INVERSE IMAGES OF SOME CLOSED MAPPINGS

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I) INTRODUCTION

Our purpose is to generalize some results due to K. Morita, [2], and to Y. Tanaka [3]. These results are by themselves generalizations of Stone-Morita-Hanai's theorem. We mean, by generalizations the *extension* of the results to any regular cardinal.

II) PRELIMINARY DEFINITIONS

m-infinite cardinal

DEFINITION 1 - A topological space X is m-compact if any subset of X having cardinality m has a cluster point.

DEFINITION 2 - A topological space X verifies G_m-property if the intersection of a family of cardinality m of open subsets is still open.

m-regular cardinal

DEFINITION 3 - A topological space X is n-Fréchel if given $Y \subset X$ and $p \in \bar{Y}$ there exists a n-sequence $(p_\beta)_{\beta<n}$, $p_\beta \in Y$, $\forall \beta < n$, converging to p.

III) THEOREMS

THEOREM 1 - Let f be a closed continuous function from a paracompact, locally m-compact Hausdorff space X onto a space Y. Let Y_0 be the following subset

$$Y_0 = \left\{ y \in Y \mid f^{-1}(y) \text{ is not m-compact} \right\},$$

where m ia a regular cardinal.

Then, if X satisfies property G_p, for any $p < m$, Y_0 is a closed discrete subset of Y.

PROOF - It is enough to show that $\{f^{-1}(y) \mid y \in Y_0\}$ is a discrete collection of closed subsets of X. For this, we are going to show that given C, a m-compact subset, C intersects fewer than m subsets $f^{-1}(y)$, $y \in Y_0$. This is enough having in mind that X satisfies G_p, for any $p < m$. Let us assume the contrary, that there exists a family $(x_\alpha)_{\alpha<m}$ such that:

$$x_\alpha \in C \cap f^{-1}(y_\alpha),\ y_\alpha \in Y_0,\ \alpha < m,$$

$$y_\beta \neq y_\alpha, \text{ for } \beta \neq \alpha$$

C is m-compact, so there exists a cluster point x_* for the family $(x_\alpha)_{\alpha<m}$.

Obviously we can assume $f(x_*) \neq y_\alpha$, $\forall \alpha < m$, otherwise if $f(x_*) = y_{\bar{\alpha}}$ for $\bar{\alpha} < m$ we could change the family $(x_\alpha)_{\alpha<m}$ to $(x_\alpha)_{\alpha<m,\ \alpha\neq\bar{\alpha}}$.

Being $f(x_*) = y_*$, we have

(1) $$y_* \in Y_0,\ y_* \neq y_\alpha \text{ for } \alpha < m.$$

Let us prove (1). For that assume the contrary that $y_* \in Y - Y_0$. Then $f^{-1}(y_*)$ is m-compact. X being m-compact, there exists an open subset L such that $\bar{L}$ is m-compact and $f^{-1}(y_*)\subset L$. Here it is important the condition of regularity on m to garantee the existence of such L. Let M be defined as $M = Y - f(X-L)$, then M is an open subset of Y and let

$$x_* \in f^{-1}(y_*) \subset f^{-1}(M) \subset L.$$

x_* is a cluster point of $(x_\alpha)_{\alpha<m}$ and so $x_{\bar{\alpha}} \in f^{-1}(M)$ for some $\bar{\alpha}$, $\bar{\alpha} < m$. Therefore for $\bar{\alpha}$, $f^{-1}(y_{\bar{\alpha}}) \subset f^{-1}(M) \subset L$, and $f^{-1}(f_{\bar{\alpha}})$ is m-compact and this contradicts the hypothesis that $y_\alpha \in Y_0$, $\forall \alpha<m$, and (1) is proved.

Being X paracompact and locally m-compact there exists an open covering of X, locally finite, $\{G_\beta | \beta \in \Omega\}$ such that $\overline{G_\beta}$ is m-compact for each β. Define $\Gamma = \{\beta | G_\beta \prod f^{-1}(y_*) \neq \emptyset\}$, then $|\Gamma| \geq m$ because $f^{-1}(y_*)$ is not m-compact. Let

$$G = \coprod_{\beta \in \Gamma} G_\beta,$$

G is open.

