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EQUIVARIANT NIELSEN COINCIDENCE THEORY

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Abstract

In this paper we develop a equivariant Nielsen coincidence theory for G-maps. We consider G-maps f , h : $V \longmapsto M$ defined on an open invariant subset V of an oriented, connected, closed G-manifold, M, where G is a compact Lie group and the G-action on V is not necessarily free, so that: The G-action preserves orientation, for each isotropy type (H_i) of V, M^{H_i} is orientable and the WH_i-maps f_{H_i}, $h_{H_i} : V_{H_i} \longmapsto M^{H_i}$ preserve the dimensions of the connected components of V_{H_i} and M^{H_i}. We also study the question of minimizing the number of coincidence's orbits.

Resumo

Neste trabalho desenvolvemos uma teoria de Nielsen equivariante para coincidência de G-aplicações. Consideramos G-aplicações $f, h : V \mapsto M$, definidas num subconjunto aberto invariante V de uma G-variedade conexa, fechada, orientável, M, onde G é um grupo de Lie compacto, a G-ação em V é, não necessariamente livre, e tal que: a) A ação de G em M preserva orientação; b) Para cada tipo de isotropia (H_i) de V, M^{H_i} é orientável; c) Para cada tipo de isotropia (H_i) de V, as WH_i-aplicações $f_{H_i}, h_{H_i} : V_{H_i} \mapsto M^{H_i}$ preservam as dimensões das componentes conexas de V_{H_i} e de M^{H_i}. Estudamos, também, a questão de minimizar o número de órbitas de coincidência.

1. Introduction

In the Nielsen fixed point theory, it is well known that if X is a simply connected manifold then $L(f) = 0$ implies that f is deformable to be fixed point free, where $L(f)$ is the Lefschetz number. For the non-simply connected case it follows from

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a classical result of Wecken that $N(f) = 0$ is sufficient to deform f to a fixed point free map when X is a manifold of dimension $\dim X \geq 3$.

Let G a compact Lie group. The problem of equivariantly deforming a G-map to be fixed point free is more complicated.

Fadell and Wong [FW] showed that, some codimension hypothesis, $N(f^H) = 0$ for all $H \leq G$ with Weyl group finite implies that f is G-homotopic to a fixed point free G-map, where $X^H = \{x \in X/hx = x, \forall h \in H\}$ and $f^H = f|_{X^H}$. This result was also proven by Borsari and Gonçalves [BG]. Wilczynski [W] and independently Vidal [Vi] showed this result when X^H is simply connected. Nielsen fixed point theory for equivariants maps was studied in [Wo] and developed in [W1]. On the other hand, Nielsen fixed point theory has been generalized to coincidence theory by Schirmer [H].

It is the purpose of this paper to extend the equivariant Nielsen fixed point theory of [Wo] for coincidence of G-maps $f, h : V \to M$, where V is an open invariant subset of an oriented, connected, closed G-manifold M.

Wong introduced in [Wo] the notion of an equivariant Nielsen number $N_G^c(f, V)$ which is an ordered k-tuple that depends on the isotropy types (H_1)... (H_k) of V, where $f : V \to X$ is a G-map, V is an open invariant subset of a $G - ENR\ X$ and G is a compact Lie group. When G is finite, $N_G^c(f, V)$ gives a lower bound for the minimal number of fixed points in the G-homotopy class of f (by G-compactly fixed homotopies) and this lower bound is sharp when the G-action on V is free. (The hypotesis on G being finite is not true restrictive because of lemma 3.3 of [W]).

This paper is divided into three sections. The first section contains basic concepts of group actions and the definition of G-compactly coincident maps. In the second section we define the WH_i-Nielsen classes of coincidence points and we show its relationship with the ordinary Nielsen classes. Also, we define the WH-Nielsen number of coincidence of $N_{WH}(f, h; V)$, which is a k-tuple $(N_{WH_1}(f, h, V_{H_1}), ..N_{WH_k}(f, h; V_{H_k}))$ where $(H_1), ..., (H_k)$ are the isotropy types of V. We show that $N_{WH}(f, h; V)$ is a lower bound for the minimal number of WH_i-orbits of coincidence of the maps f_{H_i}, h_{H_i}, by G-compactly coincident homotopies for the isotropy types (H_i) of V with $|WH_i|$ finite. The hypothesis of $|WH_i|$ being finite is not too restrictive, because in section 3, we have extended lemma 3.3 of [W] for coincidences. In section 3 we will study the minimality. When G is finite, acting freely on V and M is a connected, oriented, triangulable, compact G-manifold without boundary and of dimension $n \geq 3$, we show that $N_G(f, h; V)$ is realized in the class of the G-maps G-homotopic to f and h by compactly coincident homotopies. We also showed that, given a compact Lie group G with $dimG > 0$ acting freely on V and G-maps $f, h : V \to M$, then there exists $f' : V \to M$, $G-\epsilon$-homotopic to f such that $Coin(f', h) = \emptyset$, which extends the lemma 3.3 of [W]. Finally, we proved an equivariant version of Hopf construction for coincidence and we showed that given G-maps $f, h : V \to M$ compactly coincident, there exists a G-map $f' : V \to M$, $G - \epsilon$-homotopic to f such that the isotropy subgroup of each coincidence point in $Coin(f', h)$ has finite Weyl group.

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1. Preliminaries. Let G be a topological group and X a G-space. For any subgroup H of G, we denote by NH the normalizer of H in G and by NH/H, the Weyl group of H in G. The conjugacy class of H denoted by (H) is called the orbit type of H. A subgroup H_1 of G is subconjugate of H, if exists $g \in G$ so that gH_1g^{-1} is a subgroup of H, then we write $(H_1) \leq (H)$. If $x \in X$, in which case G_x denotes the isotropy subgroup of x. i.e., $G_x = \{g \in G| \ gx = x\}$. For each subgroup H of G, $X^H = \{x \in X| \ hx = x, \ \forall h \in H\}$ and $X_H = \{x \in X| \ G_x = H\}$, we denote by $X^{(H)} = GX^H = \{x \in X^S| \ S \in (H)\}$ and by $X_{(H)} = GX_H = \{x \in X| \ (G_x) = (H)\}$. An orbit type (H) is called an isotropy type of X if H appears as an isotropy subgroup of some $x \in X$. Suppose X has a finite set of isotropy types denoted by $\{(H_i)\}$, we can choose an admissible ordering on $\{(H_i)\}$ so that $(H_j) \leq (H_i)$ implies $i \leq j$. Then we have a filtration of G-subspaces $X_1 \subset X_2 \subset ... \subset X_k = X$, where $X_i = \{x \in X| \ (G_x) =$

(H_j) *for some* $j \leq i\}$. Also, $X_{(H_i)} = X_i - X_{i-1}$. By a free G-subset of X, we mean a G-invariant subset on which the action is free.

A cell complex (X, K) is called a G-cell complex if:

a) The orbit space X/G is a Hausdorff space.

b) G acts cellularly, that is, $e \in K$ implies $ge \in K$ for every $g \in G$.

c) Every point x of a cell e has the same isotropy subgroup, which is denoted by G_e and, in particular, each boundary point is fixed by G_e.

d) If g is not contained in G_e, then ge is disjoint from e.

e) The topology of the subspace $G\bar{e}$ is the identification topology determined by the induced G-characteristic maps, $G_\sigma(= \mu_0(1_G)) : G \times \Delta^n \to G\bar{e} \subset X$.

For more details on G-cell complex see [M1].

