HIGHER ORDER DERIVATIVES OF ANALYTIC FAMILIES OF BANACH SPACES

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Abstract. We show that the Rochberg spaces generated by complex interpolation form themselves complex interpolation scales and obtain their new interpolated spaces and associated derivations. We present our results in the context of analytic families of Banach spaces and study the problem of determining the Rochberg spaces induced by these new families.

1. Introduction

This paper studies certain analytic families of Banach spaces that arise naturally in the context of complex interpolation of families [13]. We will work in the context of *admissible* spaces $\mathscr{F}$ of vector-valued analytic functions (Definition 2.1) over a complex domain $\mathbb{U}$ as formalized by Kalton and Montgomery-Smith [24]. Starting with such an $\mathscr{F}$ we will consider, for $n \in \mathbb{N}$ and $z \in \mathbb{U}$, the spaces introduced by Rochberg [30], formed by the arrays of the truncated sequence of the Taylor coefficients of the elements of $\mathscr{F}$, namely

$$\mathscr{F}_z^{(n)} = \left\{ \left(\frac{f^{(n-1)}(z)}{(n-1)!}, \ldots, f'(z), f(z) \right) : f \in \mathscr{F} \right\}$$

endowed with the natural quotient norm. The space $\mathscr{F}_z = \mathscr{F}_z^{(1)}$ of arrays of length one (the values of the functions of $\mathscr{F}$ at z) correspond, in the suitable context, to classical interpolation spaces, while arrays of length two (the pair formed by the values of the derivative of the functions and the values of the functions at z) constitute the so-called first derived space and correspond, in the suitable context, to twisted sums of the spaces $\mathscr{F}_z$.

Admissible spaces $\mathscr{F}$ emerge from complex interpolation schemas in different ways. If one has a compatible couple (X_0, X_1) and works on the complex unit strip then $\mathscr{F}$ could be the classical Calderón space $\mathscr{C}(X_0, X_1)$ associated to the couple [6]. If one has a suitable family $\mathcal{X} = \{X_u : u \in \partial\mathbb{U}\}$ of Banach spaces then the complex interpolation method for families [13] can be applied to generate the space $\mathscr{F}$. In general, complex interpolation applied to a family $\mathcal{X}$ of Banach spaces on the boundary of $\mathbb{U}$ generates what is called an analytic family $\{X_z : z \in \mathbb{U}\}$ of Banach spaces on $\mathbb{U}$, and these are the *first* Rochberg spaces $X_z = \mathscr{F}_z$ for the suitably obtained admissible space $\mathscr{F}$. Then, one can *also* form all subsequent families of Rochberg spaces $\mathscr{F}_z^{(n)}$ for $n > 1$ and $z \in \mathbb{U}$. With this setting, our paper orbits around two axis.

The first one is the fact, implicit in Rochberg [30] and made explicit in [5], that Rochberg spaces arrange into exact sequences

$$0 \longrightarrow \mathscr{F}_z^{(n)} \longrightarrow \mathscr{F}_z^{(n+k)} \longrightarrow \mathscr{F}_z^{(k)} \longrightarrow 0\,. \tag{1.1}$$

2010 *Mathematics Subject Classification.* 46B70, 46M18, 46H99.

The research of the first and second authors was supported by Projects MINCIN MTM2016-76958-C2-1-P, PID2019-103961GB-C21 and Junta de Extremadura project IB20038. The third author was supported by FAPESP, processes 2016/25574-8, 2018/03765-1 and 2021/13401-0, and by CAPES, PDSE program, grant 88881.134107/2016-0.

Rochberg also observed that these sequences can be constructed by means of certain "unbounded nonlinear operators Ω" that we will call *differentials*. We are interested in identifying the differentials $\Omega_z^{k,n}$ that generate the sequences (1.1) and in using those differentials to derive information about the Rochberg spaces. The crucial example to see these ideas in action is that of the interpolation couple (ℓ_∞, ℓ_1), treated in Section 4. In classical Banach space theory, the middle space B in an exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is usually called a twisted sum of A and C, and corresponds [25, 22, 23] to a certain type of nonlinear maps called quasilinear maps. Thus, the existence of diagram (1.1) connects the theory of Rochberg families with the theory of twisted sums of Banach spaces and Rochberg's "unbounded nonlinear operators" with quasilinear maps.

The second axis of the paper is the connection between analytic families of Banach spaces and complex interpolation. At this point, observe that the distinction between what occurs at the border and the interior of the domain $\mathbb{U}$ is fundamental: the derived spaces only exist on the interior. Thus, given $n > 1$, it is not granted the existence of an admissible space $\mathscr{T}$ so that $\mathscr{T}_z^{(1)} = \mathscr{F}_z^{(n)}$ for every $z \in \mathbb{U}$. Due to this obstruction we introduce the more general notion of *acceptable* space of analytic functions (Definition 2.2) and prove in Theorem 6.8 one of our main results: given an acceptable (in particular, admissible) space $\mathscr{F}$ of analytic functions on $\mathbb{U}$ and $n > 1$ there exists an acceptable space of analytic functions $\mathscr{T}$ such that $\mathscr{T}_z^{(1)} = \mathscr{F}_z^{(n)}$ with equivalence of norms. Once that is proved, Rochberg spaces form themselves new "acceptable families" and thus they are bound to form interpolation families. This is interesting in itself even for practical reasons since, according to Kalton and Montgomery-Smith [24, p.1151], "One of the drawbacks of the complex method is that in general it seems relatively difficult to calculate complex interpolation spaces. There is one exception to this rule, which is the case when one has a pair of Banach lattices". Rochberg spaces are not Banach lattices and yet we can calculate the spaces obtained by complex interpolation between them.

