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Local equilibrium approximation in free turbulent flows: Verification through the method of differential constrains

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We present a full version of the results obtained in Grebenev et al [Doklady Physics 47(7), 518–521 (2002)] wherein the closure formula, that is, the local equilibrium approximation of second-order moments for modeling free turbulent flows was justified by the method of differential constrains. The proposed analysis provides us a point of view from the modern theory of symmetry analysis on the closure problem in turbulence. Specifically, closure relationships in the physical space are interpreted as the (differential) equations of invariant sets (manifolds) in a phase-space. We demonstrate how this concept can be applied for verification of the local equilibrium approximations (LEA) of second-order moments. With this, we obtain the equivalence of LEA and vanishing the Poisson bracket for the defect of the longitudinal velocity component and of the turbulent energy. Numerical experiments carried out in a far turbulent wake confirm this conclusion.

KEYWORDS
differential constrains, free turbulent flows, local equilibrium approximation

MSC (2020)
76F05; 35K55; 35K65; 35B40

1 | INTRODUCTION

This paper is mainly motivated by the fact that, in turbulence, numerous investigations assume often implicitly that the closure relationships are additional constrains coupled with differential transport equations for the moments of hydrodynamic quantities, resulting to an overdetermined system of equations. As it was noted by Chorin in [11], closure formulas in turbulence are, as a rule, derived using empirical hypotheses and assumptions which are often poorly justified. With this, the method of differential constrains suggested by Yanenko in [3] provides a symmetry analysis approach to investigation of overdetermined systems on their compatibility. It is worth observing that the fundamental results on overdetermined systems were obtained by Cartan quite long ago [12]. We can associate closure relationships with the (differential) constrains which present equations of invariant manifolds introduced in [13] for an arbitrary system of evolution equations. This is a nontrivial generalization of the notion of invariant set which appears in systems of ordinary differential equations. The difficulties in propounding such a theory are that a straight-forward transfer of the known scheme of group analysis to finding invariant manifolds of PDEs is not impossible in general [15]. Several approaches to this problem were worked out during a long history of studying invariant of the highest symmetries of differential equations and in some cases it leads to successful attempts in deriving the invariant manifolds, see for details [13]. However, an important question is whether a differential constrain to be an invariant manifold of PDEs is very effective verified by the direct methods [13].

To the author's knowledge the approach based on the method of differential constrains which leads to the explicit justification of the closures of several models of turbulence has not received due attention. Unlike the first works [1], [14], which proposed this approach, were published in the form of short communications. In [14], we considered a shear-free mixing turbulent layer. It was established that the tensor-invariant Hanjalic-Launder (Zeman-Lumley) closure for an unstratified (stratified) flow is a compatible differential constrain of the model. Local equilibrium approximations of second-order moments were analyzed in [1] wherein the problem of dynamics of a far plane turbulent wake is considered. It was announced that the application of the LEA is associated with vanishing the Poisson bracket, that is, $\{e, U_1\} = 0$ for the deviation of the averaged longitudinal velocity component U_1 and of the turbulence energy e. The results of numerical experiments carried out in a far turbulent wake were presented in the form of graphs and table.

The aim of this paper is the presentation of a full version of the results obtained in [1]. Namely, we give the background of the method of differential constrains adopted to the closure problem in turbulence. Then, we prove the results obtained in [1], that provide the correctness of replacing the corresponding second-order moment by the LEA. It will be based on the following elements: (a) To obtain the conditions which guarantee the compatibility of the closure in the model under consideration. (b) To verify that the condition obtained (vanishing the Poisson bracket $\{e, U_1\} = 0$) in the item a) is realized in the model and it will be done by numerical experiments. A discussion and summary of results are given in the conclusion Section 5.

2 | BACKGROUND

This section has the purpose to summarize the basic definitions from the symmetry analysis of differential equations to define several constructions of the method of differential constrains.