Let Y be a G-space and $f, h : X \to Y$ G-maps. We denote by $Coin(f, h)$ the set of coincidence points of f and h, i.e. $Coin(f, h) = \{x \in X |\ f(x) = h(x)\}$.

Definition 1.1. The maps f and h are called *compactly coincident* if the set of coincidence points of f and h, $Coin(f, h)$, is a compact subset of X.

Definition 1.2. The maps f and h are called *G-compactly coincident* if for each isotropy type (H_i) of X the maps $f_{H_i}, h_{H_i} : X_{H_i} \to Y^{H_i}$ are compactly coincident, where $H_i \in (H_i)$.

If f and h are G-compactly coincident then they are compactly coincident but the converse is not true. See [Wo], 2.3.

Definition 1.3. Two G-homotopies $F, T : X \times I \to Y$, where G act on I trivially, are called *compactly coincident* if $\bigcup_s Coin(F_s, T_s)$ is a compact subset of X. The homotopies F and T are called *G-compactly coincident* if for each isotropy type (H_i) of X, $\bigcup_s Coin(F_{H_i} \times \{s\}, T_{H_i} \times \{s\})$ is a compact subset of X_{H_i}, where $H_i \in (H_i)$.

Unless otherwise specified in this paper, the word "manifold" will refer to a C^∞-manifold without boundary. Let G be a Lie group and M a manifold. By a smooth action of G on M we mean an action $\theta : G \times M \to M$ which is a smooth map. A manifold M together with such an action will be called a G-manifold. In general, M/G is not a differentiable manifold with the structure induced by the orbit map $p : M \to M/G$, see [Bd] p. 301, as a matter of fact, not even a manifold see [tD], I.2.19, ex. 3.

Proposition 1.4. *Let G be a compact Lie group. A compact G-manifold has finite orbit type.*

Proof. See [tD], I.5.11.

□

Proposition 1.5. *Let G be a compact Lie group and M a G-manifold. Let H be any isotropy subgroup of G. Then $M_{(H)}$ is a submanifold of M (which may have components of different dimensions).*

Proof. See [tD], I.5.13.

□

It follows from 1.5 that M^G is always a closed submanifold of M.

Proposition 1.6. *Let V be an open invariant subset of a G-manifold M, where G is a compact Lie group, and let H be an isotropy type of V. Then V_H is an open subset of M^H.*

Proof. It follows from corollary II.5.5 of [Bd].

□

Let M_1 and M_2 be connected, oriented n-manifolds with M_1 compact. Denotes by $z_1 \in H_n(M_1)$ be the fundamental class of M_1 and $U_2 \in H^n(M_2^\times)$ the Thom class of M_2, where $M_2^\times = (M_2 \times M_2, M_2 \times M_2 - \Delta(M_2))$. Suppose W is an open set in M_1 and $f, h : W \to M_2$ are maps for which $Coin(f, h)$ is a compact subset of W. Since M_1 is normal, there exists an open set V in M_1

with $Coin(f,h) \subset V \subset \bar{V} \subset W$.

Define the *Coincidence index* of the pair (f,h) on W to be the integer $I^W_{f,h}$ given by the image of the class z_1 under the composition:

$$H_n(M_1) \longmapsto H_n(M_1, M_1 - V) \stackrel{exc.^{-1}}{\longmapsto} H_n(W, W - V) \stackrel{(f,h)_*}{\longmapsto} H_n(M_2^{\times}) \approx Z.$$

Here the map $(f,h) : W \to M_2 \times M_2$ is given by $(f,h)(x) = (f(x), h(x))$ and the identification $H_n(M_2 \times M_2, M_2 \times M_2 - \Delta(M_2)) \approx Z$ is given by sending a class α to the integer $< U_2, \alpha >$. For more details on $I^W_{f,h}$ see [V] chapter 6.

Proposition 1.7. *Let M_1, M_2, M_1' and M_2' be connected, oriented manifolds of dimension n with M_1 and M_1' compact. Let $\sigma_1 : W \to W' = \sigma_1(W)$ be a homeomorphism, where W is an open subset of M_1. Let $f,h : W \to M_2$ and $f',h' : W' \to M_2'$ be maps compactly coincident and suppose that the following diagram is commutative:*

$$\begin{array}{ccc} W & \stackrel{f,h}{\longrightarrow} & M_2 \\ \downarrow \sigma_1 & & \downarrow \sigma_2 \\ W' & \stackrel{f',h'}{\longrightarrow} & M_2' \end{array}$$

where $\sigma_2 : M_2 \to M_2'$ is a local homeomorphism.

i) If both σ_1 and σ_2 preserve or reverse the orientations, then $I^W_{f,h} = I^{W'}_{f',h'}$.

ii) If σ_1 preserves the orientation and σ_2 reverses it, then $I^W_{f,h} = -I^{W'}_{f',h'}$.

Proof. The first part follows from [O], p. 16 and the second follows from the first, by changing the orientation of M_2'.

□

2. Equivariant Nielsen numbers

Throughout this section, G will denote a compact Lie group.

Let X and Y be G-spaces, where G acts freely on X, and $f,h : X \to Y$ compactly coincident G-maps. Suppose $Coin(f,h) \neq \emptyset$.

Definition 2.1. Two points $x, y \in Coin(f,h)$ are said to be *G-Nielsen equivalent*, denoted by $x \sim_G y$, if either (i) $x \in G(y)$ or (ii) there exists a path

$\alpha : I \to X$ such that $\alpha(0) = x$, $\alpha(1) = gy$ for some $g \in G$ and $f \circ \alpha$ is homotopic to $h \circ \alpha$, relative to endpoints in X.

It is easy to see that $\sim_G$ is an equivalence relation on $Coin(f, h)$.

Definition 2.2. Let V be a free G-subset of X and $f, h : V \to Y$ compactly coincident G-maps. The equivalence classes on $Coin(f, h) \subset V$, given by the above relation, will be called *G-Nielsen classes (or G-classes)* of coincidence of f and h on V.

Now, we will show how an ordinary Nielsen class relats to the G-class of coincidence that contains it. Let $x_0 \in Coin(f, h)$ and R be the (ordinary) Nielsen class so that $x_0 \in R$. It is easy to see that $G_R = \{g \in G | \; gx_0 \in R\}$ is a subgroup of G and does not depends of the point x_0 chosen. If R_1 and R_2 are two Nielsen classes contained in the same G-class $\hat{R}$ it is easy to show that G_{R_1} and G_{R_2} are conjugate to each other.

Definition 2.3. G_R will be called *the isotropy subgroup of R.*

Consider the set of equivalence classes on $G/G_R = \{H_1, H_2, ..., H_i, ...\}$, where $H_1 = G_R$ and denote by $h_i \in H_i$ a representative of the class H_i, $i = 1, 2, ...$ It is easy to see that for each i=1,2..., the class h_iR does not depend on the representative $h_i \in H_i$ chosen. In this way we will denote the class h_iR simply by R_i.

Proposition 2.4. *Let $\hat{R}$ be the G-class containing x_0. Then $\hat{R} = \bigsqcup_i R_i$ (disjoint union).*

Proof.

i) $\bigcup_i R_i \subset \hat{R}$, follows from 2.1.

ii) $\hat{R} \subset \bigcup_i R_i$ Let $y \in \hat{R}$, if $y \notin G(x_0)$, then y is Nielsen equivalent to gx_0 for some $g \in G$, so $y \in g.R = R_j$, for some $j = i, 2, ...$

iii) Se $R_i \cap R_j \neq \emptyset$, we have $R_i = R_j$ and then $i = j$.