Therefore $V_0 = Y - f(X-G)$ is open and $f^{-1}(y_*) \subset f^{-1}(V_0) \subset G$. The set of the points $(x_\alpha)_{\alpha<m}$ that belong to $f^{-1}(V_0)$ has cardinality m, X being T_1.

Let us denote this family $(x_{\alpha_\beta})_{\beta<m}$. Then x_* is a cluster of $(x_{\alpha_\beta})_{\beta<m}$.

Let $D = \{y_{\alpha_\beta} | y_{\alpha_\beta} = f(x_{\alpha_\beta}),\ \beta<m\}$. We have $y_* \in \bar{D} - D$.

Since $x_{\alpha_\beta} \in f^{-1}(V_0) \Longrightarrow y_{\alpha_\beta} \in V_0$, $\forall \beta < m$.

Therefore $f^{-1}(y_{\alpha_\beta}) \subset f^{-1}(V_0) \subset G$, $\forall \beta < m$.

By transfinite induction, we can find points

$$x'_{\alpha_1} \in f^{-1}(y_{\alpha_1}) \sqcap G_{\alpha_1}$$
$$\vdots$$
$$x'_{\alpha_p} \in f^{-1}(y_{\alpha_p}) \sqcap (X - \bigsqcup_{\beta<p} G_{\gamma_\beta}), \qquad p < m$$

Since $f^{-1}(y_{\alpha_p})$ is not m-compact we can guarantee that:

$$f^{-1}(y_{\alpha_p}) \sqcap (X - \bigsqcup_{\beta<p} G_{\gamma_\beta}) \neq \emptyset, \qquad p < m$$

for any number p of sets G_γ, $\gamma \in \Gamma$. Since $x'_{\alpha_p} \in G_\gamma$ and $\gamma_p \neq \gamma_q$ for $p \neq q$ and $\{G_\gamma | \gamma \in \Gamma\}$ is locally finite. Therefore its image D, by is f, closed. But $y_0 \in \bar{D} - D$. that is impossible. So theorem 1 is proved.

THEOREM 2 - Under the same hypothesis of theorem 1, let

$$Y_1 = \left\{ y \in Y \mid \partial f^{-1}(y) \text{ is not m-compact} \right\}$$
$$(\partial f^{-1}(y) = \overline{f^{-1}(y)} \sqcap \overline{\complement f^{-1}(y)}).$$

Then $Y - Y_1$ is locally m-compact.

PROOF - Let $y \in Y - Y_1$. Then $\partial f^{-1}(y)$ is m-compact. X is locally m-compact so there exists an open subset L of X such that:

$$\partial f^{-1}(y) \subset L \text{ and } \bar{L} \text{ is m-compact.}$$

Let $U = f^{-1}(y) \sqcup L = \overset{\circ}{\widehat{f^{-1}(y)}} \sqcup L$, therefore U is open. Being $V = Y - f(X-U)$, V is open in Y and $f^{-1}(y) \subset f^{-1}(V) \subset U$ that implies $y \in V \subset f(U) \subset f(L) \sqcup \{y\}$. We conclude that $\bar{V}$ is m-compact.

THEOREM 3 - Under the same hypothesis of theorem 2, we can prove that the closure of every neighborhood of y is not m-compact for any $y \in Y_1$.

PROOF - Let us suppose that $y_1 \in Y_1$ is such that has a neighborhood W, with $\bar{W}$ m-compact. $\partial f^{-1}(y_1)$ is not m-compact. According to theorem 1, the set F, defined as below is closed in X.

$$F = \bigsqcup \left\{ f^{-1}(y) \mid y \in Y_0 - \{y_1\} \right\}$$

$F \sqcap f^{-1}(y_1) = \emptyset$ then y belongs the open set $V = Y - f(Y)$. Let V_1 be the following neighborhood of Y_1

$$V_1 = V \sqcap W$$

$\bar{V}_1 \subset \bar{W}$, so $\bar{V}_1$ is m-compact.

