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# ON THE IMAGE OF THE GENERALIZED GAUSS MAP OF A COMPLETE MINIMAL SURFACE IN $\mathbb{R}^4$

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The generalized Gauss map of an immersed oriented surface M in $\mathbb{R}^4$ is the map which associates to each point of M its oriented tangent plane in $G_{2,4}$, the Grassmannian of oriented planes in $\mathbb{R}^4$. The Grassmannian $G_{2,4}$ is naturally identified with $Q_2$, the complex hyperquadric $\{[z_1,z_2,z_3,z_4] \mid \sum_{k=1}^{4} z_k^2=0\}$ in $P^3(\mathbb{C})$. The normalized Fubini-Study metric on $P^3(\mathbb{C})$ with holomorphic curvature 2 induces an invariant metric on $Q_2 \cong G_{2,4}$, which corresponds exactly to the metric on the canonical representation of $S^2(1/\sqrt{2})\times S^2(1/\sqrt{2})$ in $\mathbb{R}^6$ as $\{X\in\mathbb{R}^6 \mid x_1^2+x_2^2+x_3^2 = \frac{1}{2},\ x_4^2+x_5^2+x_6^2 = \frac{1}{2}\}$. The product representation above allows us to associate with any map g in $Q_2$ two canonical projections $g_1$, $g_2$. In the case where g is complex analytic map defined on some Riemann surface $S_0$, the projections $g_1$, $g_2$ are complex analytic also. Detailed treatment can be found in the recent work of Hoffman and Osserman [5].

The study of the image of the Gauss map of a complete minimal surface in $\mathbb{R}^3$ was motivated in one way to generalize a classical theorem of S. Bernstein [1], and was initiated by Osserman [7,8,9] And the value distribution of the generalized Gauss map of a complete

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minimal surface in $\mathbb{R}^4$, due to the product representation of $Q_2$, can therefore be studied in a similar manner. In fact, results treating the case in $\mathbb{R}^3$ have been extended to that in $\mathbb{R}^4$ by Chern [3], Osserman [9], Hoffman and Osserman [5]. Very recently, Xavier [10] has made a remarkable improvement in the study of the case in $\mathbb{R}^3$. Therefore its' quite natural to extend it to the case in $\mathbb{R}^4$, which will be shown in the following theorem.

THEOREM 1 - Let S be a complete minimal surface in $\mathbb{R}^4$ with g its generalized Gauss map and $g_1$, $g_2$, the corresponding projections. Then S must be a plane if

(i) both $g_1$ and $g_2$ omit more than 10 points, or

(ii) one projection is constant and the other omits more than 6 points.

PROOF - Let S be given by

$$X: S_0 \longrightarrow \mathbb{R}^4 \tag{1}$$

where $S_0$ is a Riemann surface. Its generalized Gauss map can be expressed by

$$g = [\phi_1(\zeta), \phi_2(\zeta), \phi_3(\zeta), \phi_4(\zeta)] \tag{2}$$

where

$$\phi_k(\zeta) = 2\frac{\partial x_k}{\partial \zeta} \tag{3}$$

with $\zeta$ a local complex parameter. And the projection $g_1$, $g_2$ are expressed by

$$g_1 = \frac{\phi_3 + i\phi_4}{\phi_1 - i\phi_2}, \quad g_2 = \frac{\phi_3 - i\phi_4}{\phi_1 - i\phi_2}. \tag{4}$$

The induced metric is given by

(5) $$ds^2 = \frac{1}{4}|f|^2(1+|g_1|^2)(1+|g_2|^2)|d\zeta|^2$$

where $f(\zeta) = \phi_1 - i\phi_2$. For detailed explanation, see Osserman [9].

Without loss of generality, we may assume $S_0$ to be simply connected. Combining our hypothese in (i), (ii) with the Koebe uniformization theorem and the Picard's theorem, we may assume further that $S_0$ is the unit disk $D = \{\zeta\in\mathbb{C} \mid |\zeta|<1\}$.