□

Corollary 2.5. *If X is a compact ANR and Y is a compact ANR or an ENR, then $[G : G_R]$ is finite.*

Proof. It follows from 2.4 and corollary 1.5.1 of [F].

□

Let M_1 and M_2 be connected, oriented G-manifolds with M_1 compact. Let V be a free G-subset of M_1 and $f, h : V \to M_2$ compactly coincident G-maps. By proposition 2.4, each G-class $\hat{R}$ is an open subset of $Coin(f, h)$, then there exists an open subset U of V such that $\hat{R} = U \cap Coin(f, h)$.

We may now, define the index of a G-class.

Definition 2.6. The *coincidence index* of the G-class $\hat{R}$, denoted by $I(\hat{R})$ is the coincidence index of the pair (f, h) on U.

Proposition 2.7. *$I(\hat{R}) = \Sigma_{i=1}^{r} I(R_i)$, where $r = [G : G_R]$ and R is a Nielsen class contained in $\hat{R}$.*

Proof. It follows from proposition 2.4 and lemma 6.6 of [V].

□

Proposition 2.8. *If the action preserves orientation on M_1 and M_2, (or reverses the orientation on M_1 and M_2), then $I(\hat{R}) = [G : G_R].I(R)$, where $R \subset \hat{R}$ is a Nielsen class of f and h.*

Proof. By 2.7, it suffices to show that $I(f, h; R) = I(f, h; g.R)$, $\forall g \in G$. Denote by ϕ_g the homeomorphism $\phi_g : M_1 \to M_1$ defined by $\phi_g(x) = gx$. Let U be an open subset of W so that $R = U \cap Coin(f, h)$ and $\sigma_g = \phi_g |_U : U \to \phi_g(U) = V$, then $gR = \sigma_g(R)$. It is easy to see that $V \cap Coin(f, h) = gR$.

We have the following commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{f,h} & M_2 \\ \downarrow \sigma_g & & \downarrow \mu_g \\ V & \xrightarrow{f,h} & M_2, \end{array}$$

where the map $\mu_g : M_2 \to M_2$ is given by $\mu_g(x) = gx$.

Since the action preserves (reverses) the orientations, so do σ_g and μ_g accordingly preserve (reverse) too. The result follows from 1.7.

□

Definition 2.9. A G-class $\hat{R}$ is called *essential* if $I(\hat{R}) \neq 0$. The number of essential G-classes, denoted by $N_G(f, h; V)$, will be called the *G-Nielsen number* of coincidence of f and h on V.

Definition 2.10. Let $F, T : V \times I \to M_2$, be G-homotopies, where the G-action on I is trivial. A G-class $\hat{R} \subset Coin(F_0, T_0)$ is said to be *FT-related* to a G-class $\hat{P} \subset Coin(F_1, T_1)$, denoted by $\hat{R}FT\hat{P}$, if there exist Nielsen classes $R \subset \hat{R}$ and $P \subset \hat{P}$ so that $RFTP$, i.e. there exists $x \in R$, $y \in P$ and a path $\alpha : I \to V$ with $\alpha(0) = x$, $\alpha(1) = y$ so that $< F, \alpha > \sim < T, \alpha >$, where $< F, \alpha > (t) = F(\alpha(t), t)$.

Proposition 2.11. *If $RFTP$, then $G_R = G_P$.*

Proof. Let $g \in G_R$ so $gR = R$. Since F and T are G-maps and $RFTP$ we have that $gRFTgP$ and so $RFTgP$, so $g.P = P$, see proposition 1.5.6, of [F], and then $g \in G_P$.

Similarly, we show that $G_P \subset G_R$.

□

Proposition 2.12. *If the G-homotopies F and T are compactly coincident and $\hat{R}FT\hat{P}$, then $I(\hat{R}) = I(\hat{P})$.*

Proof. Since $I(R) = I(P)$ if $RFTP$, where $R \subset \hat{R}$ is a local Nielsen class of F_0 and T_0 and $P \subset \hat{P}$ is a local Nielsen class of F_1 and T_1, the result follows from 2.7 and 2.11.

□

Our next objective is to verify the invariance under compactly coincident G-homotopy of $N_G(f, h; V)$.

Proposition 2.13. *If the G-homotopies F and T are compactly coincident, then* $N_G(F_0, T_0; V) = N_G(F_1, T_1; V)$.

Proof. Let $\hat{R}$ be a G-class of F_0 and H_0. i) If $\hat{R}$ is FT-related to some G-class $\hat{P}$ of F_1 and H_1, then by 2.12, $I(\hat{R}) = I(\hat{P})$. ii) If $\hat{R}$ is not FT-related to a G-class of F_1 and H_1, then $I(\hat{R}) = 0$.

So, for each essential G-class $\hat{R}$ of F_0 and H_0, there exist a essential G-class $\hat{P}$ of F_1 and H_1 with $\hat{R}FH\hat{P}$.

□

Corollary 2.14. i) *If* $N_G(f, h; V) \neq 0$*, then f and h have at least one orbit of unremovable coincidences.*

ii) If G is a finite group, then $|G|N_G(f, h; V) \leq \#Coin(f', h')$*, for all pair of G-maps* f'*,* h' *G-homotopic to f and h by compactly coincident G-homotopies.*

□

Let X and Y be G-spaces, and $f, h : X \to Y$ compactly coincident G-maps and suppose $Coin(f, h) \neq \emptyset$.

Definition 2.15. Two points $x, y \in Coin(f, h)$ are said to be *G-Nielsen equivalent*, denoted by $x \approx_G y$, if (i) For some $H \leq G$ x and y lie in X_H, and (ii) $x \sim_{WH} y$.

It is easy to see that $\approx_G$ is an equivalence relation to $Coin(f, h)$.

Definition 2.16. Let V be an invariant open subset of X and $f, h : V \to Y$ G-compactly coincident maps. The equivalence classes on $Coin(f, h) \subset V$, given by the above relation, will be called *WH-Nielsen classes* (or *WH-classes*) of coincidence of f and h on V.

Remark. The number of $WH - classes$, where $H \in (H)$ is finite iff $\#(\frac{G}{NH})$ is finite.

Let M_1 and M_2 be connected, oriented, G-manifolds with M_1 compact. Let V be an invariant open subset of M_1 and $f, h : V \to M_2$ G-compactly coincident maps.

In order to define the index of a WH-class, the idea is to apply the previous theory to the functions $f_H, h_H : V_H \to M_2^H$, since WH acts freely on V_H and by proposition 1.6, V_H is an open subset of M_1^H. But there exist two problems:

i) If M_1 is oriented, even if the G-action preserves the orientation, M_1^H could be non-orientable. *Exemple: Let $N = S^1 \times S^1 \times S^1 \times I$ with the Z_2-action, $t(x,y,z,a) = (x,\bar{y},\bar{z},a)$. Then $N^{Z_2} = S^1 \times \{\pm 1\} \times \{\pm 1\} \times I$. Let M be the manifold obtained from N by identifying the points $(x,y,z,0)$ with $(\bar{x},\bar{y},z,1)$. It is easy to see that the Z_2-action defined on N induces an action on M that preserves the orientation, but M^{Z_2} is the disjoint union of 4 copies of Klein bottle, therefore non-orientable.*

ii) Even in the case where M_1^H and M_2^H are connected, they can have different dimensions. *Exemple: Let $N = S^5$ with the Z_2-action $t(x_1,x_2,x_3,x_4,x_5,x_6) = (x_1,x_2,-x_3,-x_4,-x_5,-x_6)$. This action induces an action on RP^5 such that $(RP^5)^{Z_2} = S^1 \cup RP^3$. It is enough to consider the constant map $f : RP^5 \to RP^5$, on one point $P \in S^1$.*

From now on we will consider the case where V is an open invariant subset of a comapct, connected, oriented, G-manifold M so that all connected components of M^H are orientable. Moreover, denoting by $(M^H)^d$ the union of all connected components of M^H of dimension d and by $V_H^d = V_H \cap (M^H)^d$, we will always assume that $f_H(V_H^d) \subset (M^H)^d$ and $h_H(V_H^d) \subset (M^H)^d$, for all isotropy types (H) of V.