Since $\partial f^{-1}(y_1) \subset f^{-1}(V_1)$ and $\partial f^{-1}(y_1)$ is not m-compact there exists a locally finite family $(G_\alpha)_{\alpha \in \Gamma}$ of open subsets of X such that:

a) $\partial f^{-1}(y_1) \subset \bigsqcup_{\beta \in \Gamma} G_\alpha$

b) $|\Gamma| \geq m$ (If every Γ has cardinality $< m$, $\partial f^{-1}(y_1)$ would be m-compact, impossible)

c) $G_\alpha \subset f^{-1}(V_1)$, $\forall \alpha \in \Gamma$

d) $G_\alpha \subset \sqcap \partial f^{-1}(y_1) \neq \emptyset$, $\forall \alpha \in \Gamma$.

We can choose then, for any $\alpha \in \Gamma$

$$x_\alpha \in (X - f^{-1}(y_1)) \sqcap G_\alpha .$$

The family $(G_\alpha)_{\alpha \in \Gamma}$ is locally finite, then each point of $\bigsqcup_{\alpha \in \Gamma} G_\alpha$ belongs to a finite number of G_α's.

We can choose, if necessary, a subfamily of $(x_\alpha)_{\alpha \in \Gamma}$, that we denote $(x_\alpha)_{\alpha \in \Gamma'}$ such that $|\Gamma'| = m$, and $\beta \neq \beta'$ implies $x_\beta \neq x_{\beta'}$,

β and β' in Γ. So $A = \bigsqcup_{\alpha\in\Gamma'} \{x_\alpha\}$ is closed discrete and has cardinality m. Let us prove f(A) discrete and $f(A)\subset\bar{V}_1$, so f(A) must have cardinality $p < m$. So $|A| < m$, which is impossible.

THEOREM 4 - Let $f: X \to Y$ be a continuous closed mapping from X onto Y. X is paracompact and Y is locally m-compact. Then for every $y\in Y$, $\partial f^{-1}(y)$ is m-compact.

REMARK - Before we start to prove it is good to observe that X is not necessary locally m-compact. Let X be a space that is not m-compact. Let $Y = \{p\}$ a unitary set with unique is possible topology and let $f: X \to Y$ be a constant map. We have all the conditions from theorem 4.

PROOF - Assuming the contrary, there exists $y_0\in Y$ such that $\partial f^{-1}(y_0)$ is not m-compact. Hence there exists a family $(x_\alpha)_{\alpha<m}$, $x_\alpha\in\partial f^{-1}(y_0)$ $\forall\alpha<m$, that does not have a cluster point. Being X paracompact there exists a discrete family of open sets $(A_\alpha)_{\alpha<m}$ such that $x_\alpha\in A_\alpha$, $\forall\alpha<m$.

Let V be an open neighborhood of y_0 such that $\bar{V}$ is m-compact. Let $V_\alpha = A_\alpha \sqcap f^{-1}(V)$.

We are going to choose by transfinite induction a family $(x'_\alpha)_{\alpha<m}$, $x'_\alpha\in A$, $\forall\alpha<m$ such that

1) $\{f(x'_\alpha) \mid \alpha<m\}$ is closed discrete and is contained in $\bar{V}$

2) $f(x'_\alpha) \neq f(x'_\beta)$ for $\alpha\neq\beta$, α and $\beta < m$

3) $x'_\alpha\in V_\alpha \sqcap (X-f^{-1}(y_0))$, $\forall\alpha<m$

For $\alpha = 0$, as $x_0\in\partial f^{-1}(y_0)$ let us choose x'_0 such that

$$x'_0\in V_0 \sqcap (X-f^{-1}(y_0)).$$

Suppose we have choosen every x_α, $\alpha < \delta < n$. Then

$$F_\alpha = \{f(x'_\beta) \mid \beta \le \alpha\}$$

is closed discrete and is defined having in mind that

$$x'_\beta \in V_\beta \sqcap (X - f^{-1}(y_0))$$

and all $f(x'_\beta)$ are distinct. $F = \bigsqcup_{\alpha<\delta} F_\alpha$ is closed in Y, and $y_0 \notin F$. Hence there exists a neighborhood W of y_0 disjoint from F. So we can choose

$$x'_\delta \in V_\delta \sqcap (X - f^{-1}(y_0) \sqcap f^{-1}(W)$$

therefore $x'_\delta \in V_\delta \sqcap (X - f^{-1}(y_0))$ and $f(x'_\delta) \in F$.