A crucial lemma used by Xavier [10] can be adapted easily in our case as:

LEMMA - Let $g_1 : D \longrightarrow \mathbb{C} - \{0,a\}(a\neq 0)$, $g_2 = D \longrightarrow \mathbb{C} - \{0,b\}(b\neq 0)$ be holomorphic functions. Then

$$\int_D \left|\frac{g_1'(\zeta)}{g_1(\zeta)}\right|^p \left|\frac{g_2'(\zeta)}{g_2(\zeta)}\right|^p d\xi d\eta < \infty$$

for $0 \leq p < \frac{1}{4}$, where $\zeta = \xi + i\eta$.

Now we proceed our proof. Suppose S is not a plane. Under the hypothese in (i) or (ii), we may assume that both $g_1$ and $g_2$ are holomorphic.

For the case (i), suppose $g_1$ omits $a_1,\ldots,a_{10}$ in $\mathbb{C}$ and $g_2$ omits $b_1,\ldots,b_{10}$ in $\mathbb{C}$. Consider the function

(6) $$h = g_1' g_2' f^{-9} \prod_{i=1}^{10} (g_1 - a_i)^{-1} \prod_{j=1}^{10} (g_2 - b_j)^{-1}.$$

For the case (ii), suppose $g_1$ constant and $g_2$ omits $b_1,\ldots,b_6$ in $\mathbb{C}$. And consider the function

(7) $$h = g_2' f^{-32/7} \prod_{j=1}^{6} (g_2 - b_j)^{-1}$$

In both cases, using the same arguments in [10], we can see that from one hand, essentially due to a theorem of Yau [11].

(8) $|h| \notin L^{2/9}(S_0)$ for case (i) and $|h| \notin L^{7/16}(S_0)$ for case (ii)

and from the other hand, by direct calculation, we get

(9) $|h| \in L^{2/9}(S_0)$ for case (i) and $|h| \in L^{7/16}(S_0)$ for case (ii).

which is impossible. Q.E.D.

Next we shall extend a theorem of Gackstatter [6] on complete abelian minimal surfaces in $\mathbb{R}^3$ to those in $\mathbb{R}^4$.

THEOREM 2 - Let S be a complete abelian minimal surface in $\mathbb{R}^4$, and g its generalized Gauss map. Then S must be a plane if either

a) one projection, say $g_1$, omits more than 4 points and the other projection $g_2$ omits more than 3 points, or

b) $g_1$ is constant and $g_2$ omits more than 3 points.

PROOF - By a complete abelian minimal surface S in $\mathbb{R}^4$. We mean that S can be constructed out of a meromorphic differential $f d\zeta$ and two meromorphic functions $g_1$, $g_2$ on a compact Riemann surface $\bar{M}$ with the metric

$$ds^2 = \frac{|f|^2}{4}(1+|g_1|^2)(1+|g_2|^2)|d\zeta|^2$$

which never vanishes. And the construction is made in the sense of L.Bers [2] such that the immersion is given by the formula

(10) $$x = \mathrm{Re} \int \frac{f}{2}(1+g_1 g_2, i(1-g_1 g_2), g_1 - g_2, -i(g_1+g_2)) d\zeta$$

on a covering space M over $\bar{M} - \{p \mid ds^2(p) = \infty\}$ as long as (10) is well-defined. The boudary points to the metric $ds^2$ are those finitely many points $p_1, \ldots, p_r$ in $\bar{M}$ where $ds^2 = \infty$.

By a rotation of S, we may assume that

i) both $g_1$ and $g_2$ have only simple poles, and they don't have poles in common,

ii) the poles of $g_1$, $g_2$ don't fall into the boundary points $p_1, \ldots, p_r$, and hence,

iii) at each pole of $g_1$ or $g_2$, f must have a simple zero,

iv) f has no other zeros, and

v) at each $p_j$, f must have a pole of order $m_j \geq 1$.