Let (H_i) be an isotropy type of V and V_s a connected component of $V_{(H_i)}$. It is easy to see that $WH_{i_{V_s}} = \{g \in WH_i |\ gV_s = V_s\}$ is a subgroup of WH_i and does depend on the connected component V_s choosen.

Definition 2.17. Let $f, h : V \to M$ be G-compactly coincident maps. We define the *coincidence index of the WH_i-class $\tilde{R}$*, as follows:

Let V_1 be a connected component of V_{H_i} and M_s the connected component of M^{H_i} so that $f_{H_i}(V_1) \subset M_s$ and $h_{H_i}(V_1) \subset M_s$. Choose an orientation on V_1 and an orientation on M_s, let U be an open subset of V_{H_i} so that $U \cap Coin(f_{H_i}, h_{H_i}) =$

$\tilde{R}$. So $U \cap V_1$ is an open set containing $\tilde{R} \cap V_1$ and we can compute the coincidence index of $\tilde{R} \cap V_1$, i.e, $I^{U\cap V_1}_{f_{H_i}, h_{H_i}}$. We define $I(\tilde{R}) = k.I^{U\cap V_1}_{f_{H_i}, h_{H_i}}$, where $k = [WH_i : WH_{i_{V_1}}]$.

If the connected component V_s contains only one coincidence class R whith is contained in the WH_i-class $\tilde{R}$, then $WH_{i_{V_s}} = WH_{i_R}$ the isotropy subgroup of R in WH_i, so, in this case, $I(\tilde{R}) = [WH_i : WH_{i_R}]I(R)$.

Definition 2.18. A WH_i-class $\tilde{R}$ will be called *essential* if $I(\tilde{R}) \neq 0$. The number of essential WH_i-classes, denoted by $N_{WH_i}(f, h; V_{H_i})$, will be called *the WH_i-Nielsen number of f and h on V_{H_i}.*

Definition 2.19. The k-tuple $(N_{WH_1}(f, h; V_{H_1}), ..., N_{WH_k}(f, h; V_{H_k}))$ will be called *the WH-Nielsen number of f and h on V* and we will denote it by $N_{WH}(f, h; V)$, where $\{(H_i),\ i = 1...k\}$ is the set of isotropy types of V.

Remark. For every isotropy type (H_i) of V, the WH_i-Nielsen number, $N_{WH_i}(f, h; V_{H_i})$, of f and h on V_{H_i} is finite and since V_{H_i} is homeomorphic to $V_{H'}$ (also, M^{H_i} is homeomorphic to $M^{H'}$)if $H' \in (H_i)$, $N_{WH_i}(f, h; V_{H_i})$ is independent of the choice of the representative of (H_i) and hence $N_{WH}(f, h; V)$ is well defined.

Theorem 2.20. *If $F, T : V \times I \to M$ are G-compactly coincident homotopies, where the action on I is trivial, then $N_{WH}(F_0, T_0; V) = N_{WH}(F_1, T_1; V)$.*

Proof. Since V is a disjoint union of $V_{(H_i)}$, where H_i appears as an isotropy subgroup, it suffices to show that for each $i = 1, ..., k$ $N_{WH_i}(F_0, T_0; V_{H_i}) = N_{WH_i}(F_1, T_1; V_{H_i})$.

By 2.18, $N_{WH_i}(F_0, T_0; V_{H_i})$ is the number of essential WH_i-classes of F_0 and T_0 on V_{H_i}, by 2.17, an essential WH_i-class contains, at least one, essential Nielsen class R. Consider the restriction of the homotopies F and T to $V_s \times I$, where V_s is the connected component of V_{H_i} such that $R \subset V_s$. Since F and T are G-compactly coincident homotopies, for each isotropy type (H_i) of V, F_{H_i}

and T_{H_i} are compactly coincident G-homotopies and the result follows from 2.13.

□

Corollary 2.21. *Let $f, h : V \to M$ be G-compactly coincident maps. If $f', h' : V \to M$ are homotopic to f and h by a G-compactly coincident homotopy then, for each isotropy type (H_i) of V with $|WH_i|$ finite we have, $N_{WH_i}(f, h; V_{H_i}) \leq \frac{1}{|WH_i|}.\#Coin(f'_{H_i}, h_{H_i})$.*

□

3. Minimal number of coicidence orbits

In this section, we study the minimal number of coincidence orbits in the G-compactly coincident homotopy class of a pair of G-compactly coincident maps. When G is finite, the G-action on V is free and M is a compact, connected, oriented, triangulable G-manifold of dimension $n \geq 3$, any pair of compactly coincidente G-maps $f, h : V \to M$ can be equivariantly deformable to a pair of G-maps with exactly $N_G(f, h; V)$ coincidence orbits by compactly coincident G-homotopies.

We give an equivariant analogous to the Hopf construction for the case of coincidence and we show that if G is a compact Lie group with $dim(G) > 0$ and (N, A) is a relative $G - CW$-complex such that the G-action on $N - A$ is free and the coincidence points in A are isolated, then f can be equivariantly deformable (relative to A) to a G-map f' such that $Coin(f', h) \cap (N - A) = \emptyset$.

Also, we show that given compactly coincident G-maps $f, h : V \to M$ and $\epsilon > 0$, there exist a G-map $f' : V \to M$, $G - \epsilon$-homotopic to f such that, the isotropy subgroup of each coincidence point in $Coin(f', h)$ has finite Weyl group.

Let M_1 and M_2 be compact, connected, oriented, triangulables n-manifolds and (Γ, μ) a triangulation of M_1.

We will consider the sequence, $0 < \epsilon_0 \leq \epsilon_1 \leq ... \leq \epsilon_{n+1}$ given on [H] p. 24.

Proposition 3.1. *Let $h : D^d \to \mathbb{R}^n$ be a continuous map and $f : S^{d-1} \to \mathbb{R}^n$ a continuous map such that, $Coin(f,h)$ is finite and $d(f,h) < \epsilon_{d-2}$. Then there exist a continuous extension f' of f on D^d such that:*

i) *If $d < n$, then $Coin(f',h) \cap intD^n = \emptyset$ and $d(f',f) < \epsilon_{d-1}$.*

ii) *If $d = n$ we have:*

a) *If $Coin(f,h) = \emptyset$, then $Coin(f',h) \cap intD^n = \emptyset$, or contains at most one point.*

b) *If $Coin(f,h) \neq \emptyset$, then $Coin(f',h) \cap intD^n = \emptyset$.*

In the cases i) and ii), we have $d(f',f) < \epsilon_n$.

Proof. The case i) for $Coin(f,h) = \emptyset$ see lemma 1 of [H], the case ii.a) see lemma 2 of [H]. We shall go prove the case i) for $Coin(f,h) \neq \emptyset$ and the case ii.b).

Let $g(x) = f(x) - h(x)$, $\forall x \in S^{d-1}$, then $Coin(f,h) = g^{-1}(0)$.

