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ABSTRACT. Let L be an RA loop, that is, a loop whose loop ring in any characteristic is an alternative, but not associative, ring. We show that every central unit in the integral loop ring $\mathbb{Z}L$ is the product $\ell\mu_0$ of an element $\ell \in L$ and a loop ring element μ_0 whose support is in the torsion subloop of L, and use this result to determine when all central units of $\mathbb{Z}L$ are trivial.

1. INTRODUCTION

A *loop* is a set L together with a binary operation denoted by juxtaposition and an identity element 1 for which the right and left translation maps $R(x)$ and $L(x)$, $x \in L$, defined by

$$aR(x) = ax, \quad aL(x) = xa \quad \text{for } a \in L$$

are bijections. For $x, y \in L$, define maps $T(x)$, $R(x,y)$, $L(x,y)$ by

$$T(x) = R(x)L(x)^{-1}$$
$$R(x,y) = R(x)R(y)R(xy)^{-1}$$
$$L(x,y) = L(x)L(y)L(yx)^{-1}.$$

The subgroup $\mathrm{Inn}(L)$ of the symmetric group on L generated by all such maps is called the *inner mapping* group of L and the elements of $\mathrm{Inn}(L)$ are called *inner maps*. In this paper, all loops are *Moufang*, that is, they satisfy the identity $(xy \cdot z)y = x(y \cdot zy)$. In a Moufang loop, $L(x,y) = R(x^{-1}, y^{-1})$ [GJM96, Theorem II.3.3], so the inner mapping group is generated just by maps of the form $T(x)$ and $R(x,y)$. A subloop K of a loop L is *normal* if $K\theta \subseteq L$ for all inner maps θ.

For any $x, y, z \in L$, we use (x,y) and (x,y,z), respectively, to denote the *commutator* of x and y and the *associator* of x, y, and z. These elements are defined by the equations

$$xy = (yx)(x,y) \quad \text{and} \quad xy \cdot z = (x \cdot yz)(x,y,z).$$

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If L is a loop and R is a commutative, associative ring with identity, the *loop ring* RL is defined in precisely the way the group ring is defined. Thus, an element α of RL is a finite sum $\alpha = \sum \alpha_\ell \ell$, $\alpha_\ell \in R$, $\ell \in L$. We shall have occasion to refer to the *support* of α, this being $\{\ell \in L \mid \alpha_\ell \neq 0\}$.

An *RA loop* is a loop whose loop rings, in any characteristic, are alternative, but not associative, rings. A good reference for the theory of RA loops (and the loop rings they determine) is [GJM96], though it is convenient to record here a few properties of RA loops that are of particular relevance to this paper.

Let L denote an RA loop. Then L is Moufang and an extension of its centre $\mathcal{Z}(L)$ by an elementary Abelian 2-group. This means that the square of any element of L is central, as is any element of odd order. Moreover, L has a unique nonidentity commutator, always denoted s, which is also a unique nonidentity associator. Thus, if a, b, and c are elements of L, the commutator (a, b) must be 1 or s and the associator (a, b, c) must be 1 or s. It is easy to see that s must be central and of order 2.

Perhaps the most important and useful property of an RA loop is *diassociativity*: Subloops generated by two elements are always associative. More generally, if a, b, c are elements of an RA loop which associate (in any order), then they generate a group.

2. Main Results

A *torsion* element of an RA loop L is an element x with the property that $x^n = 1$ for some positive integer n. The set T of all torsion elements of L is a locally finite normal subloop called the *torsion subloop* of L. If L is finitely generated, then T is finite [GJM96, Lemma VIII.4.1].

We denote by $\mathcal{U}(\mathbb{Z}L)$ the set of *units* (invertible elements) in the integral loop ring $\mathbb{Z}L$. Under certain conditions, any unit of $\mathbb{Z}L$ can be factored as the product of an element of ℓ and an element of the loop ring whose support is in the torsion subloop of L [GJM96, §XII.1]. As with group rings [MS], it turns out that any central unit can always be so factored.

Theorem 2.1. *Let L be an RA loop with torsion subloop T. If $\mu \in \mathbb{Z}L$ is a central unit, then $\mu = \mu_0 \ell = \ell \mu_0$ can be factored as the product of an element $\ell \in L$ and a unit $\mu_0 \in \mathbb{Z}T$.*

Proof. We show that $\mu = \mu_0 \ell$ and note that centrality of μ gives $\mu = \ell \mu_0$ too.

