algebraic equations:

$$\mathbf{f}(\mathbf{x}, \mathbf{p}) = \mathbf{0}\,, \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is a vector of electrical variables (generally voltage phasors) and $\mathbf{p} \in \mathbb{R}^m$ is a vector of system parameters. A solution $\mathbf{x}$ of equation (1) represents an equilibrium point of the differential equations that model the dynamical behavior of the power system.

Function $\mathbf{f}$ usually describes the balance between the injection of power in the system and the power flow in the transmission system:

$$\mathbf{f}(\mathbf{x}, \mathbf{p}) = \mathbf{S}_{\text{spec}}(\mathbf{x}, \mathbf{p}) - \mathbf{S}_{\text{calc}}(\mathbf{x})\,, \tag{2}$$

where the ith element of vetor $\mathbf{S}_{\text{spec}}$ is the complex power injected at bus i of the system, specified by the load and generation models. Similarly, the ith element of vector $\mathbf{S}_{\text{calc}}$ is the complex power injected at bus i of the system, calculated by the transmission system equations. Here, we assume that the transmission system equations do not explicitly depends on vector $\mathbf{p}$.

We emphasize that the number of equations in (1) is equal to the number of variables in $\mathbf{x}$. Thus, assuming that the system operates at the point $(\hat{\mathbf{x}}, \hat{\mathbf{p}})$ satisfying $\mathbf{f}(\hat{\mathbf{x}}, \hat{\mathbf{p}}) = \mathbf{0}$, the Implicit Function Theorem guarantees the existence and uniqueness of a function $\mathbf{x}(\mathbf{p})$ defined in the neighborhood of $\hat{\mathbf{p}}$ satisfying $\mathbf{f}(\mathbf{x}(\mathbf{p}), \mathbf{p}) \equiv \mathbf{0}$), as long as function $\mathbf{f}$ is continuously differentiable and the Jacobian $\partial_{\mathbf{x}}\mathbf{f}$ of $\mathbf{f}$ on $\mathbf{x}$ is nonsingular.

Therefore, in order to a static bifurcation occurs in the system (in which the equilibrium point $\mathbf{x}$ satisfying (1) disappears), the assumptions of the Implicit Function Theorem cannot be satisfied. Hereinafter, we will assume that function $\mathbf{f}$ is continuously differentiable in the entire domain (e.g. by means of a smooth formulation). Thus, every static bifurcation point in the system is a point where $\partial_{\mathbf{x}}\mathbf{f}$ is singular. Thus, any poing satisfying

$$\mathbf{f}(\mathbf{x}, \mathbf{p}) = \mathbf{0}\,, \tag{3a}$$

$$\mathbf{w}^T \partial_{\mathbf{x}}\mathbf{f}(\mathbf{x}, \mathbf{p}) = \mathbf{0}\,, \tag{3b}$$

$$\|\mathbf{w}\| = 1 \tag{3c}$$

is a bifurcation point in the power system. Note that $\partial_{\mathbf{x}}\mathbf{f}$ is singular due to the null eigenvalue associated with left eigenvector $\mathbf{w}$. Also note that transversality conditions for a bifurcation are assumed to be satisfied.

On the m-dimensional space of vetor $\mathbf{p}$, the singularity condition of $\partial_{\mathbf{x}}\mathbf{f}$ defines a $(m-1)$-dimensional surface which we will simply call *bifurcation surface*. On one side of this surface, every vetor $\mathbf{p}$ has a respective $\mathbf{x}$ such that $\mathbf{f}(\mathbf{x}, \mathbf{p}) = \mathbf{0}$. For $\mathbf{p}$ on the other side of this surface, there is no $\mathbf{x}$ that satisfies $\mathbf{f}(\mathbf{x}, \mathbf{p}) = \mathbf{0}$.

3 Closest bifurcation

Suppose the power system operates initially at point $(\mathbf{x}, \mathbf{p} = \mathbf{0})$ (this may be attained by simply shifting the origin of the $\mathbf{p}$-space to the current operating point) and that $\partial_{\mathbf{x}}\mathbf{f}$ is nonsingular at this point (therefore, this point is not a bifurcation point). The closest bifurcation point is the bifurcation point $(\mathbf{x}, \mathbf{p})$ for which $\|\mathbf{p}\|$ is minimal.

In order to compute the closest bifurcation point, we will initially describe the method proposed in [3], which is based on a geometric approach of the problem. Subsequently, we will write the closest bifurcation point as the solution of an optimization problem. Finally, we will prove the equivalence between these two approaches of computing the closest bifurcation point.