Every dynamic system of ordinary differential equations:

$$x_t^j = f^i(\mathbf{x}), \tag{1}$$

where $1 < i < n$ and $\mathbf{x} = (x_1, \dots, x_n)$, generates the local one-parameter group G with the vector field

$$V_f = f^1 \frac{\partial}{\partial x_1} + \dots + f^n \frac{\partial}{\partial x_n}. \tag{2}$$

A manifold M given by the equations

$$g^1(\mathbf{x}) = \dots = g^k(\mathbf{x}), \quad k < n, \tag{3}$$

is an invariant manifold of the group G if the following equalities

$$V_f(g^i)\Big|_M = 0, \quad 1 \le i \le k \tag{4}$$

hold. Invariant sets of the dynamical system (1) are defined as invariant manifolds of the one-parameter group G.

Consider the system F of evolution equations

$$u_t^i = F^i(t, \mathbf{x}, u^1, \dots, u_\lambda^j, \dots), \quad 1 \le i \le m, \tag{5}$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$, $u_\lambda^k = \partial^{|\lambda|} u^k / \partial x_1^{\lambda_1} \dots \partial x_n^{\lambda_n}$. Let L be a system of differential equations of the variables $t, x_1, \dots, x_n$. Denote by $[L]$ the set generated by the system L and all its differential consequences. Let $[L]_0$ denote the system and all its consequences in the variables $x_1, \dots, x_n$. We use the notations $D_t, D_{x_1}, \dots, D_{x_n}$ for the operators of total differentiation with respect to $t, x_1, \dots, x_n$.

Consider the formal vector field generated by the system (5)

$$V_F = \frac{\partial}{\partial t} + \sum_{i=1}^{m} F^1 \frac{\partial}{\partial u^i} + \sum_{i=1}^{m} D^\alpha(F^i) \frac{\partial}{\partial u^i_\alpha}, \tag{6}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $D^\alpha = D^{\alpha_1}_{x_1} \dots D^{\alpha_n}_{x_n}$.

A manifold H given by the equations

$$h_j(t, \mathbf{x}, u^1, \dots, u^k_\lambda, \dots) = 0, \quad 1 \le j \le l \tag{7}$$

is an invariant manifold (compatible differential constrain) of the system (5) if

$$V_F(h^j)\big|_{[H]_0} = 0, \quad 1 \le j \le l. \tag{8}$$

The definition of invariant manifold is given for systems of evolution equations. Sometimes, we can rewrite nonevolution equations in the form of evolution equations. Exemplarily, we consider the equation

$$\phi_{\xi\eta} = \exp(\phi) + \exp(-2\phi), \tag{9}$$

by the change of variables

$$t = \eta + \xi, \quad x = i(\eta - \xi) \tag{10}$$

and introducing new dependent variable $\psi = \phi_t$, the nonevolution equation (9) is reduced to the evolution system

$$\phi_t = \psi, \quad \psi_t = -\phi_{xx} + \exp(\phi) + \exp(-2\phi). \tag{11}$$

The invariance condition can be written in the following equivalent form:

$$D_t(h^j)\big|_{[F]_0}\big|_{[H]_0} = 0, \quad 1 \le j \le l, \tag{12}$$

where F is the system (5).

For example, the Kolmogorov–Petrovskii–Piskunov type equation

$$u_t = u_{xx} - 2u^3 + 2u \tag{13}$$

admits an invariant manifold of the form

$$h(x, u_x, u_{xx}) \equiv u_{xx} + 3uu_x - 2u^3 + 2u = 0 \tag{14}$$

that follows immediately by the direct verification of the criterion (12).