Replacing L by the support of μ and, if necessary, three elements which do not associate, we may assume that L is finitely generated, thus T is finite and the group algebra $\mathbb{Q}T = \sum A_i$ is the direct sum of unique simple algebras A_i with identity elements e_i which are primitive central idempotents. (This is classical if T is associative and Corollary VI.4.3 in [GJM96] if T is an RA loop.) Writing $e_i \sim e_j$ if $e_j = e_i\theta$ for some inner map θ, we obtain an equivalence relation and, partitioning the e_i into equivalence classes, we have $\mathbb{Q}T = \sum_{h=1}^{m} R_h$, each R_h a sum of those A_j whose identities form an equivalence class. In particular, each R_h is *normal* in QT in the sense that $R_h\theta = R_h$ for any $\theta \in \text{Inn}(L)$. This implies

$$R_h\alpha = \alpha R_h, \quad (R_h\alpha)\beta = R_h(\alpha\beta),$$
$$(\alpha R_h)\beta = \alpha(R_h\beta), \text{ and } \alpha(\beta R_h) = (\alpha\beta)R_h$$

for any $\alpha, \beta \in QT$.

Let Q be a *transversal* for T in L (that is, a set of representatives of the cosets of T in L) and write $\mu = \sum_{i=1}^{n} \mu_i q_i$, $\mu_i \in ZT$, $q_i \in Q$. Writing $\mu_i = \sum_{h=1}^{m} \mu_{i,h}$, $\mu_{i,h} \in R_h$, we obtain

$$\mu = \nu_1 + \nu_2 + \cdots + \nu_m, \qquad \nu_h = \sum_{i=1}^{n} \mu_{i,h} q_i, \quad h = 1, 2, \ldots, m,$$

and argue that each ν_h is central. For this, it suffices to show that ν_h commutes elementwise with L [GJM96, Corollary III.4.2], so take $x \in L$. Since μ is central, $\mu x = x\mu$, so

$$\nu_1 x + \nu_2 x + \cdots + \nu_m x = x\nu_1 + x\nu_2 + \cdots + x\nu_m. \tag{2.1}$$

We claim that the supports of $\nu_h x$ and $\nu_k x$ are disjoint if $h \neq k$ (and similarly for the supports of $x\nu_h$ and $x\nu_k$). For this, consider the possibility that $(\alpha q_i)x = (\beta q_j)x$ for some i, j, some $\alpha \in R_h$, and some $\beta \in R_k$. This implies $\alpha q_i = \beta q_j$, so $i = j$ and $\alpha = \beta$, which cannot be because $R_h \cap R_k = \{0\}$. Furthermore, the supports of $\nu_h x$ and $x\nu_h$ are the same for any h. To see why, look at $\nu_h x = \sum_{i=1}^{n} (\mu_{i,h} q_i)x$. Normality of R_h implies

$$\begin{aligned}(\mu_{i,h} q_i)x \\ &= \mu_1(q_1 x) = s^\epsilon \mu_1(x q_i) = s^\epsilon(\mu_2 x)q_i == s^\epsilon(x\mu_3)q_i = s^\epsilon x(\mu_4 q_i),\end{aligned} \tag{2.2}$$

for certain $\mu_1, \mu_2, \mu_3, \mu_4 \in R_h$ and $\epsilon \in \{0, 1\}$. Since $s^2 = 1$, the projection s_i of s in each simple algebra A_i is $\pm e_i$. In particular, $s_i A_i \subseteq A_i$ and $sR_h \subseteq R_h$ for each R_h. Thus, (2.2) shows that the supports of $\nu_h x$ and $x\nu_h$ are the same, as asserted, and then (2.1) shows that $\nu_h x = x\nu_h$. Thus each ν_h is indeed central.

Now let $t \in T$ and write $\mu_{i,h} = \sum_{j=1}^{r} \alpha_j t_j$, $\alpha_j \in \mathbb{Q}$. We have

$$\begin{aligned} t\nu_h &= \sum_{i=1}^{n} t(\mu_{i,h} q_i) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{r} \alpha_j t(t_j q_i) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{r} \alpha_j s^{\epsilon_j}(tt_j)q_i, \quad s^{\epsilon_j} = (t, t_j, q_i), \quad \epsilon_j \in \{0, 1\}, \end{aligned} \tag{2.3}$$

and

$$\nu_h t = \sum_{i=1}^{n} (\mu_{i,h} q_i) t = \sum_{i=1}^{n} \sum_{j=1}^{r} \alpha_j (t_j q_i) t.$$