We are thus ready to describe the organization of the paper. In Section 2, *Spaces of analytic functions and complex interpolation*, we recall the definition of an admissible space of analytic functions, its connection with the complex interpolation method for families and introduce the notion of an acceptability. The definition of an acceptable space requires the using of a Fréchet algebra of analytic functions, whose construction is presented in the Appendix. In Section 3, *Rochberg spaces and their entwining exact sequences*, we do exactly as the title says; the results can be considered a reformulation of [5] in the context of this paper. Section 4, *The cornerstone example*, presents a detailed study of the higher order Rochberg spaces $\mathscr{Z}_n$ generated by the interpolation couple (ℓ_∞, ℓ_1) at $z = \frac{1}{2}$. The first of these spaces is $\mathscr{Z}_1 = \ell_2$ and the second $\mathscr{Z}_2 = Z_2$ is the celebrated Kalton–Peck *twisted Hilbert space* [25]. We will obtain very precise two sided estimates for the type 2 constants of the finite-dimensional subspaces of each of the spaces $\mathscr{Z}_n$ from which we deduce, for instance, that $\mathscr{Z}_m$ is not isomorphic to a subspace of $\mathscr{Z}_k$ if $m > k$. In Section 5, *Duality issues*, we introduce, given an admissible space $\mathscr{F}$, a kind of admissible space $\mathscr{F}^\star$ so that $(\mathscr{F}^\star)_z^{(n)}$ can be interpreted as $(\mathscr{F}_z^{(n)})^*$ maintaining the entwining exact sequences between the spaces. To some extent, this section is the natural extension of duality results from Kalton and Peck [25], Rochberg [30] and Cwikel [17]. Section 6, *Analytic families of Rochberg spaces and interpolation* is the central section of the paper, where acceptable families enter the game with a substantial role. We show that if $\mathscr{F}$ is an acceptable space then, for each $n \geq 2$, there is an acceptable space $\mathscr{T}$ such that $\mathscr{T}_z = \mathscr{F}_z^{(n)}$. As we mentioned above, this is interesting in itself due to the difficulty in calculating complex interpolation spaces other than Banach lattices. Since an acceptable space of analytic functions depends on the complex domain $\mathbb{U}$ on which it is based, it is necessary for technical reasons to move between $\mathbb{U}$ and the unit disc $\mathbb{D}$. In particular, the difference between working on the unit strip (classical interpolation with two spaces), on the unit disc (classical interpolation for families) and on a general domain conformally equivalent to them has to be considered. We present a preparatory version on the unit strip (Proposition 6.1) and a general result on the unit disc (Proposition 6.7). Then, after a Chain and a Leibniz rule, useful in translating results from the disc to other domains, we state and prove our main result (Theorem 6.8). In Section 7, *Derivation of Rochberg families*, we use a bit homologically oriented techniques to describe the intertwining exact sequences of Rochberg spaces in "low dimensions" and the way in which differentials are interlaced.

In Section 8, *Applications*, we solve a few problems in the literature. One of the conclusions that we draw is that the higher Rochberg spaces provide the right perspective to understand analytic families, even if one is interested only in the first derived spaces. Finally, the Appendix, *A Fréchet algebra of analytic functions*, displays the construction of the Fréchet algebra of analytic functions required to sustain the notion of an acceptable space.

1.1. **Notation.** Domains of the complex plane are displayed in "blackboard" fonts: $\mathbb{C}$ is the complex plane, $\mathbb{D}$ is the unit disk and $\mathbb{S}$ is the unit strip. The border of a domain $\mathbb{U}$ will be called $\partial\mathbb{U}$, even if the unit circle will be always $\mathbb{T}$. Spaces of vector-valued analytic functions are displayed in "mathscript" fonts: $\mathscr{F}, \mathscr{G}$ and so on. Spaces and algebras of complex valued analytic functions follow a standard notation: $H_p, N^+, A, A^\infty, W^+$, etc. The superscript $^{(n)}$ is always related to derivatives while X^n denotes the product of n copies of X. We use the following notation for lists of Taylor coefficients. If A is an ordered subset of the nonnegative integers and f is analytic in a neighbourhood of $z \in \mathbb{C}$, then

$$\tau_A(f) = \left(\frac{f^{(n)}}{n!}\right)_{n\in A} \qquad \text{and} \qquad \delta_z\tau_A(f) = \tau_A(f)(z) = \left(\frac{f^{(n)}(z)}{n!}\right)_{n\in A}.$$

In particular,

$$\tau_n(f) = \frac{f^{(n)}}{n!}, \quad \tau_{[n,0]}(f) = \left(\frac{f^{(n)}}{n!}, \dots, f\right), \quad \tau_{(n,0]}(f) = \left(\frac{f^{(n-1)}}{(n-1)!}, \dots, f\right).$$

Given a (commutative, unital) topological algebra B and a Banach space X, we say that X is a B-module if there is a jointly continuous outer product $B \times X \longrightarrow X$ satisfying the usual algebraic requirements. Note that in this case, for each fixed $a \in B$, the map $x \longmapsto ax$ is a bounded operator on X whose norm will be denoted by $\|a\|_{L(X)}$ if necessary. Also note that B need not to be normed: actually the Fréchet algebras $A^\infty_{\mathbb{U}}$ introduced in the Appendix play a role in this paper.

2. Spaces of analytic functions and analytic families

This section introduces the spaces of analytic functions that we shall use along the paper. First, we recall the standard notion of an admissible space of analytic functions taken from Kalton and Montgomery-Smith [24]:

Definition 2.1. Let $\mathbb{U}$ be an open set of $\mathbb{C}$ conformally equivalent to the disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let Σ be a complex Banach space. A Banach space $\mathscr{F}$ of analytic functions $f : \mathbb{U} \longrightarrow \Sigma$ is said to be admissible provided:

(a) For each $z \in \mathbb{U}$, the evaluation map $\delta_z : \mathscr{F} \longrightarrow \Sigma$ is bounded.
(b) For every conformal equivalence $\varphi : \mathbb{U} \longrightarrow \mathbb{D}$ and every analytic function $f : \mathbb{U} \longrightarrow \Sigma$ we have $f \in \mathscr{F}$ if and only if $\varphi f \in \mathscr{F}$ and $\|\varphi f\|_{\mathscr{F}} = \|f\|_{\mathscr{F}}$.

Condition (b) is basically a boundary condition and implies that for every $z \in \mathbb{U}$ the space $\mathscr{F}$ is isometric to the subspace $\ker \delta_z = \{f \in \mathscr{F} : f(z) = 0\}$, the isometry being given by multiplication by any fixed conformal map $\varphi : \mathbb{U} \longrightarrow \mathbb{D}$ such that $\varphi(z) = 0$.

It turns out that admissibility is a too rigid notion for our present purposes and so we need to introduce a weak version that we have called (for which we apologize in advance) *acceptable spaces*. This notion requires using the algebras $A^\infty_{\mathbb{U}}$, whose definition and properties can be found in the Appendix.