Invariance of manifolds is closely connected with the existence of solutions to overdetermined systems. The following result holds:

Theorem 2.1 [13]. *Assume that the system F with sufficient smooth F^i has an invariant manifold H which is solved with respect to higher derivatives. Then there exists a smooth solution of F at least locally.*

Notice that invariant manifolds are generated not only by symmetries of differential equations but also by conservation laws. Namely, the symmetries and the variational derivatives of the densities of conservation laws satisfy the determining equations of the form

$$D_t h^i = \sum_{k=1}^{m} a^{ik}_n D^n_x(h^k) + \dots a^{ik}_0 h^k, \quad i = 1, \dots, m, \tag{15}$$

where a_j^{ik} depend on t, x, the functions u^l, and their derivatives. With this, Equations (15) have to be hold due to the system (5). It is obvious that if h^i satisfy the determining equations (15) then

$$D_t(h^i)\big|_{[(5)]_0}\big|_{[H]_0} = 0, \quad 1 \le i \le m, \tag{16}$$

where $[H]_0$ denotes the equations

$$h^1 = \cdots = h^m = 0. \tag{17}$$

Consider now a system of equations which are nonevolution in general

$$F^i(x_0, \mathbf{x}, u^1, \dots, u^j_\lambda, \dots) = 0, \quad 1 \le i \le m, \quad \mathbf{x} = x^1, \dots, x^n. \tag{18}$$

Arbitrary m functions $h^1, \dots, h^m$ depending on $x_0, \mathbf{x}, u^1, \dots, u^m, u^j_\alpha, \dots$ define the differential operator

$$H^* = \sum_{i=1}^{m}\left(h^i \frac{\partial}{\partial u^i} + \sum_{|\alpha|>1} D^\alpha(h^i)\frac{\partial}{\partial u^i_\alpha}\right), \tag{19}$$

where $D^\alpha = D^{\alpha_0}_{x_0} \dots D^{\alpha_0}_{x_0}$ and

$$h^1 = \cdots = h^m = 0. \tag{20}$$

Equations (20) are compatible differential constrains of the system (18) if

$$H^*(F) + Bh|_{(18)} = 0, \tag{21}$$

where $F = (F^1, \dots, F^m)$.

If equations (18) is a system of evolution equations then the conditions (21) takes the form

$$D_t h^i + L^i(h)\big|_{(18)} = 0, \tag{22}$$

where

$$L^i(h)\big|_{(20)} = 0. \tag{23}$$

It is evident that in this case the equations H describe an invariant manifold. For example, the nonlinear Poisson equation

$$u_{xx} + u_{yy} = au\ln u \tag{24}$$

admits the following invariant manifold (compatible differential constrain)

$$u_y + S(y)u = 0, \tag{25}$$

where $S(y)$ satisfies the equation

$$S_{yy} - 2SS_y - aS = 0. \tag{26}$$

In the next section, we apply the concept of compatible differential constrain or invariant manifold to a closure procedure for the dynamics of a far plane turbulent wake.

3 | LOCAL EQUILIBRIUM APPROXIMATION: A DIFFERENTIAL CONSTRAIN OF THE MODEL

In this section, we consider the local equilibrium approximation, the relationship between the second-order correlations and the mean-flow gradients which is applied for a closure of the models at the level of the equations for the second-order correlations. This method for calculating turbulent flow has been pursued in [8–10] by Donaldson, Rodi, Hanjalic, Launder (see more references in these papers) for different transport modeling. Sometimes, the name of this modeling, an invariant model [2] that refers to the constrains imposed on the choice of model terms used for closure. It means that the model term has to exhibit at least the same symmetry properties and dimensionality as the term it replaces.

We consider the dynamics of a far plane turbulent wake [8, 16] and the following simplest mathematical model will be used in the forthcoming analysis [5–7, 17]:

$$U_0 \frac{\partial U_1}{\partial x} = \frac{\partial}{\partial y} \langle u'v' \rangle, \tag{27}$$

$$U_0 \frac{\partial e}{\partial x} = \frac{\partial}{\partial y} \nu_{t_1} \frac{\partial e}{\partial y} + P - \epsilon, \tag{28}$$

$$U_0 \frac{\partial \epsilon}{\partial x} = \frac{\partial}{\partial y} \nu_{t_2} \frac{\partial \epsilon}{\partial y} + \frac{\epsilon}{e} (C_{\epsilon_1} P - C_{\epsilon_2} \epsilon), \tag{29}$$