Now t_j, q_i, and t associate if and only if they generate a group, and this occurs if and only if t, t_j, and q_i associate. Thus $(t_j, q_i, t) = (t, t_j, q_i)$ and

$$\begin{aligned} \nu_h t &= \sum_{i=1}^{n} \sum_{j=1}^{r} \alpha_j s^{\epsilon_j} t_j (q_i t), \quad s^{\epsilon_j} = (t_j, q_i, t), \\ &= \sum_{i=1}^{n} \sum_{j=1}^{r} \alpha_j t_j (t q_i) \end{aligned}$$

since if t and q_i commute, $s^{\epsilon_j} = 1$ while, otherwise, $s^{\epsilon_j} = s$ and $q_i t = s t q_i$. (Remember that $s^2 = 1$.) Thus

$$\nu_h t = \sum_{i=1}^{n} \sum_{j=1}^{r} \alpha_j s^{\epsilon_j} (t_j t) q_i. \tag{2.4}$$

Comparing the coefficients of q_i in (2.3) and (2.4), we have

$$\sum_{j=1}^{r} \alpha_j s^{\epsilon_j} (t t_j) = \sum_{j=1}^{r} \alpha_j s^{\epsilon_j} (t_j t) = \alpha, \quad \text{say.}$$

This equation shows that if $t_j t \neq t t_j$ (and hence $t t_j = s t_j t$), then α contains terms $s^{\epsilon_j} s t_j t$ and $s^{\epsilon_j} t_j t$, with the same coefficient. After cancelling t, it follows that

$$\mu_{i,h} = \sum_{j=1}^{r} \alpha_j t_j = \sum_{t_j \in S_1} \alpha_j t_j + (1+s) \sum_{t_j \in S_2} \alpha_j t_j, \tag{2.5}$$

where S_1 is the set of those t_j which commute with t and the t_j in S_2 do not commute with t. Since $(1+s)\mathbb{Z}L$ is central [GJM96, Corollary III.3.5], the element $\mu_{i,h}$ commutes with t. Since t was arbitrary, $\mu_{i,h}$ is central in QT. Writing $R_h = A_{i_1} + A_{i_2} + \cdots + A_{i_k}$ and $\mu_{i,h} = a_1 + a_2 + \cdots + a_k$, $a_j \in A_{i_j}$, centrality of $\mu_{i,h}$ implies centrality of each a_j.

With $\mu_{i,h} = \sum_{j=1}^{r} \alpha_j t_j$ as before, and $x \in T$, we have

$$\mu_{i,h} T(x) = \sum_{j=1}^{r} \alpha_j t_j T(x) = \sum_{j=1}^{r} \alpha_j s^{\epsilon_j} t_j, \quad \epsilon_j \in \{0,1\},$$

since $t_j T(x) = x^{-1} t_j x = t_j$ or $s t_j$. Thus $\mu_{i,h} T(x) - \mu_{i,h} = (1-s)\omega_1$, $\omega_1 \in QT$. Similarly, if $x, y \in L$,

$$\mu_{i,h} R(x,y) = \sum_{j=1}^{r} \alpha_j t_j R(x,y) = \sum_{j=1}^{r} \alpha_j s^{\epsilon_j} t_j, \quad \epsilon_j \in \{0,1\},$$

since $t_jR(x,y) = (t_jx \cdot y)(xy)^{-1} = t_j$ or st_j according as $(t_j,x,y) = 1$ or s. Thus $\mu_{i,h}R(x,y) - \mu_{i,h} = (1-s)\omega_2$, $\omega_2 \in QT$, and so $\mu_{i,h}\theta = \mu_{i,h} + (1-s)\omega_{i,h}$ for any inner map θ. Now $\mu_{i,h} \in R_h$, which is the direct sum $A_{i_1} + A_{i_2} + \cdots + A_{i_k}$ of certain simple algebras A_{i_j}. Writing $\mu_{i,h} = (a_1, a_2, \ldots, a_k)$, $a_j \in A_{i_j}$, and choosing $\theta \in \mathrm{Inn}(L)$ such that $e_j\theta = e_1$, we have $\mu_{i,h}\theta = (a_j, \ldots)$, so $a_j = a_1 + (1-s_j)\omega_j$, where s_j the projection of s and ω_j is the projection of $\omega_{i,h}$ on A_{i_j}. It follows that $\mu_{i,h} = (a_1, a_1, \ldots, a_1) + (1-s)\omega_{i,h}$. We next show that $\mu_{i,h} \notin (1-s)QL$ so that $a_1 \neq 0$.