Let $F_\delta = F \sqcup \{f(x'_\delta)\}$.

F_δ satisfies the conditions and it is possible to construct a discrete closed subset $\{f(x'_\alpha) \mid \alpha < m\} \subset \bar{V}$. This is impossible because $\bar{V}$ is m-compact. □

Let m, n and p be infinite cardinals such that $m<n<p$ and n is regular.

THEOREM 5 - Let $f: X \rightarrow Y$ be a continuous closed mapping from X onto Y. X is paracompact and n-Fréchet and there exists a family $\mathcal{B}$ of closed subsets of Y such that: if $K \subset U$, K m-compact and U open subset of Y then $K \subset \bigsqcup_{F \in \mathcal{F}} F$ where $\mathcal{F} \subset \mathcal{B}$, $|\mathcal{F}| = m$ and $\mathcal{B}$ is n-puntually.

Under those conditions $\partial f^{-1}(y)$ is p-compact, for any $y \in Y$.

PROOF - Let us suppose that there exists y_0 such that $\partial f^{-1}(y_0)$ is not p-compact . There exists a family $(x_\alpha)_{\alpha \in A}$, $|A| = p$, discrete, $x_\alpha \in \partial f^{-1}(y_0)$ for any $\alpha \in A$. X is paracompact so we can find a discret family of open sets $(U_\alpha)_{\alpha\in A}$, $x_\alpha \in U_\alpha$ for any $\alpha\in A$. Since $x_\alpha \in \partial f^{-1}(y_0)$, for any $\alpha\in A$, this implies $x_\alpha \in U_\alpha - f^{-1}(y_0)$, for any $\alpha\in A$. X is n-Frechet, so for any $\alpha\in A$ there exists a n-sequence $(x_{\alpha\beta})_{\beta<n}$ and

$$x_{\alpha\beta} \longrightarrow x_\alpha, \qquad \alpha\in A.$$

For any $\alpha\in A$, let $C_\alpha = \{x_{\alpha\beta}, \beta<n\} \bigsqcup \{x_\alpha\}$ Hence C_α is n-compact, for any $\alpha\in A$. $(C_\alpha)_{\alpha\in A}$ is a discrete family of closed sets, hence $X_0 = \bigsqcup_{\alpha\in A} C_\alpha$ is closed in X. (X_0 is paracompact of course).

Let $g = f|_{X_0}$, $g: X_0 \longrightarrow Y_0$ being $Y_0 = f(X_0)$. g is a continuous closed mapping from a paracompact locally n-compact space X_0 onto Y_0. Let $Y_1 \subset Y_0$ defined as following:

$$Y_1 = \{y\in Y_0 \mid g^{-1}(y) \text{ is not n-compact}\}$$

Y_1 is closed and discrete in Y_0, by theorem 1. C_α intersects at most m elements $g^{-1}(y)$, $y\in Y_1$. Let C'_α be a subsequence of $(x_{\alpha\beta})_{\beta<n}$ from which we took of the points of $g^{-1}(y_1)$. C'_α converges to x_α, $\alpha\in A$. Let $A_\alpha = g(D_\alpha)$, D_α being the set of the points of C'_α. $\mathcal{U} = \{A_\alpha\}_{\alpha\in A}$, $\mathcal{U}$ is locally finite in $Y_0 - Y_1$ (and hence pontually finite in $Y_0 - Y_1$). $|A_\alpha| \le n$ and if we fix $\alpha\in A$. $A(\alpha) = \{\beta\in A \mid A_\alpha \bigcap A_\beta \neq \emptyset\}$ is such that $|A(\alpha)| \le n$. Hence there exists $A' \subset A$, $|A'| = p$ such that

$$\mathcal{U}' = \left\{A_\alpha \mid \alpha\in A'\right\} \text{ is pairwise disjoint.}$$

$$g = \Big|_{\bigsqcup_{\alpha\in A'} D_\alpha} = f\Big|_{\bigsqcup_{\alpha\in A'} D_\alpha - \{x_\alpha\}}$$

is one-to-one.