$$U_0 \frac{\partial}{\partial x} \langle u'v' \rangle = \frac{\partial}{\partial y} \nu_{t_3} \frac{\partial}{\partial y} \langle u'v' \rangle - C_{\phi_1} \langle u'v' \rangle \frac{\epsilon}{e} + C_{\phi_2} e \frac{\partial U_1}{\partial y}, \tag{30}$$

where U_0 is the remote velocity, $U_1 = U_0 - U$ is the defect of the averaged longitudinal velocity component, the angle brackets $\langle\, . \,\rangle$ mean averaging. The coefficients of turbulent viscosity ν_{t_1}, ν_{t_2} and ν_{t_3} have the form

$$\nu_{t_1} = C_\mu \frac{e^2}{\epsilon}, \quad \nu_{t_2} = \frac{\nu_{t_1}}{\sigma_\epsilon}, \quad \nu_{t_3} = C_s C_\mu^{-1} \nu_{t_1}, \tag{31}$$

where e is the turbulence energy, ϵ is the rate of dissipation of turbulence energy into heat. Here

$$P = \langle u'v' \rangle \frac{\partial U_1}{\partial y} \tag{32}$$

describes the generation of turbulence energy. The quantities σ_ϵ, C_{ϵ_1}, C_{ϵ_2}, C_μ, C_{ϕ_1}, C_{ϕ_2}, and C_s are empirical constants wherein $C_{\phi_2} = C_{\phi_1} C_\mu$. Dimensionless variables are introduced by using the body diameter D and the velocity U_0 as scales of length and velocity.

The LEA means that there is no time evolution or spatial diffusion of the second-order correlations (see [2]). With this, applying this concept to the model (27)–(30), we get the following representation for the tangential Reynolds stress $\langle u'v' \rangle$, see [5–7, 9, 10, 17]:

$$\langle u'v' \rangle = C_\mu \frac{e^2}{\epsilon} \frac{\partial U_1}{\partial y} \equiv C_\mu e \tau \frac{\partial U_1}{\partial y}, \tag{33}$$

where $\tau = e/\epsilon$. This relationship between the second-order correlation $\langle u'v' \rangle$ and the mean-flow gradient forms a first-order closure. As it was noted in [2], this will be a valid approximation, provided (a) any changes in the mean flow are very slow compared with the characteristic time of the turbulence and (b) spatial variations in the turbulence are small over the scale length of turbulent motion. A particular region where both conditions are satisfied is in the constant flux region of the boundary layer.

The relatively simple model presented here does suitable results over turbulent wakes. More general models for free turbulent flows together with their invariant modeling can be found in [2, 5–7, 9, 10].

We believe that the verification of the LEA in general can be done by mathematical calculations based on the method of differential constrains. We demonstrate it for the model prediction (33). As the basic result that uses this mathematical tool, we prove that the set

$$M = \left(e, \tau, U_1, \langle u'v' \rangle \ : \ \mathcal{H}(e, \tau, U_1, \langle u'v' \rangle) \equiv \langle u'v' \rangle - C_\mu e \tau \frac{\partial U_1}{\partial y} = 0 \right) \tag{34}$$

is invariant under the flow generated by system (27)–(30). More exactly, the following result provides the criterion of the invariance of M.

Theorem 3.1. *Let $(e, \tau, U_1, \langle u'v' \rangle)$ be a sufficiently smooth solution of system (27)–(30). Let $\sigma_\epsilon = C_{\epsilon_1} = 1$ and suppose that $C_{\phi_2} - C_{\phi_1} C_\mu = C_\mu (C_{\epsilon_2} - 1)$. Then the set M is an invariant manifold of system (27)–(30) at $\tau = \tau_h = U_0^{-1}(C_{\epsilon_2} - 1)(x + x_0)$ if and only if the Poisson bracket $\{e, U_1\} = 0$.*