3.1 Geometric approach

The method described in [3] to compute the closest bifurcation is based in successive linearizations of the bifurcation surface to iteratively update vector $\mathbf{p}$. This method is better explained graphically.

Suppose we have an initial estimate $\mathbf{u}_1$ of the direction of change in $\mathbf{p}$ that leads to the closest bifurcation point, as shown in Figure 1. By executing a continuation power flow [1, 2] in this direction, we continuously change $\mathbf{p} = \mu\mathbf{u}_1$ by increasing μ from $\mu = 0$ until $\mu = \hat{\mu}_1$. For $\mu > \hat{\mu}_1$, there is no equilibrium $\mathbf{x}$ satisfying $\mathbf{f}(\mathbf{x}, \mu\mathbf{u}_1) = \mathbf{0}$. Therefore, vector $\mathbf{p}_1 = \hat{\mu}_1\mathbf{u}_1$ lies on the bifurcation surface. In order to update our current estimate $\mathbf{u}_1$ of the direction that leads to the closest bifurcation, we linearize the bifurcation surface at point $\hat{\mu}_1\mathbf{u}_1$ and choose the new vector $\mathbf{u}_2$ as the direction that minimizes the distance from the current operating point to this hyperplane. This procedure is shown in Figure 1, where $\mathbf{n}_1$ is the vector normal to the bifurcation surface at $\mathbf{p}_1 = \hat{\mu}_1\mathbf{u}_1$. It can be easily demonstrated that vector $\mathbf{u}_2$ must also be normal to the linearized surface, therefore $\mathbf{u}_2$ and $\mathbf{n}_1$ are collinear.

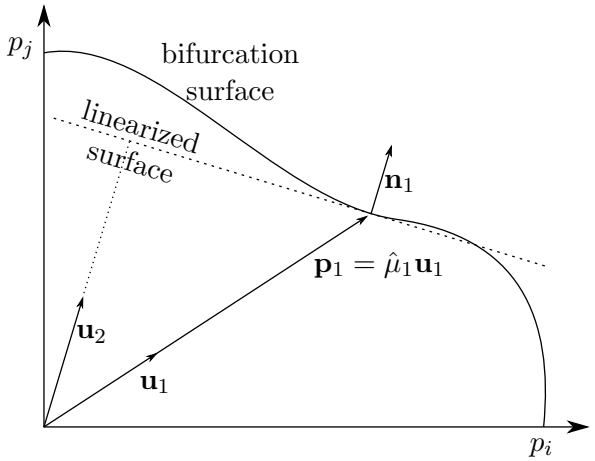

Figure 1: Update on vector $\mathbf{u}$ proposed in [3].

To compute vector $\mathbf{n}_1$, we explore the singularity of $\partial_{\mathbf{x}}\mathbf{f}$. By differentiating (1) at point $(\mathbf{x}_1, \mathbf{p}_1)$, we obtain:

$$\partial_{\mathbf{x}}\mathbf{f}(\mathbf{x}_1, \mathbf{p}_1)d\mathbf{x} + \partial_{\mathbf{p}}\mathbf{f}(\mathbf{x}_1, \mathbf{p}_1)d\mathbf{p} = \mathbf{0}\,. \tag{4}$$

By pre-multiplying (4) by $\mathbf{w}_1^T$, we obtain:

$$\mathbf{w}_1^T\partial_{\mathbf{x}}\mathbf{f}(\mathbf{x}_1, \mathbf{p}_1)d\mathbf{x} + \mathbf{w}_1^T\partial_{\mathbf{p}}\mathbf{f}(\mathbf{x}_1, \mathbf{p}_1)d\mathbf{p} = \mathbf{0}\,. \tag{5}$$

If $\mathbf{w}_1$ is the left eigenvector associated with the null eigenvalue of $\partial_{\mathbf{x}}\mathbf{f}(\mathbf{x}_1, \mathbf{p}_1)$, the first term on (5) vanishes:

$$\mathbf{w}_1^T\partial_{\mathbf{p}}\mathbf{f}(\mathbf{x}_1, \mathbf{p}_1)d\mathbf{p} = \mathbf{0}\,. \tag{6}$$

Thus, the normal vector can be computed by $\mathbf{n}_1 = \partial_{\mathbf{p}}\mathbf{f}(\mathbf{x}_1, \mathbf{p}_1)^T\mathbf{w}_1$.