Suppose that $g^{-1}(0) = \{x_0\}$. Given a point $x \in intD^d$, let x_1 be the other point obtained by the intersection of the straight line through x_0 and x with S^{d-1}. Let $t \in I$ such that, $x = tx_0 + (1-t)x_1$ and define $\tilde{g}$ on x by, $\tilde{g}(x) = (1-t)g(x_1)$. So $\tilde{g}$ is a continuous map and extend g. If for some point $x \in D^d$, $\tilde{g}(x) = 0$, then $(1-t)g(x_1) = 0$, where x_1 is the other point obtained by the intersection of the straight line through x_0 and x with S^{d-1}, since $g(x_1) \neq 0$ we have, $1 - t = 0$, then $t = 1$ and $x = x_0 \in S^{d-1}$.

Now suppose the result is true for $\#(g^{-1}(0)) = k-1$, it is easy to see that is true for $\#(g^{-1}(0)) = k$.

Let $\tilde{f} = h + \tilde{g}$. Then $\tilde{f}$ extend f, $Coin(\tilde{f},h) \cap intD^d = \emptyset$ and we have $d(\tilde{f},h) \leq d(f,h) \leq \epsilon_{d-2} < \epsilon_{d-1} < \epsilon_n$.

□

The next lemma is fundamental for Theorem 3.3.

Lemma 3.2. *For any G-space Y and $\epsilon > 0$, there is a $\delta > 0$ such that, if $f,h : Y \to X$ are equivariant maps and $d(f,h) < \delta$, then f and h are* *equivariantly ϵ-homotopic through a homotopy constant on the coincidence set of f and h.*

Proof. See [W], 2.3.

□

In what follows, given $\epsilon > 0$ we will assume that $2\epsilon_{n+1} \leq \delta$, where δ is given by lemma 3.2.

The next theorem, is an equivariant version of the Hopf construction for coincidence when G is finite and the G-action on V is free.

Theorem 3.3. *Let M be a compact, connected, oriented, triangulable G-manifold with $dim M = n$ and V a free G-subset of M. Let $f, h : V \to M$ compactly coincident G-maps. Then, there exists a G-map $f' : V \to M$, $G - \epsilon$-homotopic to f by a G-homotopy $F : V \times I \to M$, so that:*

i) $F_t(x) = f(x),\ \forall x \in V - K,\ t \in I.$

ii) *$Coin(f', h)$ is finite, each coincidence point of f' and h lies in the interior of some maximal simplex of K and each maximal simplex of K contains at most one coincidence point.*

iii) *$\bigcup_{t\in I} Coin(F_t, h)$ is a compact subset of V.*

Proof. By III.1.1 of [Bd] we can assume that the triangulation of M is regular for the action of G. It is easy to see that there exists a finite homogeneous $n - G$-subcomplex of Γ, $K \subset V$, with $Coin(f, h) \subset int K$.

We can suppose $diam(f(\sigma^n)) < \epsilon_0/2,\ diam(h(\sigma^n)) < \epsilon_0/2,\ \forall \sigma^n \in K$, where ϵ_0 is given on [H] p. 24.

If $\sigma \in K$ let $St_K(\sigma) = St(\sigma) \cap K$, since the action on V is free, we have $St_K(\sigma) \cap St_K(g.\sigma) = \emptyset,\ \forall g \in G,\ g \neq e.$

In K, $d(f, h) < \epsilon_0$. For any $y \in K$ there exists $\sigma^n \in T$ so that, $y \in \overline{\sigma}^n \in K$, thus exists $x \in \overline{\sigma}^n$ with $f(x) = h(x)$ then, $d(f(y), h(y)) \leq d(f(y), f(x)) + d(f(x), h(x)) + d(h(x), h(y)) < \epsilon_0$. In $U - K$, define $f' = f$ and in K define f' as follows:

For $\sigma^0 \in Coin(f,h)$, define f' such that, $0 < d(f'(\sigma^0), h(\sigma^0)) < \epsilon_0$ and for all $g \in G$ define f' on $g.\sigma^0$ by $g.f'(\sigma^0)$. Repeat this procedure for all coincidence orbits of 0-simplex on $K - G(\sigma^0)$.

For the other 0-simplexes of K define $f' = f$.

Let $\sigma^1 \in K$, such that $\sigma^1 \cap Coin(f,h) \neq \emptyset$. We have defined f' on $\partial\sigma^1$ such that $f'(x) \neq h(x)$, $\forall x \in \partial\sigma^1$. This way, we can extend f' continuously from σ^1 using 3.1 such that $f'(x) \neq h(x)$, $\forall x \in \sigma^1$, $d(f', f) < \epsilon_1$. For all $g \in G$, define f' on $g.\sigma^1$ as follows: Given $y \in g.\sigma^1$, there exists only one point $x \in \sigma^1$ with $y = g.x$, then define $f'(y) = g.f'(x)$.

Repeat this construction for all coincidence orbits contained in the interior of some 1-simplex of $K - G.(\overline{\sigma}^1)$.

For the other 1-simplexes of K, define $f' = f$.

Repeat this procedure for the $2, 3, ..(n-1)$-simplexes of K which contains some coincidence point and in the others $2, 3, ...(n-1)$-simplexes of K define $f' = f$.

Now let $\sigma^n \in K$, with $\sigma^n \cap Coin(f,h) \neq \emptyset$.

We have defined f' on $\partial\sigma^n$ with $f'(x) \neq h(x)$, $\forall x \in \partial\sigma^n$ and $d(f', h) < \epsilon_{n-1}$.

Using [H], lemma 2, we extend f' to σ^n such that $Coin(f', h) \cap \sigma^n = \emptyset$ or it contains exactly one coincidence point when $I(f', h; \sigma^n) \neq 0$.

For all $g \in G$, define f' on $g.\sigma^n$ as follows: Given $y \in g.\sigma^n$, there is only one point $x \in \sigma^n$ with $y = g.x$, then define $f'(y) = g.f'(x)$.

Repeat this procedure for all coincidence orbits contained in the interior of some n-simplex of $K - G(\overline{\sigma}^n)$.

So, we have a G-map $f' : V \to M$, such that, f' and h only have isolated coincidences each of which lies in the interior of n-simplex σ^n of K with $I(f', h; \sigma^n) \neq 0$ and $f' \neq h$ for the other n-simplex of K.

In K, $d(f,h) < \epsilon_0 < \epsilon_{n+1}$, $d(f',h) < \epsilon_{n+1}$ and $f = f'$ in $V - K$, then $d(f', f) < 2\epsilon_{n+1} \leq \delta$, so by 3.2, f' and f are G-homotopic by a G-homotopy $F : V \times I \to M$, which is constant on the coincidence points of f' and f.

It is easy to see that $\bigcup_{t \in I} Coin(F_t, h)$ is a compact subset of V.

□

Let M be a compact, connected, oriented, triangulable G-manifold of dimension $n \geq 3$, G a finite group and V a free G-subset of M.

Next, we show how to coalesce coincidence orbits in the same G-class.