Definition 2.2. Let $\mathbb{U}$ and Σ be as before. An acceptable space is a Banach space of analytic functions $f : \mathbb{U} \longrightarrow \Sigma$ having the following properties:

(a) The evaluation maps $\delta_z : \mathscr{F} \longrightarrow \Sigma$ are bounded.
(b) For each conformal mapping $\varphi : \mathbb{U} \longrightarrow \mathbb{D}$ there is a constant $K[\varphi]$ such that, if $f : \mathbb{U} \longrightarrow \Sigma$ is analytic then $\varphi f \in \mathscr{F}$ if and only if $f \in \mathscr{F}$ and $K[\varphi]^{-1}\|\varphi f\|_{\mathscr{F}} \leq \|f\|_{\mathscr{F}} \leq K[\varphi]\|\varphi f\|_{\mathscr{F}}$.
(c) $\mathscr{F}$ is a module over the algebra $A^\infty_{\mathbb{U}}$ under pointwise multiplication, that is, the pointwise product $A^\infty_{\mathbb{U}} \times \mathscr{F} \longrightarrow \mathscr{F}$ is jointly continuous.

Lemma 2.3. *Every admissible space of analytic functions is acceptable.*

Proof. It clearly suffices to check (c). Assume $\mathscr{F}$ is admissible on $\mathbb{U}$ and let us fix a conformal map $\psi : \mathbb{U} \longrightarrow \mathbb{D}$. Then, for $f \in \mathscr{F}$ we have $\psi f \in \mathscr{F}$, with $\|\psi f\|_{\mathscr{F}} = \|f\|_{\mathscr{F}}$. Thus, if $(c_n)_{n\geq 0}$ is absolutely summable and $g(u) = \sum_{n\geq 0} c_n \psi(u)^n$, then $gf \in \mathscr{F}$, with $\|gf\|_{\mathscr{F}} \leq \left(\sum_{n\geq 0}|c_n|\right)\|f\|_{\mathscr{F}}$. This implies that $\mathscr{F}$ is a "contractive" module over $\psi^*[W^+]$ under the pointwise multiplication. The definition of $\psi^*[W^+]$ is given in the Appendix. But $\psi^*[W^+]$ contains $A^\infty_{\mathbb{U}}$ with continuous inclusion (see Lemma 9.4); therefore $\mathscr{F}$ is an $A^\infty_{\mathbb{U}}$-module as well. □

The following result and its proof have been included here at the request of the referee.

Lemma 2.4. *Let $\mathscr{F}$ be an acceptable space of analytic functions on $\mathbb{U}$. For every integer $n \geq 0$ and every $z \in \mathbb{U}$, the map $\delta_z^{(n)} : \mathscr{F} \longrightarrow \Sigma$ given by $\delta_z^{(n)}(f) = f^{(n)}(z)$ is a bounded operator.*

Proof. If $n = 0$ the conclusion is contained in the definition of an acceptable space. Let us see that the maps $\delta_z^{(1)}$ are bounded. Pick $z \in \mathbb{U}$ and then a sequence $(z_n)_{n\geq 1}$ in $\mathbb{U}\backslash\{z\}$ converging to z. Given $f \in \mathscr{F}$ we have

$$\delta_z^{(1)}(f) = f'(z) = \lim_{n\to\infty} \frac{f(z_n) - f(z)}{z_n - z}.$$

According to the Banach–Steinhaus theorem, $\delta_z^{(1)}$ is bounded since it is the pointwise limit of a sequence of bounded operators, namely of $(\delta_{z_n} - \delta_z)/(z_n - z)$. Now proceed by induction on n. □

2.1. **Calderón spaces.** The simplest examples of admissible spaces are the spaces associated to couples introduced by Calderón in his seminal paper [6]. Interpolation of couples is usually done in the unit strip $\mathbb{S} = \{0 < \Re(z) < 1\}$. In this paper we need to be careful with the spatial variable which is used to differentiate functions and thus the size of the strip where the spaces are placed needs to be taken in consideration; see Section 6.4.1. So, given real numbers $a < b$ we put $\mathbb{S}_{a,b} = \{a < \Re(z) < b\}$. Now suppose that (X_a, X_b) is a compatible couple: this just means that X_a and X_b are Banach spaces linear and continuously embedded into a third Banach space Σ.

The most popular Calderón space is $\mathscr{C} = \mathscr{C}(X_a, X_b)$, which consists of those bounded analytic functions $f : \mathbb{S}_{a,b} \longrightarrow \Sigma$ that extend continuously to the closure of $\mathbb{S}_{a,b}$ and, denoting again by f the extension, satisfy the boundary condition that for $j = a, b$ the restriction $t \in \mathbb{R} \longmapsto f(j + it) \in X_j$ is continuous and bounded. The norm of the space $\mathscr{C}$ is defined by $\|f\|_{\mathscr{C}} = \sup\{\|f(j + it)\|_{X_j} : t \in \mathbb{R}, j = a, b\}$. A useful variant is the space

$$\mathscr{C}_0 = \{f \in \mathscr{C} : \|f(j + it)\|_{X_j} \to 0 \text{ as } |t| \to \infty, j = a, b\},$$

which is a closed subspace of $\mathscr{C}$. It is easy to prove that, if $f \in \mathscr{C}$, then for every $z \in \mathbb{S}_{a,b}$ the function $w \longmapsto e^{(w-z)^2} f(w)$ belongs to $\mathscr{C}_0$. Moreover, if Δ is *any* dense subset of $X_a \cap X_b$, then the functions of the form

$$f(z) = e^{\delta z^2} \sum_{1 \leq i \leq k} e^{\lambda_k z} x_k \qquad (x_k \in \Delta, \lambda_k, \delta \in \mathbb{R}, \delta > 0) \tag{2.1}$$

are dense in $\mathscr{C}_0$; see [27, Chapter IV, Theorem 1.1, p. 220] or [2, Lemma 4.2.3]. We shall denote the space of such functions as $\mathscr{C}_{00}$.

2.2. **Interpolation families.** A basic source of admissible spaces is the complex interpolation method for families. The method we present here, which is that of [14], is a slight modification of the method from [13].