This iterative procedure converges as shown in Figure 2. Note that this method converges to a vector $\mathbf{p}$ that is normal to the bifurcation surface. This can be written as:

$$\mathbf{p} = k\mathbf{n} = k\partial_{\mathbf{p}}\mathbf{f}^T\mathbf{w}\,. \tag{7}$$

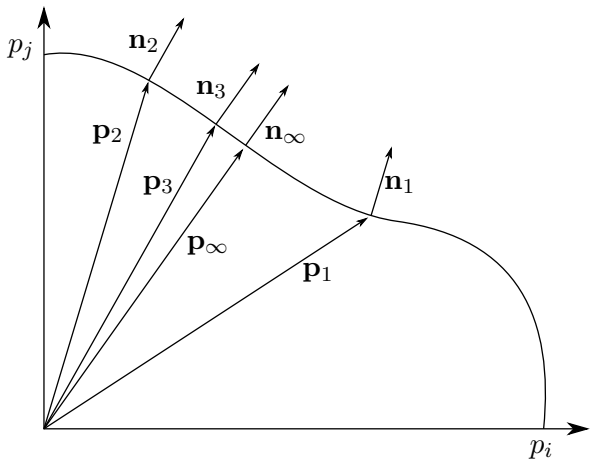

Figure 2: Iterative procedure proposed in [3].

By combining (3) to (7), we formulate a system of nonlinear equations that characterize the closest bifurcation point:

$$\mathbf{f}(\mathbf{x}, \mathbf{p}) = \mathbf{0}\,, \tag{8a}$$

$$\mathbf{w}^T\partial_{\mathbf{x}}\mathbf{f}(\mathbf{x}, \mathbf{p}) = \mathbf{0}\,, \tag{8b}$$

$$\|\mathbf{w}\| = 1\,, \tag{8c}$$

$$\mathbf{p} = k\partial_{\mathbf{p}}\mathbf{f}(\mathbf{x}, \mathbf{p})^T\mathbf{w}\,. \tag{8d}$$

This system has the same number of variables and equations, since (8a), (8b), (8c) and (8d) have the same size as $\mathbf{x}$, $\mathbf{w}$, k and $\mathbf{p}$, respectivelly. The "direct method" proposed in [3] consists of solving this system of equations directly, without recurring to the iterative procedure shown in Figure 2.

3.2 Optimization approach

The problem of computing the closest bifurcation is naturally written as an optimization problem. Since we are searching for the minimum $\|\mathbf{p}\|$ such that $\mathbf{p}$ lies on the bifurcation surface, we are searching for the minimum $\|\mathbf{p}\|$ that satisfies (3). Thus, we want to solve the following problem:

$$\begin{array}{ll} \underset{\mathbf{x},\mathbf{p},\mathbf{w}}{\text{minimize}} & \|\mathbf{p}\| \\ \text{subject to} & \begin{array}{c} \mathbf{f}(\mathbf{x},\mathbf{p}) = \mathbf{0} \\ \partial_{\mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{p})^T\mathbf{w} = \mathbf{0} \\ \|\mathbf{w}\| - 1 = 0 \end{array} \end{array} . \tag{9}$$

The *Lagrangian* for this problem is:

$$\mathcal{L}(\mathbf{x},\mathbf{p},\mathbf{w},\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\lambda_3) = \|\mathbf{p}\| + \boldsymbol{\lambda}_1^T\mathbf{f}(\mathbf{x},\mathbf{p}) + \boldsymbol{\lambda}_2^T\partial_{\mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{p})^T\mathbf{w} + \lambda_3(\|\mathbf{w}\| - 1)\,. \tag{10}$$

where $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$ and λ_3 are the Lagrange multipliers of the first, second and third constraints of problem (9), respectively.

The Karush–Kuhn–Tucker condition [4] for this problem is:

$$\partial_{\mathbf{x}}\mathcal{L} \equiv \boldsymbol{\lambda}_1^T\partial_{\mathbf{x}}\mathbf{f} + \boldsymbol{\lambda}_2^T\partial_{\mathbf{x}}\left(\partial_{\mathbf{x}}\mathbf{f}^T\mathbf{w}\right) = \mathbf{0}\,, \tag{11a}$$

$$\partial_{\mathbf{p}}\mathcal{L} \equiv \partial_{\mathbf{p}}\|\mathbf{p}\| + \boldsymbol{\lambda}_1^T\partial_{\mathbf{p}}\mathbf{f} + \boldsymbol{\lambda}_2^T\partial_{\mathbf{p}}\left(\partial_{\mathbf{x}}\mathbf{f}^T\mathbf{w}\right) = \mathbf{0}\,, \tag{11b}$$

$$\partial_{\mathbf{w}}\mathcal{L} \equiv \boldsymbol{\lambda}_2^T\partial_{\mathbf{x}}\mathbf{f}^T + \lambda_3\partial_{\mathbf{w}}\|\mathbf{w}\| = \mathbf{0}\,, \tag{11c}$$

where we have defined $\partial_{\mathbf{x}}\mathbf{f} \equiv \partial_{\mathbf{x}}\mathbf{f}(\mathbf{x},\mathbf{p})$ and $\partial_{\mathbf{p}}\mathbf{f} \equiv \partial_{\mathbf{p}}\mathbf{f}(\mathbf{x},\mathbf{p})$ to simplify notation.