Proposition 3.4. *Let $f, h : V \to M$ be compactly coincident G-maps, K the $n - G$-subcomplex given by 3.3, and $\mathcal{O}_1$, $\mathcal{O}_2$ two coincidence orbits in the same G-class $\hat{R}$. Then there exists a G-neighborhood U of $\mathcal{O}_1 \cup \mathcal{O}_2$ on V and G-maps $f', h' : V \to M$, G-homotopic to f and h by compactly coincident G-homotopies F and H so that:*

i) $F_t(x) = f(x)$ and $H_t(x) = h(x), \forall x \in V - U,\ t \in I$.

ii) *f' and h' have no coincidences in U, or have at most one coincidence orbit on U, if $I^U_{f',h'} \neq 0$.*

Proof. By 3.3, we can suppose that $Coin(f, h)$ is finite and lies in the interior of the n-simplexes of K, with each n-simplex containg at most one coincidence point.

Since $\mathcal{O}_1$ and $\mathcal{O}_2$ are in the same G-classe, there exist $x_1 \in \mathcal{O}_1, x_2 \in \mathcal{O}_2$ and a path $\lambda : I \longmapsto K$ of x_1 to x_2 with $f \circ \lambda \sim h \circ \lambda$.

Consider $\overline{\lambda}$ the image of λ under the orbit map $p : V \to V/G$.

By lemma 7 of [H], $\overline{\lambda}$ is homotopic to a polygonal path $\overline{\beta}$ such that the interior of each segment lies inside some maximal simplex and each endpoint lies in the interior of some simplex of one less dimension. Moreover, by general position we may assume that $\overline{\beta}$ is simple.

By lemma 9 of [H], there exist a closed ϵ-neighborhood $\overline{U}(\overline{\beta})$ of $\overline{\beta}$ on V/G which, have no coincidence point of $\overline{f}$ and $\overline{h}$ excluding $p(x_1)$ and $p(x_2)$, where $\overline{f}$ and $\overline{h}$ are induced by f and h on V/G.

Lifting this homotopy, we have a path β of x_1 to x_2 homotopic to λ.

Since $\overline{\beta}$ is simple, for all $g \in G,\ g \neq e,\ g.\beta \cap \beta = \emptyset$ and since $g.\beta$ is closed in V, $\forall g \in G$, there exists an open subset U_β of V containing β such that:

i) For all $g \in G$, $g.\beta \subset g.U_\beta$ and,

ii) $g_1.U_\beta \cap g_2.U_\beta = \emptyset$, if $g_1 \neq g_2$.

Let $\overline{U}(\beta) = p^{-1}(\overline{U}(\overline{\beta})) \cap \overline{U}_\beta$.

We then coalesce the coincidence points x_1 and x_2 along β inside $U(\beta)$ by the lemma 9 of [H], where $U(\beta) = int\overline{U}(\beta)$, so we obtain a pair of maps f' and h', homotopic to f and h by homotopies $F', H' : V \times I \longmapsto M$, respectively, such that:

a) $F'_t(x) = f(x),\ H'_t(x) = h(x),\ \forall x \in V - U(\beta), t \in I$.

b) either f' and h' have no coincidences in $U(\beta)$, or have exactly one coincidence point in $U(\beta)$ with $I(f', h'; U(\beta)) \neq 0$.

For all $g \in G$, define F and H on $g.U(\beta) \times I$ as follows:

If $y \in g.U(\beta)$, there exists only one $x \in U(\beta)$ with $y = g.x$, then define $F_t(y) = g.F'_t(x)$ and $H_t(y) = g.H'_t(x),\ \forall t \in I$.

Let $U = \bigcup_{g\in G} g.U(\beta)$, for each $x \in V - U$ define $F_t(x) = f(x)$ and $H_t(x) = h(x),\ \forall t \in I$. Let $f' = F_1$ and $h' = H_1$.

Since f' and h' coincide with f and h outside a small contractible neighborhood of $G\beta$ these G-homotopies are compactly coincident.

□

Theorem 3.5. *(Minimality) Let $f, h : V \to M$ be compactly coincident G-maps. Then there exists G-maps $f', h' : V \to M$, G-homotopic to f and h by compactly coincident G-homotopies so that, $\#Coin(f', h') = |G|.N_G(f, h; V)$.*

Proof. By 3.3, f is G-homotopic to a G-map f' via a homotopy F such that f' and h have only isolated coincidence which lies in the interior of some n-simplexes of K with index non zero. Moreover $\bigcup_{t\in I} Coin(F_t, h)$ is a compact subset of V.

Applying 3.4 finitely many times, each G-class contains only one coincidence orbit with non-zero index. So we obtain G maps $f_1,\ h_1$ G-homotopic to f' and h by compactly coincident G-homotopies such that $Coin(f_1, h_1)$ have only essential G-class, each of which contains only one coincidence orbit. Therefore, there are $N_G(f_1, h_1; V)$ coincidence orbits. Since the G-homotopies are compactly coincident, it follows from 2.13 that $\#Coin(f_1, h_1) = |G|.N_G(f_1, h_1; V) = |G|.N_G(f, h; V)$.

□

Following Matumoto (4.4 of [M2]), if G is a compact Lie group, any compact G-manifold M has a $G-CW$-complex structure which induces a triangulation on the orbit space.

Lemma 3.6. *Let M be a compact, connected G-manifold and V an open G-invariant subset of M. Let $f, h : V \to M$ be compactly coincident G-maps. Then there exist a structure of $G-CW$-complex on M, which induces a triangulation on M/G and a G-subcomplex $K \in V$ with $Coin(f, h) \subset intK$ such that $p(K) \subset V/G$ is finite, where $p : M \to M/G$ is the orbit map.*

Proof. Consider on M a structure of $G-CW$-complex that induces a triangulation on M/G, given by 4.4 of [M2].

Let $A = \{\sigma^d \in M/G \mid \overline{\sigma^d} \cap p(Coin(f, h)) \neq \emptyset\}$ and $K' = \bigcup_{\sigma^d \in A} \overline{\sigma^d}$.

Consider a sufficiently fine subdivision of the triangulation of M/G, such that $K' \subset V/G$ and consider on M a structure of $G-CW$-complex which induces this new triangulation of M/G.

Let $K = p^{-1}(K')$.

□

The next theorem, is an equivariant version of the Hopf construction for coincidence when G is a compact Lie group and the G-action on V is not necessarily free.

Theorem 3.7. *Let M be a compact, connected G-manifold and V a G-invariant open subset of M and suppose that $|WH_i|$ is finite for any isotropy type (H_i) of V. Let $f, h : V \to M$ be compactly coincident G-maps. Then there exist a G-subcomplex $K \in V$ with $Coin(f, h) \subset intK$, $p(K) \subset V/G$ finite and a G-map $f' : V \to M$, $G-\epsilon$-homotopic to f such that, $Coin(f'_{H_i}, h_{H_i})$ is finite* *and $p(Coin(f'_{H_i}, h_{H_i}))$ lies in the interior of some maximal simplexes of $p(K)$ for all isotropy type (H_i) of V.*

Proof. Let K be the G-subcomplex given by 3.6, and $\{(H_1), (H_2), ...(H_r)\}$ the isotropy types of V with associated filtration $V_1, V_2, ...V_r = V$ and denote by $K_i = K \cap V_i$, for all $i = 1, 2, ...r$.

In $V - K$ define $f' = f$, we will construct f' on K.

Let $K^d_{H_1} = K_{H_1} \cap (M^{H_1})^d$. Since $p(K)$ is finite, for $d = 0$ define $f' = f$.

Now, suppose $d = 1$.