Let $\mathbb{U}$ be a domain of the complex plane conformally equivalent to the disc and let $\varphi : \mathbb{D} \longrightarrow \mathbb{U}$ be a fixed conformal map. Conformal maps belong to the Smirnov class N^+ (see [18, Theorem 2.1] and note that "our" Smirnov class N^+ corresponds to the "Nevanlinna class N" in Duren's book) and so they have nontangencial limits for almost every $z \in \mathbb{T}$. Let us assume from now on that φ extends to a surjective continuous function $\tilde{\mathbb{D}} \longrightarrow \overline{\mathbb{U}}$ (there is no need to relabel), where $\tilde{\mathbb{D}}$ is a subset of the closed disc which contains $\mathbb{D}$ together with almost every point of $\mathbb{T}$. In particular φ maps $\mathbb{T} \cap \tilde{\mathbb{D}}$ onto $\partial\mathbb{U}$ (up to a null set). Note that this is actually a property of the domain, and domains such as a spiral of infinite turns approaching the unit circle lacks it — we owe this remark to the referee of an earlier version of the paper. When $\mathbb{U} = \mathbb{S}$ one can use the conformal equivalence given by the formula

$$\varphi(z) = \frac{1}{2} + \frac{2i}{\pi} \log \frac{z+1}{1-z}$$

which extends to the closed disc, except $z = \pm 1$.

Definition 2.5. A family $\mathcal{X} = \{X_\omega : \omega \in \partial\mathbb{U}\}$ of Banach spaces is an interpolation family with containing space Σ, intersection space Δ and containing function k if:

- Σ is a Banach space for which there are linear continuous embeddings $X_\omega \longrightarrow \Sigma$. We will identify X_ω with its image in Σ from now on.
- Δ is a subspace of $\bigcap_{\omega \in \partial\mathbb{U}} X_\omega$ such that for every $x \in \Delta$ the function $z \in \mathbb{T} \bigcap \tilde{\mathbb{D}} \longmapsto \|x\|_{\varphi(z)}$ is measurable and
$$\int_{\mathbb{T}} \log^+ \|x\|_{\varphi(z)}\, d|z| < \infty,$$
where $\log^+ t = \max(0, \log t)$ for $t > 0$.
- $k : \partial\mathbb{U} \longrightarrow (0, \infty)$ is a measurable function such that
$$\int_{\mathbb{T}} \log^+ k(\varphi(z))\, d|z| < \infty,$$
and $\|x\|_\Sigma \leq k(u)\|x\|_u$ for every $u \in \partial\mathbb{U}$ and every $x \in \Delta$.

If no risk of confusion arises we will simply say that $\mathcal{X}$ is an interpolation family. Given an interpolation family $\mathcal{X}$, we define $\mathscr{G} = \mathscr{G}(\mathcal{X})$ as the space of all functions on $\mathbb{U}$ of the form $g = \sum_{j=1}^n g_j x_j$, where $g_j \circ \varphi$ is in the Smirnov class N^+, $x_j \in \Delta$ for all j, and

$$\|g\| = \operatorname*{ess\,sup}_{u \in \partial\mathbb{U}} \|g(u)\|_u < \infty. \tag{2.2}$$

Here, $\partial\mathbb{U}$ carries the image of the measure $d|z|$ under the map φ. Notice that this is well-defined because functions in the Smirnov class N^+ have almost everywhere nontangential limits on $\mathbb{T}$ and it does not depend of φ, because if $\psi : \mathbb{D} \longrightarrow \mathbb{U}$ is another conformal map, then $\varphi \circ \psi^{-1}$ is an automorphism of the disc.

Let us briefly explain how these spaces fit into the general framework described earlier. We just state the basic facts and refer the reader to [13, 14, 15] for more details. First, the evaluations $\delta_u : \mathscr{G} \longrightarrow \Sigma$ are bounded. This fact depends on the hypotheses made on the containing function k. Indeed, by a result of Szegő [13, Proposition 1.1], (any measurable extension of) the function $k \circ \varphi : \mathbb{T} \longrightarrow [0, \infty)$ has an associated "outer" function in the Smirnov class, which means that there is $K \in N^+$ such that $|K(z)|k(\varphi(z)) = 1$ for almost everywhere $z \in \mathbb{T}$, where the extension of K to $\mathbb{T}$ is defined taking nontangential limits. It is now easy to check that for each $u \in \mathbb{U}$ one has $\|\delta_u : \mathscr{G} \longrightarrow \Sigma\| \leq |K(z)|$, where $\varphi(z) = u$; see [15, Proposition 2.3.52].

As a rule, the space $\mathscr{G}$ will fail to be complete; however it always fulfils conditions (a) and (b) in Definition 2.1: Let $\mathscr{F} = \mathscr{F}(\mathcal{X})$ be its completion and observe that the continuity of the evaluations at points of $\mathbb{U}$ allows us to identify $\mathscr{F}$ as a Banach space of analytic functions $\mathbb{U} \longrightarrow \Sigma$, on which the point evaluations remain bounded with the same norm (see [13, Proposition 2.3]). About condition (b), keeping an eye in Definitions 2.1 and 2.2 let us observe the trivial fact that $\mathscr{G}$, and therefore $\mathscr{F}$, are contractive modules over $H^\infty(\mathbb{U})$, since every bounded analytic function on the disc belongs to the Smirnov class N^+. In particular, if $h : \mathbb{U} \longrightarrow \mathbb{D}$ is a conformal map and $f \in \mathscr{F}$, then $hf \in \mathscr{F}$ and $\|hf\|_{\mathscr{F}} = \|f\|_{\mathscr{F}}$. It then remains to prove that $f \in \mathscr{F}$ whenever $hf \in \mathscr{F}$. This is related to the coincidence of the interpolation spaces associated to $\mathscr{G}$ and $\mathscr{F}$. To explain this, and following [13], let us fix $z \in \mathbb{U}$ and consider the following two spaces: the first one, often denoted by $X\{z\}$, is the completion of the intersection space Δ equipped with the norm $x \in \Delta \longmapsto \inf\{\|g\|_{\mathscr{F}} : g \in \mathscr{G} \text{ and } x = g(z)\}$. The definition makes sense because for every $x \in \Delta$ there is $g \in \mathscr{G}$ such that $x = g(z)$. The other space is

$$X[z] = \{x \in \Sigma : x = f(z) \text{ for some } f \in \mathscr{F}\},$$

equipped with the quotient norm. To see how these spaces are related, have a look at the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker\delta_z & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{F}/\ker\delta_z & \longrightarrow & 0 \\
 & & \uparrow & & \| & & \uparrow{\scriptstyle Q} & & \\
0 & \longrightarrow & \overline{\ker\delta_z \cap \mathscr{G}} & \longrightarrow & \overline{\mathscr{G}} & \longrightarrow & \mathscr{F}/\overline{\ker\delta_z \cap \mathscr{G}} & \longrightarrow & 0
\end{array}$$