If every element in the support of $\mu_{i,h}$ is central in T, then $\mu_{i,h}$ is central in $R_h = A_{i_1} + A_{i_2} + \cdots + A_{i_k}$ and hence a scalar in each simple component A_{i_j}. If $\mu_{i,h} = (1-s)\omega$ for some ω, then $\mu_{i,h}$ is a zero divisor, so the projection of $\mu_{i,h}$ in each A_{i_j} is a zero divisor, hence 0. Thus $\mu_{i,h} = 0$, which is not true. On the other hand, suppose there exists $t_j \in \mathrm{supp}(\mu_{i,h})$ which is not central in T. Choose t with $tt_j \neq t_jt$ and write $\mu_{i,h} = \sum_{t_j \in S_1} \alpha_j t_j + (1+s)\beta$ as in (2.5). Suppose $\mu_{i,h} = (1-s)\omega$. Multiplying

$$\sum_{t_j \in S_1} \alpha_j t_j + (1+s)\beta = (1-s)\omega$$

by $1+s$ gives $(1+s)\sum \alpha_j t_j + 2(1+s)\beta = 0$, a contradiction, because the nonzero element β contains in its support no elements that commute with t while, for $t_j \in S_1$, both t_j and st_j do. All this proves that $\mu_{i,h} \notin (1-s)QL$ and $a_1 \neq 0$. As a consequence, $A_{i_1}a_1$ is nonzero and a two-sided ideal of the simple algebra A_{i_1}. (Recall that a_1 is central.) So $A_{i_1}a_1 = A_{i_1}$, which shows that a_1 is invertible. It follows that $a_j = a_1 + (1-s_j)\omega_j \neq 0$ for any j (otherwise, $a_1 = -(1-s_j)\omega_j$ is invertible and a zero divisor), hence $A_{i_j} = A_{i_j}a_j$. So each a_j is invertible, hence $\mu_{i,h}$ is invertible.

Recall that $\nu_h = \sum_{i=1}^n \mu_{i,h}q_i$. Write $\nu_h^{-1} = \sum_{i=1}^N \tau_i p_i$, $p_i \in \mathcal{Q}$. Since ν_h^{-1} is a central unit, we know that the coefficients τ_i, like the $\mu_{i,h}$, are invertible. In a Moufang loop, any inner map θ is a semi-automorphism in the sense that $(xyx)\theta = (x\theta)(y\theta)(x\theta)$ [GJM96, Theorem II.3.3] so, in particular, θ maps units to units. In the calculations that follow, we make use of the inner maps $T(x)$, $R(x,y)$, and $L(x,y)$, $x,y \in L$, to move parentheses and change the order of elements in a product with the assistance of the following identities:

$$yx = x[yT(x)] \quad \text{and} \quad xy = [yT(x)^{-1}]x$$
$$xy \cdot z = [xR(y,z)](yz) \quad \text{and} \quad xy \cdot z = x \cdot yzL(y,x)^{-1}$$
$$x \cdot yz = (xy)[zL(y,x)] \quad \text{and} \quad x \cdot yz = xR(y,z)^{-1}y \cdot z.$$

Setting $\theta_1 = R(q_i, \tau_j q_j)$, $\theta_2 = T(q_j)$, $\theta_3 = L(q_j, q_i)$, $\theta_4 = T(q_iq_j)^{-1}$, and $\theta_5 = R(\tau_j\theta_2\theta_3\theta_4, q_iq_j)^{-1}$, we have

$$\begin{aligned}(\mu_{i,h}q_i)(\tau_j q_j) &= [\mu_{i,h}\theta_1][q_i \cdot \tau_j q_j] \\ &= [\mu_{i,h}\theta_1][q_i \cdot q_j(\tau_j\theta_2)] \\ &= [\mu_{i,h}\theta_1][q_iq_j \cdot \tau_j\theta_2\theta_3]\end{aligned}$$

$$= [\mu_{i,h}\theta_1][\tau_j\theta_2\theta_3\theta_4 \cdot q_iq_j]$$
$$= [(\mu_{i,h}\theta_1\theta_5)(\tau_j\theta_2\theta_3\theta_4)][q_iq_j].$$

So the equation $\nu_h\nu_h^{-1} = 1$ implies an equation of the form $1 = \sum \gamma_{i,j}q_ip_j$. Since L/T is torsion-free Abelian, it can be ordered and we may assume $q_1 < q_2 < \cdots < q_n$ and $p_1 < p_2 < \cdots < p_N$. It follows that $\sum \gamma_{i,j}q_ip_j$ can have only one summand, so $n = N = 1$. In particular, $\nu_h = \mu_{1,h}q_1$.