Let $A^s_1 = \{\overline{\sigma}^s \in p(K^1_{H_1}) | \overline{\sigma}^s \cap p(Coin(f_{H_1}, h_{H_1})) \neq \emptyset\}$. Choose $\Delta^0 \in p^{-1}(A^0_1)$, so $f(\Delta^0) \in (M^{H_1})^1$. Since all connected component of $(M^{H_1})^1$ have dimension 1, for all neighborhood U of $f(\Delta^0)$ with $diam(U) < \epsilon_0$, there exist some point $y \in U$, such that $G_y = G_{f(\Delta^0)}$. Then, define f' on Δ^0 so that, $0 < d(f'(\Delta^0), h(\Delta^0)) < \epsilon_0$. For each $g \in G$, define f' on $g.\Delta^0$ by $f'(g.\Delta^0) = g.f'(\Delta^0)$.

Repeat this procedure for all 0-cells in $p^{-1}(A^0_1) - G(\Delta^0)$. For the other 0-cells of $K^1_{H_1}$ define $f' = f$.

Let $\Delta^1 \in p^{-1}(A^1_1)$. For all $x \in \partial\Delta^1$, $f'(x) \neq h(x)$, so by lemma 2 of [H], we can extend f' to Δ^1 with, $d(f', f) < \epsilon_1$ and $Coin(f', h) = \emptyset$ or it contains exactly one coincidence point in Δ^1. For all $g \in G$ define f' on $g.\Delta^1$ as follows: If $y \in g.\Delta^1$, then there exists $x \in \Delta^1$ with $y = g.x$ and we define $f'(y) = g.f'(x)$.

Repeat this procedure for all 1-cells of $p^{-1}(A^1_1) - G(\Delta^1)$. For the other 1-cells of $K^1_{H_1}$ define $f' = f$.

Now, suppose $d = 2$.

Let $A^s_2 = \{\sigma^s \in p(K^2_{H_1}) | \overline{\sigma}^s \cap p(Coin(f_{H_1}, h_{H_1})) \neq \emptyset\}$. Consider $\Delta^0 \in p^{-1}(A^0_2)$, so $f(\Delta^0) \in (M^{H_1})^2$ and since all connected component of $(M^{H_1})^2$ has dimension 2, for any neighborhood U of $f(\Delta^0)$ with $diam(U) < \epsilon_0$, there exist some point $y \in U$ such that $G_y = G_{f(\Delta^0)}$. Then define f' on Δ^0 so that, $0 < d(f'(\Delta^0), h(\Delta^0)) < \epsilon_0$. For each $g \in G$, define f' on $g.\Delta^0$ by $f'(g.\Delta^0) = g.f'(\Delta^0)$.

Repeat this procedure for all 0-cells in $p^{-1}(A^0_2) - G(\Delta^0)$. For the other 0-cells of $K^2_{H_1}$ define $f' = f$.

Let $\Delta^1 \in p^{-1}(A^1_2)$. For all point $x \in \partial\Delta^1$, $f'(x) \neq h(x)$, then by 3.1.i), we can extend f' to Δ^1, with $d(f', f) < \epsilon_1$ and $f'(x) \neq h(x)$, for all $x \in \Delta^1$.

For all $g \in G$ define f' on $g.\Delta^1$ as follows: If $y \in g.\Delta^1$, then there exists $x \in \Delta^1$ with $y = g.x$ and we define $f'(y) = g.f'(x)$.

Repeat this procedure for all 1-cells of $p^{-1}(A^1_2) - G(\Delta^1)$. For the other 1-cells of $K^2_{H_1}$ define $f' = f$.

Now, let $\Delta^2 \in p^{-1}(A^2_2)$, then f' is defined on $\partial\Delta^2$ such that, $d(f', f) < \epsilon_1$ and $Coin(f', h) = \emptyset$ on $\partial\Delta^2$, so by lemma 2 of [H] it is possible to extend f' to Δ^2 with $d(f', f) < \epsilon_2$ and $Coin(f', h) = \emptyset$ or it contains exactly one coincidence point on Δ^2.

For all $g \in G$ define f' on $g.\Delta^2$ as above.

Repeat this procedure for all 2-cells of $p^{-1}(A^2_2) - G(\Delta^2)$. For the other 2-cells of $K^2_{H_1}$ define $f' = f$.

Similarly we define f' on $K^d_{(H_1)}$ for all $3 \leq d \leq n$, where $n = dim M$.

Then F' is defined on $K_{(H_1)} = \bigsqcup_{i=1}^n K^i_{(H_1)}$ such that $Coin(f', h)$ is finite, $p(Coin(f', h))$ lies in the interior of some maximal simplexes of $p(K_{(H_1)})$ and $d(f', f) < \epsilon_{n+1}$.

Since $K_{(H_1)} = K_1$, we have f' defined on K_1.

Now we will define f' on K_2.

Let $K_{H_2}' = \{\Delta^s \in K_{H_2} | \overline{\Delta}^s \cap K_1 = \emptyset\}$, $0 \leq s \leq m$ and $K'_2 = K_{H_2} - K_{H_2}'$

In K_{H_2}' define f' in the same as in K_{H_1}. Define f' on K'_2 as follows:

Let ${K'_2}^d = K'_2 \cap (M^{H_2})^d$, $0 \leq d \leq m$.

If $d = 0$, then ${K'_2}^0 = \emptyset$. Now, suppose $d = 1$.

Consider $\Delta^1 \in {K'_2}^1$. Then the boundary Δ^1 consists of two 0-cells, Δ^0_1 and Δ^0_2 such that:

a) $\Delta^0_1 \in K_1$ and $\Delta^0_2 \in K_{H_2}'$

We can have:

i) $\Delta^0_1 \in K^1_{H_1}$, in this case $f'(\Delta^0_1) \neq h(\Delta^0_1)$.

Since $\Delta_2^0 \in {K_{H_2}}'^1$, we may apply the lemma 2 of [H], and extend f' to Δ^1 such that, $d(f', f) < \epsilon_1$ and $Coin(f', h) = \emptyset$ or it contains exactly one coincidence point on Δ^1.

ii) $\Delta_1^0 \in K_{H_1}^0$, in this case $f'(\Delta_1^0) = h(\Delta_1^0)$.

Since $\Delta_2^0 \in {K_{H_2}}'$, by 3.1.ii.b), we may extend f' on Δ^1 such that, $d(f', f) < \epsilon_1$ and $Coin(f', h) \cap \overline{\Delta}^1 = \Delta_1^0$.

b) $\Delta_1^0, \Delta_2^0 \in K_1$ and $\Delta^1 \cap K_1 = \emptyset$.

In this case Δ_1^0 and Δ_2^0 are in $K_{H_1}^0$. Thus $f'(\Delta_1^0) = h(\Delta_1^0)$ and $f'(\Delta_2^0) = h(\Delta_2^0)$, so by 3.1.ii.b) we may extend f' on Δ^1 such that, $d(f', f) < \epsilon_1$ and $Coin(f', h) \cap \overline{\Delta}^1 = \{\Delta_1^0, \Delta_2^0\}$.

In the cases a) and b), we extend f' to $G(\Delta^1)$ by $f'(g.\Delta^1) = g.f'(\Delta^1)$, for all $g \in G$.

Repeat this procedure for all 1-cells of $K_2'^1 - G(\Delta^1)$

Proceeding this way to $d = n$, we will have f' defined on $K_{(H_2)}$ such that $d(f', f) < \epsilon_{n+1}$ and $Coin(f', h)$ is finite and each point in $p(Coin(f', h))$ lies in the interior of a maximal simplex of $p(K_{(H_2)})$.

Since $K_2 = K_1 \bigsqcup K_{(H_2)}$, (disjoint union), we have f' defined on K_2.

Now, repeat this process for all K_j, $j = 3, 4 \ldots r$.