Here, $Q\left(f + \overline{\ker\delta_z \cap \mathscr{G}}\right) = f + \ker\delta_z$ is an isometric quotient map (it maps the open unit ball of $\mathscr{F}/\overline{\ker\delta_z \cap \mathscr{G}}$ onto that of $\mathscr{F}/\ker\delta_z$). A moment's reflection suffices to realize that each nonzero element of $\ker Q$ corresponds to an element $f \in \ker\delta_z$ which is not in $\overline{\ker\delta_z \cap \mathscr{G}}$, and so $\ker Q = \ker\delta_z/\overline{\ker\delta_z \cap \mathscr{G}}$. Now observe that $X\{z\} = \mathscr{F}/\overline{\ker\delta_z \cap \mathscr{G}}$ and $X[z] = \mathscr{F}/\ker\delta_z$; thus, Q induces an isometric quotient map of $X\{z\} \longrightarrow X[z]$ "extending" the inclusion $\Delta \longrightarrow \Sigma$. This map is injective (that is, $X\{z\} = X[z]$) if and only if $\ker\delta_z \cap \mathscr{G}$ is dense in $\ker\delta_z$. On the other hand, if $h : \mathbb{U} \longrightarrow \mathbb{D}$ is a conformal map vanishing at z, we have $\ker\delta_z \cap \mathscr{G} = h\mathscr{G}$ in the sense that each function in $\mathscr{G}$ vanishing at z has the form hg, for some $g \in \mathscr{G}$. Using this, the following lemma is not hard to prove and it concludes the argument:

Lemma 2.6. *The following statements are equivalent:*

- $X\{z\} = X[z]$.

- $\ker \delta_z \cap \mathscr{G}$ *is dense in* $\ker \delta_z$.
- $\ker \delta_z = h \cdot \mathscr{F}$. □

Definition 2.7. An interpolation family $\mathcal{X} = \{X_\omega : \omega \in \partial\mathbb{U}\}$ is said to be admissible at $z \in \mathbb{U}$ if it satisfies the equivalent conditions recorded in the preceding lemma, and it is said to be admissible if it is admissible at every $z \in \mathbb{U}$

Observe that an interpolation family $\mathcal{X}$ is admissible if and only if the space $\mathscr{F}(\mathcal{X})$ obtained from it is admissible in the sense of Definition 2.1, by the third condition in the lemma. Also note that it would not make any sense to define "acceptable interpolation family", because if $X\{z\}$ and $X[z]$ agree as sets, then they carry the same norm — equivalently, if $\ker \delta_z = h \cdot \mathscr{F}$, then "multiplication by h" is an isometry.

3. Rochberg spaces and their entwining exact sequences

Let us translate the basic facts of [5] to the context of acceptable spaces. If we fix $z \in \mathbb{U}$, the map $\delta_z : \mathscr{F} \longrightarrow \Sigma$ is continuous and $\mathscr{F} / \ker \delta_z$ is a Banach space which is isometric to

$$\mathscr{F}_z = \{w \in \Sigma : w = f(z) \text{ for some } f \in \mathscr{F}\},$$

endowed with the quotient norm $\|w\|_{\mathscr{F}_z} = \inf_{w=f(z)} \|f\|_{\mathscr{F}}$. The family $(\mathscr{F}_z)_{z\in\mathbb{U}}$ will be called the analytic family of Banach spaces associated to $\mathscr{F}$, which is consistent with the traditional use when $\mathscr{F}$ is admissible (cf. [24, § 10]) and, in particular, when $\mathscr{F}$ arises from an admissible interpolation family, as in Section 2.2. In this case we have $\mathscr{F}_z = X[z] = X\{x\}$.

Since the maps $\delta_z^{(n)} : \mathscr{F} \longrightarrow \Sigma$ are bounded for all $z \in \mathbb{U}$ and all $n \in \mathbb{N}$ (see Lemma 2.4), it makes sense to consider the Banach spaces

$$\mathscr{F} \Big/ \bigcap_{i<n} \ker \delta_z^{(i)} \qquad (n \in \mathbb{N}). \tag{3.1}$$

As before these spaces are isometric to the Rochberg spaces

$$\begin{aligned} \mathscr{F}_z^{(n)} &= \{w \in \Sigma^n : w = \tau_{(n,0]} f(z) \text{ for some } f \in \mathscr{F}\} \\ &= \left\{(w_{n-1}, \dots, w_0) \in \Sigma^n : w_i = \frac{f^{(i)}(z)}{i!} \text{ for some } f \in \mathscr{F} \text{ and all } 0 \le i < n\right\}, \end{aligned} \tag{3.2}$$

endowed with the quotient norm: the norm of $w = (w_{n-1}, \dots, w_0)$ in $\mathscr{F}_z^{(n)}$ is the infimum of the norms of the functions of $\mathscr{F}$ fitting in (3.2).

For fixed z, the spaces $\mathscr{F}_z^{(n)}$ can be arranged into exact sequences in a very natural way: this is implicit in [30], even if the syntagma "exact sequence" does not appear, and a complete treatment can be found in [5]. Indeed, if for $1 \le n, k < m$ we denote by $\imath_{n,n+k} : \Sigma^n \longrightarrow \Sigma^{n+k}$ the inclusion on the left given by $\imath_{n,n+k}(x_n, \dots, x_1) = (x_n, \dots, x_1, 0 \dots, 0)$ and by $\pi_{n+k,k} : \Sigma^{n+k} \longrightarrow \Sigma^k$ the projection on the right given by $\pi_{n+k,k}(x_{n+k}, \dots, x_k, \dots, x_1) = (x_k, \dots, x_1)$, then $\pi_{n+k,k}$ restricts to an isometric quotient map of $\mathscr{F}_z^{(n+k)}$ onto $\mathscr{F}_z^{(k)}$ (this is trivial) and $\imath_{n,n+k}$ is an isomorphic embedding of $\mathscr{F}_z^{(n)}$ into $\mathscr{F}_z^{(n+k)}$. This can be proved as [5, Proposition 2(a)], just using the second condition in the definition of an acceptable space instead of the second condition in the definition of an admissible space; the third condition plays no role here and will play no role until Section 6.3. Thus, see [5, Theorem 4], for each $n, k \in \mathbb{N}$ there is an exact sequence of Banach spaces and operators

$$0 \longrightarrow \mathscr{F}_z^{(n)} \xrightarrow{\imath_{n,n+k}} \mathscr{F}_z^{(n+k)} \xrightarrow{\pi_{n+k,k}} \mathscr{F}_z^{(k)} \longrightarrow 0 \tag{3.3}$$