First, we consider the equation for τ

$$\begin{aligned} \frac{\partial \tau}{\partial x} &= \frac{1}{\epsilon}\frac{\partial \tau}{\partial x} - \frac{e}{\epsilon^2}\frac{\partial \epsilon}{\partial x} = U_0^{-1}\left(\frac{1}{\epsilon}\frac{\partial}{\partial y}\nu_{t_1}\frac{\partial e}{\partial y} + \frac{P}{\epsilon} - 1 - \frac{e}{\epsilon^2}\frac{\partial}{\partial y}\nu_{t_2}\frac{\partial \epsilon}{\partial y} - \frac{1}{\epsilon}(C_{\epsilon_1}P - C_{\epsilon_2}\epsilon) \right) \\ &= U_0^{-1}\left(\frac{1}{\epsilon}\frac{\partial}{\partial y}C_\mu \tau^2 e \frac{\partial \epsilon}{\partial y} + \frac{1}{\epsilon}\frac{\partial}{\partial y}C_\mu \tau e \epsilon \frac{\partial \tau}{\partial y} + \frac{P}{\epsilon} - 1 - \frac{1}{\epsilon}\tau \frac{\partial}{\partial y}\frac{C_\mu}{\sigma_\epsilon}\tau e \frac{\partial \epsilon}{\partial y} - \frac{1}{\epsilon}C_{\epsilon_1}P + C_{\epsilon_2} \right). \end{aligned} \tag{35}$$

This equation has a solution of the form

$$\tau(x, y) \equiv \tau_h = U_0^{-1}(C_{\epsilon_2} - 1)(x + x_0) \tag{36}$$

for $\sigma_\epsilon = 1$ (this value is recommended in [17]) and $C_{\epsilon_1} = 1$ (the recommended value equals 1.4). In order to prove that M is an invariant manifold of system (27)–(30), we have to very that the invariance condition (16) holds, that is,

$$D_x(\mathcal{H})|_{[(34)]_0}|_{[(27)-(30)]_0} = 0. \tag{37}$$

The direct calculations of the left hand side of equation (37) give for $\tau = \tau_h(x)$ the follows

$$D_x(\mathcal{H}) = -e\frac{\partial U_1}{\partial y}(\tau_{hx} U_0 C_\mu + C_{\phi_1} C_\mu - C_{\phi_2}) - \tau_h U_0 C_\mu \left(\frac{\partial e}{\partial x}\frac{\partial U_1}{\partial y} - \frac{\partial e}{\partial y}\frac{\partial U_1}{\partial x} \right). \tag{38}$$

Using the relationships between the constants

$$C_{\phi_2} - C_{\phi_1} C_\mu = C_\mu (C_{\epsilon_2} - 1), \tag{39}$$

we have

$$D_x(\mathcal{H}) = \tau_h U_0 C_\mu \left(\frac{\partial e}{\partial x}\frac{\partial U_1}{\partial y} - \frac{\partial e}{\partial y}\frac{\partial U_1}{\partial x} \right) \equiv \tau_h U_0 C_\mu \{e, U_1\}, \tag{40}$$

where $\{\,,\}$ denotes the Poisson bracket. Therefore, the invariance condition (37) holds for $\tau = \tau_h$ if the Poisson bracket $\{e, U_1\} = 0$. The example of flow with vanishing Poisson bracket is a flow with equal rates of the generation and dissipation of turbulence energy, that is, $P = \epsilon$. In the next section, we present the results of the numerical validation of vanishing the Poisson bracket $\{e, U_1\}$. Numerical experiments are carried out for a far turbulent wake, that is, for the LEA model (27)–(29), (33) or Model 1 and the classical model (27)–(30) or Model 2.

Remark 3.2. Using the constants $C_{\phi_1} = 2.8$, $C_\mu = 0.09$ and $C_{\epsilon_2} = 1.95$ recommended in [10], we find that the right-hand side of the relation $C_{\phi_2} - C_{\phi_1} C_\mu = C_\mu (C_{\epsilon_2} - 1)$ equals 0.085. Therefore C_{ϕ_2} differs only slightly from $C_{\phi_1} C_\mu = 0.252$ that is close to the value $C_{\phi_2} = C_{\phi_1} C_\mu$ recommended in [10], [17].