Now suppose $g_1$ is an $N_1$-sheet and $g_2$ is an $N_2$-sheet branching covering. Then by the Riemann relation for the differential $f d\zeta$, we have

$$(11) \qquad (N_1 + N_2) - \sum_{j=1}^{r} m_j = 2\gamma - 2$$

where $\gamma$ is the genus of $\bar{M}$.

And by the Riemann relation for $g_1$ and $g_2$, in case of non-constant, we have

$$(12) \qquad \sum_{g_1} (\ell_1 - 1) - 2N_1 = 2\gamma - 2$$

$$(13) \qquad \sum_{g_2} (\ell_2 - 1) - 2N_2 = 2\gamma - 2$$

where $\sum_{g_1} (\ell_1 - 1)$ and $\sum_{g_2} (\ell_2 - 1)$ are the total branching orders of $g_1$, $g_2$, respectively.

Now suppose S is non-flat, i.e, $g_1$, $g_2$ can't both be constant, and that

a) $g_1$ omits 5 values $a_1,\ldots,a_5$, $g_2$ omits 4 values $b_1,\ldots,b_4$ and neither one is constant. Then clearly

$$g_1^{-1}\{a_\nu \mid 1 \le \nu \le 5\} \subset \{p_1,\ldots,p_r\}, \tag{14}$$

$$g_2^{-1}\{b_\mu \mid 1 \le \mu \le 4\} \subset \{p_1,\ldots,p_r\}. \tag{15}$$

And (12), (13) can be written as

$$\sum_{g_1 \ne a_\nu} (\ell_1 - 1) + 3N_1 = 2\gamma - 2 + \sum_{g_1 = a_\nu} 1, \tag{16}$$

$$\sum_{g_2 \ne b_\mu} (\ell_2 - 1) + 2N_2 = 2\gamma - 2 + \sum_{g_2 = b_\mu} 1. \tag{17}$$

Comparing with (11), we get

$$2\sum_{j=1}^{r} m_j < \sum_{g_1 = a_\nu} 1 + \sum_{g_2 = b_\mu} 1 \tag{18}$$

which contradicts (14) and (15).

b) $g_1$ constant and $g_2$ omits 4 points $b_1,\ldots,b_4$. Clearly (15) and (17) still hold with $N_1 = 0$, $N_2 > 0$. From (11), (17), we have

$$\sum_{j=1}^{r} m_j < \sum_{g_2 = b_\mu} 1 \tag{19}$$

which contradicts (15). Q.E.D.

COROLLARY - If a) $g_1$ omits exactly 4 points and $g_2$ omits exactly 4 points, or

b) $g_1$ constant and $g_2$ omits exactly 3 points, then a) $r = 4$, $m_j = 1$ or b) $r = 3$, $m_j = 1$, respectively. Further, in neither case S can have flat points.

PROOF - Note that p is a flat point of S if and only if $g_1'(p) = 0$ and $g_2'(p) = 0$. In case a) comparing (11) with

$$\sum_{g_1 \neq a_\nu} (\ell_1 - 1) + 2N_1 = 2\gamma - 2 + \sum_{g_1 = a_\nu} 1$$

and (17), we get $r = 4$, $m_j = 1$, $\ell_1 \equiv 1$, $\ell_2 \equiv 1$.

And in case b) comparing (11) with

$$\sum_{g_2 \neq b_\mu} (\ell_2 - 1) + N_2 = 2\gamma - 2 + \sum_{g_2 = b_\mu} 1$$

and $N_1 = 0$, we get $r = 3$, $m_j = 1$, $\ell_2 \equiv 1$. Q.E.D.

For complete minimal surface with finite total curvature, it's known [4] that $M = \bar{M} - \{p_1, \ldots, p_r\}$ and $m_j \geq 2$. Thus, Theo.2 & Cor. together give an alternative proof of

THEOREM 3 (Hoffman-Osserman [5]) - Let S be a complete minimal surface in $\mathbb{R}^4$ with finite total curvature. Then S must be a plane if

a) both $g_1$ and $g_2$ omit more than 3 points, or

b) $g_1$ constant and $g_2$ omits more than 2 points.

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