To describe the sequences (3.3) as twisted sums we will use the maps $\Omega^{k,n} : \mathscr{F}_z^{(k)} \longrightarrow \Sigma^n$ defined as follows: we fix $\varepsilon \in (0,1)$, and, for each $x = (x_{k-1}, \dots, x_0)$ in $\mathscr{F}_z^{(k)}$, select $f_x \in \mathscr{F}$ such that $x = \tau_{(k,0]} f_x(z)$, with $\|f_x\| \leq (1+\varepsilon)\|x\|$, in such a way that f_x depends homogeneously on x. Then define

$$\Omega^{k,n}(x) = \tau_{(n+k,k]} f_x(z).$$

We could emphasize the fact that $\Omega^{k,n}$ depends on z by adding the subscript z, if necessary. It is clear that it also depends on the choice of f_x, but different choices of f_x only produce bounded perturbations of the same map. Any $\Omega^{k,n}$ defined in this way is a quasilinear map (see the definition below) from $\mathscr{F}_z^{(k)}$ to $\mathscr{F}_z^{(n)}$, which means that there is a constant C such that, for every $x, y \in \mathscr{F}_z^{(k)}$ the difference $\Omega^{k,n}(x+y) - \Omega^{k,n}(x) - \Omega^{k,n}(y)$, which belongs a priori to Σ^n, actually falls into $\mathscr{F}_z^{(n)}$ and obeys the estimate

$$\|\Omega^{k,n}(x+y) - \Omega^{k,n}(x) - \Omega^{k,n}(y)\|_{\mathscr{F}_z^{(n)}} \leq C\left(\|x\|_{\mathscr{F}_z^{(k)}} + \|y\|_{\mathscr{F}_z^{(k)}}\right).$$

The map $\Omega^{k,n}$ can be used to form the twisted sum space

$$\mathscr{F}_z^{(n)} \oplus_{\Omega^{k,n}} \mathscr{F}_z^{(k)} = \left\{(y,x) \in \Sigma^{n+k} : y - \Omega^{k,n}(x) \in \mathscr{F}_z^{(n)}, x \in \mathscr{F}_z^{(k)}\right\},$$

endowed with the quasinorm

$$\|(y,x)\|_{\Omega^{k,n}} = \left\|y - \Omega^{k,n}(x)\right\|_{\mathscr{F}_z^{(n)}} + \|x\|_{\mathscr{F}_z^{(k)}}. \tag{3.4}$$

It turns out that $\mathscr{F}_z^{(n)} \oplus_{\Omega^{k,n}} \mathscr{F}_z^{(k)}$ and $\mathscr{F}_z^{(n+k)}$ are the same space, and that (3.4) is a quasinorm equivalent to the norm of $\mathscr{F}_z^{(n+k)}$, see [5, Section 3.5]. Although the explicit use of quasilinear maps is marginal in this paper it will be convenient to record the definition here:

Definition 3.1. Let X and Y be quasinormed spaces and let H be a linear space containing Y. A homogeneous mapping $\Phi : X \longrightarrow H$ (not Y) is said to be quasilinear from X to Y if:

(a) $\Phi(x+y) - \Phi(x) - \Phi(y) \in Y$ for all $x, y \in X$.
(b) There is a constant C such that $\|\Phi(x+y) - \Phi(x) - \Phi(y)\|_Y \leq C(\|x\|_X + \|y\|_X)$ for all $x, y \in X$.

Condition (a) guarantees that $Y \oplus_\Phi X = \{(h,x) \in H \times X : h - \Phi(x) \in Y\}$ is a linear subspace of $H \times X$, while (b) and the homogeneous character of Φ imply that the formula $\|(h,x)\|_\Phi = \|h - \Phi(x)\| + \|x\|$ defines a quasinorm on $Y \oplus_\Phi X$, which is equivalent to a norm when Φ arises as a derivation. The map $\imath : Y \longrightarrow Y \oplus_\Phi X$ given by $\imath(y) = (y, 0)$ preserves the (quasi) norms and the map $\pi : Y \oplus_\Phi X \longrightarrow X$ given by $\pi(h,x) = x$ takes the unit ball of $Y \oplus_\Phi X$ onto that of X. These form a short exact sequence

$$0 \longrightarrow Y \xrightarrow{\ \imath\ } Y \oplus_\Phi X \xrightarrow{\ \pi\ } X \longrightarrow 0 \tag{3.5}$$

that shall be referred to as the sequence generated by Φ. We say that Φ is trivial (as a quasilinear map from X to Y) if (3) splits. In general, a short exact sequence

$$0 \longrightarrow Y \xrightarrow{\ \imath\ } Z \xrightarrow{\ \pi\ } X \longrightarrow 0$$

is said to split if there is an operator $P : Z \longrightarrow Y$ such that $P\imath = \mathbf{I}_Y$, equivalently, there is an operator $S : X \longrightarrow Z$ such that $\pi S = \mathbf{I}_X$. For the exact sequence generated by Φ this happens if and only if there is a, not necessarily continuous, linear map $L : X \longrightarrow H$ such that $\Phi - L$ is bounded from X to Y in the sense that $\|\Phi(x) - L(x)\|_Y \leq M\|x\|_X$ for some constant M and all $x \in X$.

We conclude this rugged introduction emphasizing the compatibility of the sequences (3.3) passing through a given $\mathscr{F}_z^{(m)}$ and drawing some consequences. Indeed, if $m = k + n = i + j$, with $k < i$ say, then the following diagram is commutative:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathscr{F}_z^{(j)} & = & \mathscr{F}_z^{(j)} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathscr{F}_z^{(n)} & \longrightarrow & \mathscr{F}_z^{(m)} & \longrightarrow & \mathscr{F}_z^{(k)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathscr{F}_z^{(n-j)} & \longrightarrow & \mathscr{F}_z^{(i)} & \longrightarrow & \mathscr{F}_z^{(k)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{3.6}
$$

Notation has been lightened up by understanding that unlabelled arrows $\mathscr{F}_z^{(N)} \longrightarrow \mathscr{F}_z^{(M)}$ are $\iota_{N,M}$ if $N \leq M$ and $\pi_{N,M}$ if $N \geq M$.