TABLE 1 Variation of δ^n as a function of the distance from the body

x/D	mesh#1, δ_1	mesh#2, δ_2
700	$0.427 \cdot 10^{-4}$	$0.421 \cdot 10^{-4}$
800	$0.359 \cdot 10^{-4}$	$0.353 \cdot 10^{-4}$
900	$0.305 \cdot 10^{-4}$	$0.300 \cdot 10^{-4}$
1000	$0.256 \cdot 10^{-4}$	$0.252 \cdot 10^{-4}$
1500	$0.174 \cdot 10^{-4}$	$0.171 \cdot 10^{-4}$

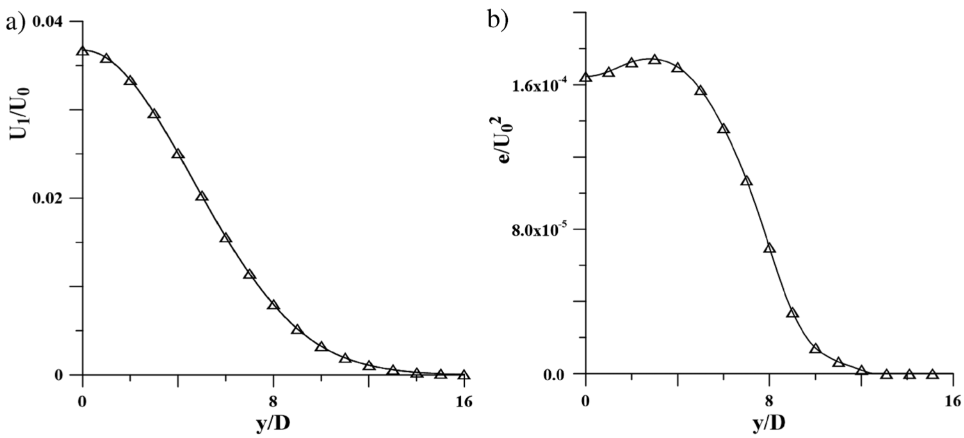

FIGURE 1 Left panel (a): The deviation of the longitudinal velocity component U_1. Right panel (b): The turbulence energy e

4 | RESULTS OF NUMERICAL EXPERIMENTS

We have carried out a series of numerical experiments. To perform such numerics, initial conditions are specified at $x_0/D = 625$ which are in agreement with experimental data of the dynamics of a plane turbulent wake behind a round cylinder [8], [16]. The initial distribution of the tangential stress $\langle u'v' \rangle$ is specified by the formula (33). A finite-difference algorithm, its test and realization, and the results of its application to the problems of free turbulence were given in [18]. The calculations are based on the model (27)–(29), (33) (Model 1) and show that the finite-difference counterpart of the Poisson bracket is close to zero for all mesh parameters. The calculations are performed with double accuracy. Table 1 presents the quantity

$$\delta^n = \frac{\max_j \left|(U^h_{1x} e^h_y)^n_j - (U^h_{1y} e^h_x)^n_j\right|}{\max_j(\left|\nabla^h e\right|^2_j, \left|\nabla^h U_1\right|^2_j)}, \tag{41}$$

which is considered as a function of the distance from the body and, at $x = x^n$, characterizes the finite-difference counterpart of the Poisson bracket. Here $(U^h_{1x})_j$, $(U^h_{1y})_j$, $(e^h_x)_j$, $(e^h_y)_j$, $(\nabla^h U_1)_j$ and $(\nabla^h e)_j$ are finite-difference approximations of the first derivatives and gradients at the points (x^n, y_j) : $x = x^n = n \cdot h_x, y = y_j, j = 1, 2, \dots, n_y - 1$. Column I shows the distance from the body x/D, and columns II, III present δ^n in the uniform meshes $1 - 2$ with the parameters $h_x = 0.25$, $h_y = 0.05$ and $h_x = 0.125$, $h_y = 0.0125$, respectively.