Let $X_2 = \coprod_{\alpha \in A'} C_\alpha$, $Y_2 = f(X_2)$ is a closed subset of Y. Y_2 is homeomorphic to X_2/F_2 that is obtained indentifying the points of $F_2 = \{x_\alpha, \alpha \in A'\}$.

Let us think about the sequences C'_α converging to $y_0 = f(F_0)$ as "rays". Let B_1 be the set of elements of B such that y_0 does not belong to them and that intersect n rays and have cardinality n, since B has puntually cardinality n.

Let $(P_\beta)_{\beta \in B_1}$ $|B_1| = n$ be a well ordering of B_1. Let $z_0 \in P_0$ and for each $\beta \in B_1$ let us choose $z_\beta \in P_\beta$, $z_\beta \neq y_0$ and z_β does not belong to any of rays to which z_β belongs, $\forall \alpha < \beta$.

Hence the unions of elements of B_2 intercets at most n rays, by Generalized Continuous Hipothesis.

So there exists a ray that does not intersect any element of B_2. Let us denote the union of this element with y_0 by K. K is m-compact. Let $G = (Y_0 - \{z_\beta\}_{\beta \in B_1}) \coprod K$. G is an open neighborhood of K in Y.

But, we can observe, that none of the elements of B insects K and is contained in G. Hence B does not have any of described properties. Hence $\partial f^{-1}(y)$ is p-compact for any $y \in Y$. □

COROLLARY - Let $f: X \to X$ be a continuous closed mapping. X is paracompact and locally m-compact. Y is locally m-compact if and only if $\partial f^{-1}(y)$ is m-compact.

PROOF - In the direction "only if" it is consequence from theorem 4. In the other way let us see the following before, if $\partial f^{-1}(y)$, is m-compact for every $y \in Y$, we can define the following open set:

$$L(y) = \begin{cases} \overset{\circ}{\overline{f^{-1}(y)}} & \text{if } \partial f^{-1}(y) \neq \emptyset \\ f^{-1}(y)-\{p_y\} & \text{if } \partial f^{-1}(y) = \emptyset, \text{ where } p_y \in f^{-1}(y) \end{cases}$$

Let $X_0 = X-L$ where $L = \coprod_{y\in Y} L(y)$. X_0 is closed in X. Let

$$\psi : X_0 \hookrightarrow X$$

and let $g = f\circ\psi$. g is closed continuous and onto.

$$g^{-1}(y) = \begin{cases} \partial f^{-1}(y) & \text{if } \partial f^{-1}(y) \neq \emptyset \\ p_y & \text{if } \partial f^{-1}(y) = \emptyset. \end{cases}$$

Hence $g^{-1}(y)$ is m-compact for any $y\in Y$.

So we can think as $f^{-1}(y)$ being m-compact for every $y\in Y$. For any $x\in f^{-1}(y)$ there exists Ω_x, open in X such that $\bar{\Omega}_x$ is m-compact and $x\in\Omega_x$.

$\{\Omega_x \mid x\in f^{-1}(y)\}$ is an open covering of $f^{-1}(y)$. Then, there exists a subcovering $(\Omega_{x_\alpha})_{\alpha\in A}$, $|A| < m$ of $f^{-1}(y)$.

If $\Omega = \coprod_{\alpha\in A} \Omega_{x_\alpha}$, then

$$y\in V = Y - f(X-\Omega) \subset \coprod_{\alpha\in A} f(\overline{\Omega_{x_\alpha}})$$

$\bar{V}$ is m-compact, because is contained in a compact set. □

EXAMPLE - A space under the conditions of theorem 5 is a space that has an open basis $B = \coprod_{\ell<n} B_i$, $|B_i| < n$ such that B_i, is a discrete covering of X and if $j > i$, B_j refines B_i. This basis B is the required family. For a concrete example see [1].

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