Since $2\epsilon_{n+1} < \delta$, follows from 3.2 that f' and f are G-ϵ-homotopic by a constant homotopy in the coincidence points of f' and f.

□

Corollary 3.8. *Let M be a compact, connected G-manifold and V a G-invariant open subset of M and suppose that $|WH_i|$ is finite for any isotropy type (H_i) of V. Let $f, h : V \to M$ be compactly coincident G-maps. Then there exists a G-map $f' : V \to M$, $G - \epsilon$-homotopic to f so that, f' and h are G-compactly coincident maps.*

Proof. It follows from 3.7, since $Coin(f', h)$ is finite.

□

Proposition 3.9. *Let G be a compact Lie group with $dimG > 0$ and (N, A) a relative $G-CW$-complex such that, the G-action on $N - A$ is free and $dimN \leq n$. Let $f, h : N \to M$ be G-maps so that, the coincidence points on A are isolated. Then there exists $f' : N \to M$, $G - \epsilon$-homotopic to f relative to A such that, $Coin(f', h) \cap (N - A) = \emptyset$.*

Proof. Consider on (N, A) the estructure of $G - CW$-complex which induces on N/G a triangulation. Since $dim(G) > 0$, $dim((N - A)/G) = m < dimN$. Since each d-cell Δ of N, $1 \leq d \leq m$ is homeomorphic to a d-simplex σ^d of N/G, denoting by ϕ_Δ this homeomorphism, we may suppose that $diam(f(\phi_\Delta^{-1}(\sigma^m)) < \epsilon_{n+1}$ and $diam(h(\phi_\Delta^{-1}(\sigma^m)) < \epsilon_{n+1}$, $\forall \sigma^m \in N/G$.

Let $K = \{\Delta^d \in N - A \mid \overline{\Delta^d} \cap Coin(f, h) = \emptyset\}$, for all $0 \leq d \leq m$ and $K' = \{\Delta^d \in N - A \mid \overline{\Delta^d} \cap A \neq \emptyset\}$, for all $1 \leq d \leq m$.

In A, define $f' = f$ and in $N - A$ define f' as follows:

In K, repeat the process of 3.7.

In K', given $\sigma^1 \in p(K')$, choose a 1-cell $\Delta^1 \in p^{-1}(\sigma^1)$, then in the boundary of Δ^1 there exist at most two coincidence points, which are in A.

If $Coin(f, h) \cap \Delta^1 \neq \emptyset$, we can extend f' by 3.1.ii.b) to Δ^1 with $d(f', f) < \epsilon_1$ and $Coin(f', h) \subset A$.

If $Coin(f, h) \cap \Delta^1 = \emptyset$, by lemma 1 of [H] we may extend f' to Δ^1 such that $d(f', f) < \epsilon_1$ and $Coin(f', h) \cap \Delta^1 = \emptyset$.

Given another 1-cell $\Delta^{1'} \in p^{-1}(\sigma^1)$, there exists $g \in G$ with $\Delta^{1'} = g.\Delta^1$ and we define $f'(\Delta^{1'}) = g.f'(\Delta^1)$.

Repeat this procedure for all 1-simplexes in $p(K') - \sigma^1$.

So we have defined f' on the 1-cells of $N - A$ such that $d(f', f) < \epsilon_1$, without coincidence with h in the interior of 1-cell and if f' have some coincidence point with h in the boundary of some 1-cell, this coincidence point is in A.

Given $\sigma^2 \in p(K')$, choose a 2-cell $\Delta^2 \in p^{-1}(\sigma^2)$, then in the boundary of Δ^2 there exist at most a finite number of coincidence points.

If $Coin(f, h) \cap \Delta^2 \neq \emptyset$, it follows from 3.1.ii.b) that we may extend f' to Δ^2 such that $d(f', f) < \epsilon_2$ and $Coin(f', h) \subset A$.

If $Coin(f,h) \cap \Delta^2 = \emptyset$, it follows from lemma 1 of [H], that we can extend f' to Δ^2 such that $d(f',f) < \epsilon_2$ and $Coin(f',h) \cap \Delta^2 = \emptyset$.

Given another 2-cell $\Delta^{2'} \in p^{-1}(\sigma^2)$, there exists $g \in G$ with $\Delta^{2'} = g.\Delta^1$ and we define $f'(\Delta^{2'}) = g.f'(\Delta^2)$.

Repeat this procedure for all 2-cell in $p(K') - \sigma^2$.

So we can extend f' to all d-cell, $2 \leq d \leq m$, such that f' and h do not have coincidence on $N - A$. Moreover $d(f',f) < \epsilon_m < \epsilon_{n+1}$.

Since $2\epsilon_{n+1} < \delta$, f' and f are G-ϵ homotopic by 3.2.

□

Theorem 3.10. *Let M be a compact, connected G-manifold and V an G-invariant open subset of M. Let $f, h : V \to M_1$ compactly coincident G-maps, where M_1 is a connected G-manifold with $dim M_1 = n$. Then there exists a G-map $f' : V \to M_1$, $G - \epsilon$-homotopic to f such that, the isotropy subgroup of each coincidence point in $Coin(f',h)$ has finite Weyl group.*

Proof. Let $(H_1),(H_2),...(H_r)$ be a admissible ordering on isotropy types of V, with associated filtration $V_1 \subset, V_2 \subset, ... \subset V_r = V$.

Since $Coin(f,h)$ is a compact in V, by 3.6 there exist a structure of $G-CW$-complex on M that induces a triangulation on M/G and a G-subcomplex $K \subset V$, with $Coin(f,h) \subset int K$.

Denote by $K_i^d = K \cap V_i^d$, where V_i^d is the union of the connected components of V_i of dimension d.

Let $i_1 \in \{1,2,...r\}$, be the first index for which $dim WH_{i_1} > 0$.

By 3.7, we can suppose that $Coin(f_{H_i}, h_{H_i})$ is finite and each coincidence lies in the interior of some maximal simplex of K_i for $1 \leq i < i_1$, where $K_i = K \cap V_{H_i}$.

Let $N_{i_1}^d = V_{H_{i_1}}^d \cap V_{i_1-1}^d \subset (M^{H_{i_1}})^d$.

The pair $(N_{i_1}^d, V_{i_1-1}^d)$ is a relative $G-CW$-complex, applying 3.9 we obtain a WH_{i_1}-map $f_{i_1}^{d'} : (N_{i_1}^d, V_{i_1-1}^d) \to (N_{i_1}^d, V_{i_1-1}^d)$, WH_{i_1}-homotopic to $f_{H_{i_1}}^d$ relative to $V_{i_1-1}^d$ and such that $Coin({f^{d'}}_{i_1}, {h_{H_{i_1}}}^d) \cap V_{i_1-1}^d = \emptyset$.

Extend ${f^{d'}}_{i_1}$ to $V_{(H_{i_1})}^d$ as follows:

For each $y \in V^d_{(H_{i_1})}$ there is one point $x \in V^d_{H_{i_1}}$ such that $y = g.x$, where $g \in WH_{i_1}$.

Define ${f^{d'}}_{i_1}(y) = g.{f^{d'}}_{i_1}(x)$.

Then we have ${f^{d'}}_{i_1}$ defined on $V^d_{i_1} = V^d_{i_1-1} \bigsqcup V^d_{(H_{i_1})}$, such that $Coin({f^{d'}}_{i_1}, h) \cap V^d_{(H_{i_1})} = \emptyset$.

Do the same for all index $j \in \{i_1, i_1+1, ...r\}$ for which $dimWH_j > 0$.

□

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