Figures 1 and 2 demonstrate the results of calculations which are carried out for Models 1 and 2. The data of defect of the longitudinal velocity component U_1 and the turbulence energy e are presented on Figure 1, respectively, the rate of energy dissipation ϵ together with the tangential Reynolds stresses $\langle u'v' \rangle$ are plotted on Figure 2. The calculations based on Model 1 are shown by solid lines. The solid lines with triangles correspond to the data of numerical experiments based on Model 2. The distance from the body x/D equals 650. The quantities $\sigma_\varepsilon = 1$, $C_{\epsilon_1} = 1$, $C_{\epsilon_2} = 1.95$, $C_\mu = 0.09$, $C_{\phi_1} = 2.8$, $C_{\phi_2} = 0.252$, and $C_s = 0.1$ are empirical constants.

It should be observed that the data wake parameters calculated for Models 1 and 2 are close to each other. This is in the complete agreement with the assertion of Theorem 3.1. Notice that for the recommended values $C_{\epsilon_1} = 1.44$ and $\sigma_\varepsilon = 1.3$ in [9] the numerical experiments also demonstrate turbulence data coinciding obtained for Models 1 and 2.

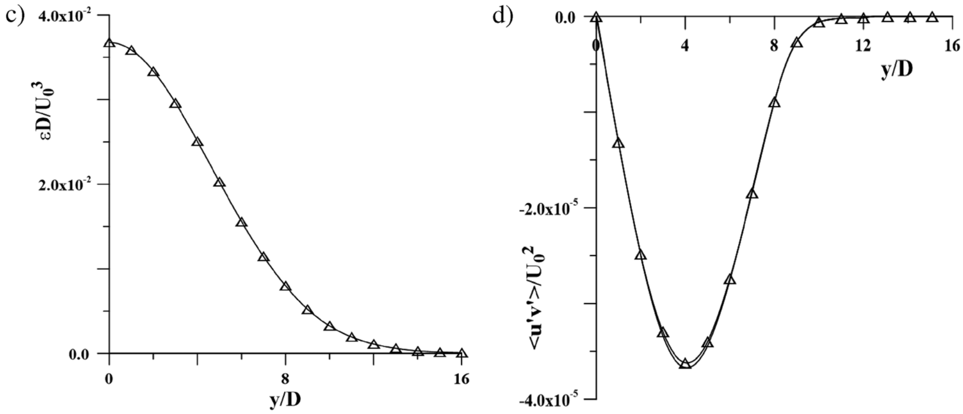

FIGURE 2 Left panel (c): The rate of energy dissipation ϵ. Right panel (d): The tangential Reynolds stresses $\langle u'v' \rangle$

Thus, Theorem 3.1 and numerical experiments conducted show that the LEA is applicable for determining the tangential Reynolds stresses $\langle u'v' \rangle$ in the problem of dynamics of a far planar turbulent wake behind a body.

5 | CONCLUSIONS

The problem of the dynamics of a planar far turbulent wake has been studied numerically. Namely, the calculations have been performed by using the hierarchy of mathematical models [2, 5–7, 9, 10, 17, 18] which also include the third-order model.

The main aim of the paper is justification of the local equilibrium approximation applied for the tangential Reynolds stresses $\langle u'v' \rangle$ by using Model 1. The main result is that a representation of $\langle u'v' \rangle$ in the form (33) presents the differential constrain of the system (27)–(30). Theorem 3.1 gives also the criterion of compatibility of the overdetermined system (27)–(30), (33) and this is vanishing the Poisson bracket $\{e, U_1\}$. Numerical experiments demonstrate that $\{e, U_1\} = 0$ within the limits of numerical accuracy. Then, we showed that the numerical solutions of Models 1 and 2 are very close to each other under the conditions of Theorem 3.1. Finally, we notice that (33) coincides with the well-known Boussinesq–Kolmogorov–Prandtl relationship, see [9].

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