Since μ is the sum of the ν_h, hence a sum of terms of the form $\mu_{i,h}q_i$, collecting terms involving the same q_i, we eventually obtain $\mu = \sum \mu_jq_j$, each μ_j a sum of $\mu_{i,h}$ for different hs, and hence invertible. (Think of the direct sum $\sum_{i=1}^{m} R_h$.) For the same reason, the product of two different μ_js is 0. So $\mu^2 = \sum \mu_j^2q_j^2 = \sum \alpha_jq_j^2$ with with $Tq_j^2 \neq Tq_k^2$ if $j \neq k$ and $\alpha_j \in ZT$ (because $\mu = \mu_jq_j \in ZT$ implies each $\mu_j \in ZT$). Since

$$t\mu^2 = \sum t\alpha_jq_j^2$$

and

$$\mu^2t = \sum \alpha_jtq_j^2$$

for $t \in T$, we have $\alpha_jt = t\alpha_j$ and so the ring R generated by the α_j is central in ZT. Let A be the (torsion-free Abelian) group generated by the q_j. Then μ^2 is a unit in the group ring RA. Let N be the (nil) radical of R. Since idempotents of R/N can be lifted to $R \subseteq ZT$, a ring which contains no nontrivial idempotents [MS02, Corollary 7.2.4], R/N contains no nontrivial idempotents. By [Seh70, Lemma 2], the group ring $[R/N]A$ has only trivial units, so $\mu^2 = \alpha q^2 + \gamma$, γ nilpotent. But γ is a sum of α_jq_j, which is invertible. So $\gamma = 0$, $\mu^2 = \alpha q^2$ has just one term, and the representation $\mu = \sum \mu_jq_j$ has just one term also. This completes the proof. □

In the proof of the next theorem, the second major result of this paper, for $\alpha = \sum \alpha_\ell \ell$ in ZL, we let α^θ denote the element $\sum \alpha_\ell \ell^{-1}$. (In group rings, α^θ is denoted α^*, notation which we avoid because α^* has an entirely different meaning when L is an RA loop [GJM96, §III.4].) It is easy to see that θ defines an antiautomorphism of ZL.

In any integral loop ring ZL, elements of the form $\pm\ell$, $\ell \in L$, are clearly units. These are called *trivial*. A classical theorem of Graham Higman describes those finite groups in whose group rings the only units are trivial. Subsequently, similar theorems concerning central units in group rings have appeared [RS90, DMS]. Here, we generalize to loop rings that are not associative.

Theorem 2.2. *Let L be an RA loop with torsion subloop T. The following are equivalent:*

(a) *All central units in ZL are trivial.*
(b) *T/L' and $\mathcal{Z}(L) \cap T$ each have exponent* 2, 3, 4, *or* 6.

Proof. Assume all central units in ZL are trivial. Since $\mathcal{Z}(L) \cap T$ is a central subloop of L, the integral group ring $Z[\mathcal{Z}(L) \cap T]$ has only trivial units, so $\mathcal{Z}(L) \cap T$ has exponent 2, 3, 4, or 6 by a generalization of Higman's theorem [GJM96, Theorem VIII.3.2]. Our argument that T/L' has one of these exponents too uses the fact that if G is a group of index two in L with $\mathcal{Z}(G) \subseteq \mathcal{Z}(L)$, then all central units in ZG are trivial. To see why, let μ be a central unit in ZG. Since the centre of a group ring is spanned by the (finite) class sums, we can write $\mu = \mu_0 + (1+s)\mu_1$ for some $\mu_1 \in ZG$, where μ_0 has support in $\mathcal{Z}(G)$. Since $\mathcal{Z}(G) \subseteq \mathcal{Z}(L)$ and $(1+s)ZG$ is central in ZL, μ is central in ZL and hence trivial.