Equations (3) and (11) can be combined into a system of equations that must be satisfied at the solution of (9). This system of equations has the same number of variables and equations, and thus can be solved using a numerical method (e.g. Newton's method). In the next subsection, we will prove that solving this system of equations is equivalent to solving (8).

3.3 Equivalence

To demonstrate the equivalence between equation (8) and the system of equations formed by (3) and (11), we must solve equation (11), eliminating the Lagrange multipliers and ending up with equation (7).

Initially, by post-multiplying equation (11c) by $\mathbf{w}$, we obtain:

$$\boldsymbol{\lambda}_2^T\partial_{\mathbf{x}}\mathbf{f}^T\mathbf{w} + \lambda_3\partial_{\mathbf{w}}\|\mathbf{w}\|\mathbf{w} = 0\,. \tag{12}$$

Since equation (3b) must hold, we eliminate the first term of (12):

$$\lambda_3\partial_{\mathbf{w}}\|\mathbf{w}\|\mathbf{w} = 0\,. \tag{13}$$

Note that $\partial_{\mathbf{w}}\|\mathbf{w}\|\mathbf{w}$ is simply the directional derivative of $\|\mathbf{w}\|$ in the direction of vetor $\mathbf{w}$, which we will denote by $\partial_{\mathbf{w}\to\mathbf{w}}\|\mathbf{w}\|$:

$$\partial_{\mathbf{w}}\|\mathbf{w}\|\mathbf{w} \equiv \partial_{\mathbf{w}\to\mathbf{w}}\|\mathbf{w}\| \equiv \lim_{\varepsilon\to 0}\frac{\|\mathbf{w}+\varepsilon\mathbf{w}\|-\|\mathbf{w}\|}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{(1+\varepsilon)\|\mathbf{w}\|-\|\mathbf{w}\|}{\varepsilon} = \|\mathbf{w}\|\,. \quad (14)$$

Since (3c) must hold, $\partial_{\mathbf{w}}\|\mathbf{w}\|\mathbf{w}$ must be nonzero, thus (13) implies $\lambda_3 = 0$.
Substituting $\lambda_3 = 0$ in (11c), we have:

$$\partial_{\mathbf{x}}\mathbf{f}\boldsymbol{\lambda}_2 = \mathbf{0}\,, \quad (15)$$

which implies that $\boldsymbol{\lambda}_2$ must lie in the null space of $\partial_{\mathbf{x}}\mathbf{f}$. This matrix generally has only one null eigenvalue (most possible bifurcations in the system have codimension one), with left eigenvector $\mathbf{w}$ and right eigenvector $\mathbf{v}$. Thus, $\boldsymbol{\lambda}_2$ must be written in the following form:

$$\boldsymbol{\lambda}_2 = c\mathbf{v}\,. \quad (16)$$

By post-multiplying equation (11a) by $\mathbf{v}$, we have:

$$\boldsymbol{\lambda}_1^T\partial_{\mathbf{x}}\mathbf{f}\mathbf{v} + \boldsymbol{\lambda}_2^T\partial_{\mathbf{x}}\left(\partial_{\mathbf{x}}\mathbf{f}^T\mathbf{w}\right)\mathbf{v} = 0\,. \quad (17)$$

Since $\mathbf{v}$ is the right eigenvector of $\partial_{\mathbf{x}}\mathbf{f}$, the first term on the above equation vanishes:

$$\boldsymbol{\lambda}_2^T\partial_{\mathbf{x}}\left(\partial_{\mathbf{x}}\mathbf{f}^T\mathbf{w}\right)\mathbf{v} = 0\,. \quad (18)$$