The commutativity of these diagrams has the following important consequence, which seems to have gone unnoticed in [5]:

Proposition 3.2. *Let $\mathscr{F}$ be an acceptable space on $\mathbb{U}$ and let $z \in \mathbb{U}$. If*

$$
0 \longrightarrow \mathscr{F}_z^{(n)} \longrightarrow \mathscr{F}_z^{(n+k)} \longrightarrow \mathscr{F}_z^{(k)} \longrightarrow 0 \tag{3.7}
$$

splits, then

$$
0 \longrightarrow \mathscr{F}_z^{(j)} \longrightarrow \mathscr{F}_z^{(j+i)} \longrightarrow \mathscr{F}_z^{(i)} \longrightarrow 0 \tag{3.8}
$$

splits whenever $i + j \leq k + n$, in which case for every $m \leq k + n$ the space $\mathscr{F}_z^{(m)}$ is isomorphic (but not necessarily "equal") to the m-fold direct sum $\mathscr{F}_z \times \cdots \times \mathscr{F}_z$. In particular, if $i + j = k + n$, then (3.7) *splits if and only if* (3.8) *splits.*

Proof. Welcome to the Club of People Who Stare at Diagrams. If you don't feel like joining, you can skip this proof. Let us begin with the following observation: In a commutative diagram of the form (please ignore the labels of the arrows for the time being)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & A & = & A & & & & \\
 & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle J} & & & & \\
0 & \longrightarrow & B & \xrightarrow{K} & C & \xrightarrow{Q} & D & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle P} & & \| & & \\
0 & \longrightarrow & E & \xrightarrow{L} & F & \xrightarrow{R} & D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{3.9}
$$

with exact rows and columns the two sequences passing through F split if and only if the two sequences passing through B split.

Indeed, if $p : F \longrightarrow C$ is a section of P (that is, $Pp = \mathbf{I}_F$) and r is a section of R, then pr is a section of Q — and $K^{-1}pL$ is a section of S. Conversely, if $k : C \longrightarrow B$ is a projection along K (that is, $kK = \mathbf{I}_B$) and $\imath$ is a projection along I, then $\imath k$ is a projection along J — and Sk defines a projection along L.

Before proceeding with the proof of the proposition, let us remark that the result is almost trivial if $(i, j) \leq (k, n)$, that is, if $i \leq k$ and $j \leq n$, so the point is to get it for $i + j = k + n$. Hence it suffices to see the "in particular" part. If $k + n = 2$ there is nothing to prove, so we may assume $n + k \geq 3$ and check that if $i + j = k + n$ and $|k - i| = |n - j| = 1$, then (3.7) splits if and only if (3.8) does. It therefore suffices to assume that $n \geq 2$ and then prove that (3.7) splits if and only if (3.8) splits if $i = k + 1$ and $j = n - 1$. Look at the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathscr{F}_z^{(n-1)} & = & \mathscr{F}_z^{(n-1)} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathscr{F}_z^{(n)} & \longrightarrow & \mathscr{F}_z^{(n+k)} & \longrightarrow & \mathscr{F}_z^{(k)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathscr{F}_z^{(1)} & \longrightarrow & \mathscr{F}_z^{(k+1)} & \longrightarrow & \mathscr{F}_z^{(k)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

which has the same form as (3.9). Assume that the middle horizontal sequence splits. Then so does the other sequence that passes through $\mathscr{F}_z^{(n)}$ since $(1, n-1) \leq (k, n)$ and we conclude that the middle vertical sequence splits. And, conversely, if the middle vertical sequence splits, then so does the other sequence that passes through $\mathscr{F}_z^{(k+1)}$ since $(k, 1) \leq (k + 1, n - 1)$ and we conclude that the middle horizontal sequence splits. □

Sometimes it is convenient to replace the starting acceptable space $\mathscr{F}$ by another one with more convenient properties. If properly done, this will have very little effect in the resulting sequences:

Lemma 3.3. *Let $\mathscr{F}$ and $\mathscr{G}$ be acceptable spaces of functions $\mathbb{U} \longrightarrow \Sigma$. Assume that $\mathscr{G} \subset \mathscr{F}$ and that the inclusion is continuous. Fix $z \in \mathbb{U}$. If $\mathscr{F}_z = \mathscr{G}_z$, necessarily with equivalent norms, then $\mathscr{F}_z^{(n)} = \mathscr{G}_z^{(n)}$, with equivalent norms, for every $n \geq 1$ and the sequences induced by $\mathscr{G}$ at z agree with those of $\mathscr{F}$.*

The proof is an easy induction argument once one realizes that, under the hypothesis of the Lemma, given $n, k \geq 1$, there is a commutative diagram, of Banach spaces and operators

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{G}_z^{(n)} & \longrightarrow & \mathscr{G}_z^{(n+k)} & \longrightarrow & \mathscr{G}_z^{(k)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{F}_z^{(n)} & \longrightarrow & \mathscr{F}_z^{(n+k)} & \longrightarrow & \mathscr{F}_z^{(k)} & \longrightarrow & 0
\end{array}$$

where the descending arrows are the corresponding formal inclusions. This implies that the derived spaces of admissible interpolation families and the corresponding exact sequences do not vary if one uses a norm of Hardy type instead of (2.2); see Pisier's comments in [28, p. 425, second column], especially the part concerning the "point often overlooked by non-specialists".

The most important application for us occurs in the context of couples, where one has $\mathscr{C}(X_0, X_1)_z^{(n)} = \mathscr{C}_0(X_0, X_1)_z^{(n)}$, up to equivalent norms, for every z in the corresponding strip and all $n \geq 1$. Actually it is easy to see that $\mathscr{C}(X_0, X_1)_z^{(n)}$ and $\mathscr{C}_0(X_0, X_1)_z^{(n)}$ are the same space, with the same norm, using functions of the form $w \longmapsto \exp(\varepsilon(w - z)^{2+4k})$, where $\varepsilon > 0$ is small and $k \in \mathbb{N}$ is large. We will use this fact without further mention.

4. The cornerstone example

Let us investigate the particularly interesting case of the couple (ℓ_∞, ℓ_1), which in a sense motivated the whole theory. We will denote by $\mathscr{X}$ the Calderón space $\mathscr{C}(\ell_\infty, \ell_1)$ on the unit strip, that is, $\mathscr{X}$ is the space of analytic functions $f : \mathbb{S} \longrightarrow \ell_\infty$ having the following properties:

(1) f extends to a continuous function on $\overline{\mathbb{S}} \longrightarrow \ell_\infty$ that we denote again by f.
(2) $\|f\|_{\mathscr{X}} = \sup\{\|f(it)\|_\infty, \|f(1 + it)\|_1 : t \in \mathbb{R}\} < \infty$.