Now let $1 \neq t \in T$. Since $t^2 \in \mathcal{Z}(L) \cap T$, t^2 has order 1, 2, 3, 4, or 6 by what we have already proven. Thus t has order one of 2, 3, 4, 6, 8, 12. If t has order any of 2, 3, 4, 6, then tL' has order one of these same numbers in T/L'. If t has order 8 or 12, then t cannot be central, so it belongs to a group G of index two in L with $\mathcal{Z}(G) = \mathcal{Z}(L)$ [GJM96, Corollary IV.2.3, Theorem IV.3.1 and Corollary III.3.5]. As observed, ZG has only trivial central units so, applying the main theorem of [DMS] to the finite normal subgroup $\langle t, s\rangle$ generated by t and s, we have t^j conjugate (in G) to either t or t^{-1} whenever j is an integer relatively prime to the order of t.[1] Using the fact that $L' = \{1, s\}$, it is easy to see that $t^4 = s$ when t has order 8 and $t^6 = s$ when t has order 12, so tL' has order 4 or 6, respectively, in T/L'.

Conversely, assume that T/L' and $\mathcal{Z}(L) \cap T$ have exponent 2, 3, 4, or 6. Let $\mu \in \mathcal{U}(ZL)$ be central. Write $\mu = \mu_0\ell$ as in Theorem 2.1. We compute $\mu\mu^\theta$ and in so doing note that μ_0 is a central multiple of ℓ (since μ is central). Thus μ_0 and μ_0^θ lie in the group generated by ℓ and the centre of ZL and no parentheses are required to indicate order of multiplication in

$$\mu\mu^\theta = \mu_0\ell\ell^{-1}\mu_0^\theta = \mu_0\mu_0^\theta.$$

Hence $\mu\mu^\theta$ is a central unit in ZL which lies in ZT. It follows that we can write $\mu\mu^\theta = \beta_0 + (1+s)\beta_1$ with $\beta_0 \in Z[\mathcal{Z}(L) \cap T]$ and the support of β_1 consisting of elements of $T \setminus \mathcal{Z}(L)$. Since the group ring of T/L' has only trivial units, we can write $\mu\mu^\theta = t + (1-s)\alpha$, $t \in T$, $\alpha \in ZT$. Multiplying by $1+s$ gives $(1+s)t = (1+s)\beta_0 + 2(1+s)\beta_1$. Using remarks made earlier about the supports of β_0 and β_1, we conclude that $(1+s)\beta_1 = 0$, so $\mu\mu^\theta = \beta_0 \in Z[\mathcal{Z}(L) \cap T]$. Since this group ring has only trivial units, it follows that $\mu\mu^\theta$ is trivial and hence equal to 1 (because it has nonzero coefficient of 1). So μ is trivial as claimed. □

As noted in the proof of Theorem 2.2, necessary and sufficient conditions for an integral group ring to have nontrivial central units were obtained in [DMS]. At first glance, the condition stated in the aforementioned paper looks different from condition (b) of our Theorem 2.2. Our final result shows that they are equivalent.

[1] In this paper, when we say ℓ is *conjugate* to k, we mean that $x^{-1}\ell x = k$ for some x.

Theorem 2.3. *Let L be an RA loop with torsion subloop T. The following conditions are equivalent.*

(a) *T/L' and $\mathcal{Z}(L)\cap T$ have exponent 2, 3, 4, or 6.*
(b) *for all $t\in T$ and all integers j relatively prime to the order of t, t^j is conjugate (in L) to t or to t^{-1};*

Proof. Assume (a) and let $t\in T$. Since T/L' has exponent 2, 3, 4, or 6 and L' has order 2, t has order 1, 2, 3, 4, 6, 8, or 12. In all cases but the last two, $t^j=t$ or $t^j=t^{-1}$ for any j relatively prime to the order of t. If t has order 8, then t is not central and $t^4=s$. Choosing $x\in L$ with $xt\neq tx$, we have $x^{-1}tx=st=t^5$, so t^5 is conjugate to t. Since t^3 is not central (because t^2 is central and t is not), we see similarly that t^3 is conjugate to t^{-1}. If t has order 12, then again t is not central and $t^6=s$, giving t^7 conjugate to t and t^5 conjugate to t^{-1}. We have proven (b).

Conversely, assume (b). If $t\in\mathcal{Z}(L)\cap T$ and j is relatively prime to the order of t, then $t^j=t$ or $t^j=t^{-1}$ and this forces t to have order 1, 2, 3, 4, or 6. If $t\in T$ is not central, then $t^2\in\mathcal{Z}(L)\cap T$, so t has order 1, 2, 3, 4, 6, 8, or 12. If t is of order 8, then t^3 must be conjugate to either t or t^7, and since any conjugate of t must be t or st (and s has order 2), it follows that t^3 is conjugate to t^7. Thus $s=t^4$ and tL' is of order 4. Similarly, if t is of order 12, then $s=t^6$ and tL' is of order 6. We have (a). □

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