Note that $\boldsymbol{\lambda}_2 = c\mathbf{v}$ and $\partial_{\mathbf{x}}\left(\partial_{\mathbf{x}}\mathbf{f}^T\mathbf{w}\right)\mathbf{v} = \partial_{\mathbf{x}\to\mathbf{v}}\left(\partial_{\mathbf{x}}\mathbf{f}^T\mathbf{w}\right)$. Thus:

$$\begin{aligned} c\mathbf{v}^T\partial_{\mathbf{x}\to\mathbf{v}}\left(\partial_{\mathbf{x}}\mathbf{f}^T\mathbf{w}\right) = c\mathbf{v}^T\partial_{\mathbf{x}\to\mathbf{v}}\left(\partial_{\mathbf{x}}\mathbf{f}\right)^T\mathbf{w} &= c\mathbf{v}^T\partial_{\mathbf{x}}\left(\partial_{\mathbf{x}\to\mathbf{v}}\mathbf{f}\right)^T\mathbf{w} \\ &= c\mathbf{w}^T\partial_{\mathbf{x}}\left(\partial_{\mathbf{x}\to\mathbf{v}}\mathbf{f}\right)\mathbf{v} = c\mathbf{w}^T\partial^2_{\mathbf{x}\to\mathbf{v}}\mathbf{f} = 0\,, \end{aligned} \quad (19)$$

where $\partial^2_{\mathbf{x}\to\mathbf{v}}\mathbf{f}$ measures the concavity of function $\mathbf{f}$ in relation to $\mathbf{x}$ in the direction of vector $\mathbf{v}$. This vector is generally nonzero since there are quadratic terms of $\mathbf{x}$ on $\mathbf{f}$.

In order to $\mathbf{w}^T\partial^2_{\mathbf{x}\to\mathbf{v}}\mathbf{f} = 0$, the left eigenvector $\mathbf{w}$ must be orthogonal to $\partial^2_{\mathbf{x}\to\mathbf{v}}\mathbf{f}$, where both $\mathbf{v}$ and $\mathbf{w}$ refer to the same eigenvalue of $\partial_{\mathbf{x}}\mathbf{f}$. This would represent an atypical situation, thus we assume that generally $\mathbf{w}^T\partial^2_{\mathbf{x}\to\mathbf{v}}\mathbf{f} \neq 0$. Therefore, equation (19) implies $c = 0$ and $\boldsymbol{\lambda}_2 = c\mathbf{v} = \mathbf{0}$.

With $\boldsymbol{\lambda}_2 = \mathbf{0}$, equation (11a) is rewritten in the following form:

$$\boldsymbol{\lambda}_1^T\partial_{\mathbf{x}}\mathbf{f} = \mathbf{0}\,, \quad (20)$$

and since we assume $\partial_{\mathbf{x}}\mathbf{f}$ has only one null eigenvalue, $\boldsymbol{\lambda}_1 = \gamma\mathbf{w}$.

With $\boldsymbol{\lambda}_1 = \gamma\mathbf{w}$ and $\boldsymbol{\lambda}_2 = \mathbf{0}$, equation (11b) becomes:

$$\partial_{\mathbf{p}}\|\mathbf{p}\| + \gamma\mathbf{w}^T\partial_{\mathbf{p}}\mathbf{f} = \mathbf{0}\,. \quad (21)$$

By considering the usual Euclidean norm on $\|\mathbf{p}\|$, $\|\mathbf{p}\| = \sqrt{\mathbf{p}^T\mathbf{p}}$ and equation (21) becomes:

$$\mathbf{p}^T/\|\mathbf{p}\| + \gamma\mathbf{w}^T\partial_{\mathbf{p}}\mathbf{f} = \mathbf{0} \Rightarrow \mathbf{p} + \gamma\|\mathbf{p}\|\partial_{\mathbf{p}}\mathbf{f}^T\mathbf{w} = \mathbf{0}\,. \quad (22)$$

By changing the variable γ to k where $k = -\gamma\|\mathbf{p}\|$:

$$\mathbf{p} = k\partial_{\mathbf{p}}\mathbf{f}^T\mathbf{w}\,, \tag{23}$$

which is identical to (7). Therefore, we have simplified the system of equations (11) to become equation (23). By combining this equation with equation (3), we have the system of equations (8). This proves the equivalence between the direct method described in subsection 3.1 and the optimization problem described in subsection 3.2.

4 Conclusion

In this work, we have demonstrated that the direct method proposed in [3] to compute the closest bifurcation is equivalent to solve an optimization problem by solving its KKT conditions. This proves that the geometric approach is in fact minimizing the distance between the current operating point and a bifurcation point.

By proving the equivalence between the geometric approach and an optimization problem, we may improve the performance of the method by exploiting the algorithmic techniques used in optimization methods. This may enable the method to fast compute of the closest bifurcation, which might to be used online on voltage stability assessment.

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