MATRIX FUNCTIONS IN THE STABILITY OF HIGHER-ORDER NUMERICAL SCHEMES

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Approximation schemes, constructed by finite differences or variational methods, for the numerical solution of linear partial differential equations of parabolic and hyperbolic type can be written in compact form

$$L(\gamma)\, U = b, \tag{1}$$

where $L(\gamma) = A_0\,\gamma^m + A_1\,\gamma^{m-1} + \ldots + A_m$ is a polynomial with nxn matrix coefficients A_j, A_0 non-singular or simply $A_0=I$ for explicit schemes, γU stands for the translation operator U_{k+1} (discrete schemes) or the derivative $U'(t)$ (semi-discrete schemes) and b contains the boundary conditions, among other things.

The numerical stability of (1) is usually analyzed by transforming (1) into a first-order scheme having the companion matrix A as its coefficient and with zero forcing term. From this, there is a departure work. Either we impose conditions on A so that bounds can be found for the matrix semigroup solution E(A,S) ($=e^{-At}$ semidiscrete, $(-A)^k$ discrete) or we assume boundary conditions that will allow to solve difference equations by the Von Neuman's technique of separation of variables. For most common numerical schemes, both ways coincide whenever A has a full basis of eigenvetors

w_j corresponding to the eigenvalues λ_j. In this case, we have at disposal a spectral decomposition which allows to refer the stability to the growth of the spectral values $E(A,\lambda_j)$. More precisely, the practical conditions Re $\lambda_j \leq K$ (semidiscrete) and $|\lambda_i| \leq 1$ (discrete).

We wish to observe in this work, that when the coefficients A_j share a common basis of eigenvectors, so that the Von Neumann technique [1] is applicable, the coefficients A_j must conmute. This latter consequence would then allow to use matrix function theory for determining stability restrictions for the time and spatial meshes. As a matter of illustration, we shall restrict ourselves to second-order schemes.

Accordingly to the theory developed in [5], the general solution of the matrix equation

$$U_{k+2} + BU_{k+1} + CU_k = 0, \tag{2}$$

can be written as $U_k = (D_{k+1} + D_k B)\, U_0 + D_k\, U_1$ where D_k is the matrix solution satisfying $D_1 = I$, $D_0 = 0$ (dynamical solution). It turns out that

$$D_k = D^{(k)}(0) = \frac{d^k}{dt^k}\left(e^{-\frac{B}{2}t}\left[\frac{\mathrm{Sen\,h}\,\sqrt{\Delta}t}{\sqrt{\Delta}}\right]\right)_{t=0}$$

$$\Delta = (B^2 - 4C)/4$$

whenever B and C conmute. Here D(t) is the dynamical solution of the equation $U'' + BU' + C\,U = 0$ and obtained in strict analogy to the scalar case. Let v be a common eigenvector for B and C with corresponding eigenvalues λ and γ. Then

$$D_k v = \frac{d^k}{dt^k}\left[2\, e^{-\frac{\lambda}{2}t}\ \mathrm{Sen\,h}[\sqrt{\lambda^2-4\gamma}\, t/2)/\sqrt{\lambda^2-4\gamma}\ \right]_{t=0} v = \sum_{j=1}^{k} \eta_+^{k-j}\, \eta_-^{j}\, v,$$

where $\eta\pm$ are the roots of characteristic equation

$$z^2 + \lambda z + \gamma = 0 \qquad (4)$$

We see from (3) that stability along the mode v will be achieved whenever $|\eta\pm|\leq 1$. This approach will be applied to second-order schemes arising for the discretization of the hyperbolic equation $u_{tt}=c^2 u_{xx}$ and the parabolic equation $u_{tt} + a^2 u_{xxxx} = 0$ on a finite spatial interval and with homogeneous boundary conditions.

The finite-difference scheme for the wave equation

$$u_j^{k+1} - 2u_j^k + u_j^{k-1} = \mu(u_{j+1}^k - 2u_j^k + u_{j-1}^k), \qquad (5)$$

where $\mu = (c\,\Delta t/\Delta x)^2$, gives rise to the stability condition

$$c\,\Delta t/\Delta x \leq 1, \qquad (6)$$

due to Courant-Friedrichs-Lewy. Let us see how this condition would be obtained by the use of matrix functions. First, we write (5) as a second-order matrix difference equation (2). We have $B = \mu A - I$, $C=I$ where A is a symmetric tridiagonal matrix with eigenvalues $\lambda_j = 2 + 2\,\mathrm{Cos}\; j\pi/N+1$, $(N+1)\,\Delta x=1$, and $U_k = \mathrm{Col}\;[U_1^k \;\ldots\; U_N^k]$. Replacing, the values $\lambda=\mu\lambda_j-2$ and $\gamma=1$ into the characteristic equation (4), it turns out $|\frac{1}{2}(-\lambda\pm\sqrt{\lambda^2-4\gamma})|\leq 1$ which is equivalent to condition (6).

Twizell and Kahliq [6] have derived a discretization for the parabolic equation $u_{tt}+a^2 u_{xxxx}=0$ in the following manner. They approximate the fourth-order derivative in such a way to obtain the semidiscrete scheme

$$U'' + a^2 A U = 0 \qquad (7)$$

where A is a symmetric banded matrix with eigenvalues $\lambda_j = 16(\Delta x)^{-4}$ $\mathrm{Sen}^4\, j\pi/2(N+1)$, $(N+1)\,\Delta x = 1$. The discretization in time of (7) is made upon the recurrence relation satisfied by the solution of (7), that is

$$U(t+h) + U(t-h) = 2D'(h)\; U(t) \tag{8}$$

where $D(t) = \mathrm{Sen}\; a\sqrt{A}\;\; t/\; a\sqrt{A}$ is the dynamical solution of (7). By using (2,2) Padē approximants for $D'(h) = \mathrm{Cos}\; a\sqrt{A}\;\; t$, it turns out again (2) with $B = R^{-1}\,(A)\; Q(A)$, $C = I$, where

$$R(\lambda) = 1 + \frac{1}{12}\, a^2(\Delta t)^2\; A + \frac{1}{144}\,(a\Delta t)^4\; A^2,$$

$$Q(\lambda) = -\,(2 - \frac{5}{6}\,(a\;\Delta t)^2\lambda + \frac{1}{72}\,(a\;\Delta t)^4\;\lambda^2).$$

We then have the characteristic equation $R(\lambda)\eta^2 + Q(\lambda)\eta + R(\lambda) = 0$. It can be shown that its satisfy $|\eta| \leq 1$ regardless of the value $r = \Delta t/(\Delta x)^2 > 0$.

It is under study the application of matrix functions to second-order, centered difference approximations to the shallow water equations [7].

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