Of course $\mathscr{X}$ is admissible and classical arguments show that $\mathscr{X}_z$ is the complex interpolation space $[\ell_\infty, \ell_1]_\theta = \ell_p$ with $\theta = \Re z$ and $p = 1/\theta$ for $\theta \in (0, 1)$; in particular, $\mathscr{X}_z = \ell_2$ for $z = \frac{1}{2}$. In the remainder of the section we fix $z = \frac{1}{2}$ as the base point and we denote $\mathscr{X}_{1/2}^{(n)}$ by $\mathscr{X}_n$ for $n \in \mathbb{N}$. If x is normalized in ℓ_2 and we set $x = u|x|$ then $f_x(z) = u|x|^{2z}$ is normalized in $\mathscr{X}$ and one has $f_x(\frac{1}{2}) = x$. Thus

$$f_x(z) = u|x||x|^{2z-1} = x|x|^{2(z-1/2)} = x \sum_{n=0}^{\infty} \frac{2^n \log^n |x|}{n!} \left(z - \tfrac{1}{2}\right)^n,$$

from which $(\tau_n f_x)(\frac{1}{2}) = \frac{2^n}{n!}\, x \log^n |x|$. For arbitrary $x \in \ell_2$ we have, by homogeneity,

$$(\tau_n f_x)(\tfrac{1}{2}) = \frac{2^n x}{n!} \log^n \left(\frac{|x|}{\|x\|_2}\right).$$

Hence,

$$\Omega^{1,n}(x) = \tau_{[n,1]} f_x(\tfrac{1}{2}) = x\left(\frac{2^n}{n!} \log^n \left(\frac{|x|}{\|x\|_2}\right), \ldots, \frac{2^2}{2!} \log^2 \left(\frac{|x|}{\|x\|_2}\right), 2 \log\left(\frac{|x|}{\|x\|_2}\right)\right), \tag{4.1}$$

which leads to a quite manageable description of the spaces $\mathscr{X}_m$. In particular, since $\mathscr{X}_1 = \ell_2$ we can use the map $\Omega^{1,1}$ to obtain that the functional $\|(y, x)\|_{\Omega^{1,1}} = \left\|y - 2x \log(|x|/\|x\|_2)\right\|_2 + \|x\|_2$ is equivalent to the norm of $\mathscr{X}_2$. This shows that $\mathscr{X}_2$ is isomorphic, but not equal, to the original Kalton–Peck space Z_2, whose quasinorm was defined by its legitimate owners as $\|(y, x)\|_{Z_2} = \left\|y - x \log(\|x\|_2/|x|)\right\|_2 + \|x\|_2$ in [25]. An isomorphism between both versions is $(y, x) \in Z_2 \longmapsto (y, -2x) \in \mathscr{X}_2$.

The paper [5] contains a proof that $\mathscr{X}_m$ is not a subspace of a twisted Hilbert space for $m \geq 3$. We show now a general result which requires the following inductive, *ad hoc* definition:

Definition 4.1. A twisted Hilbert space of order 1 is just a Banach space which is isomorphic to a Hilbert space. For $k \geq 2$, say that Z is a twisted Hilbert space of order k if for some (equivalently, for every) choice $i + j = k$ with $i, j \geq 1$ there is a short exact sequence $0 \longrightarrow A_i \longrightarrow Z \longrightarrow A_j \longrightarrow 0$ in which A_i and A_j are twisted Hilbert spaces of order i and j respectively.

The equivalence between the "for some" and the "for every" form of the definition above is not entirely straightforward and requires a judicious use of diagrams: if $n + m = i$ the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & A_n & = & A_n & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & A_i & \longrightarrow & Z & \longrightarrow & A_j & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & A_m & \longrightarrow & Z/A_n & \longrightarrow & A_j & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

shows that Z/A_n is a twisted Hilbert space of order $m + j$ and hence Z is a twisted sum of twisted Hilbert spaces of order n and $m + j$; and, analogously, if $n + m = j$ using the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & Z/B & = & A_m & \longrightarrow & 0\\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A_i & \longrightarrow & Z & \longrightarrow & A_j & \longrightarrow & 0\\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A_i & \longrightarrow & B & \longrightarrow & A_n & \longrightarrow & 0\\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The space $\mathscr{Z}_m$ is a twisted Hilbert space of order m and twisted sums of Hilbert spaces are exactly the twisted Hilbert spaces of order 2.

Theorem 4.2. *$\mathscr{Z}_m$ cannot be embedded into a twisted Hilbert space of order $k < m$. In particular $\mathscr{Z}_m$ is not a subspace of $\mathscr{Z}_k$ whenever $k < m$.*

Let us recall from [19] the definition of n-th type 2 constant: If X is a Banach space, $a_n(X)$ is the smallest positive number a such that

$$\operatorname*{Average}_{\varepsilon_i=\pm 1}\left\|\sum_{i=1}^{n}\varepsilon_i x_i\right\|^2 \leq a^2 \sum_{i=1}^{n}\|x_i\|^2$$

for all $x_1, \ldots, x_n \in X$. It is clear that the constants $a_n(X)$ satisfy:

- $a_1(X) \geq 1$;
- $a_n(X)$ is nondecreasing in n;
- If Y is isomorphic to a subspace of X then $a_n(Y)/a_n(X)$ is bounded.

Theorem 4.2 will follow straightforwardly from the next two lemmata. The first one generalizes estimates in [19, Theorem 3], [25, Theorem 6.2], [21, Theorem 7.5] that deal with the case $m = 1$. Note that the base of logarithms is irrelevant.

Lemma 4.3. *For each twisted Hilbert space Z of order $m+1$ there is a constant C, depending on Z and m, such that $a_n(Z) \leq C \log_2^m n$.*

Proof. From [19, Theorem 1, Part 1] we know that, given a subspace $Y \subset Z$, one has

$$a_{nk}(Z) \leq a_n(Y)a_k(Z) + a_n(Y)a_k(Z/Y) + a_n(Z)a_k(Z/Y)$$

We then proceed by induction. The result is trivial for $m = 0$, by the parallelogram law. Assume it is true for twisted Hilbert spaces of order m. Now let Z be a twisted Hilbert space or order $m+1$ and let $0 \longrightarrow H \longrightarrow Z \longrightarrow X \longrightarrow 0$ be a witnessing sequence, where H is a Hilbert space and X is of order m. There is no loss of generality if we assume that Z contains H isometrically and the corresponding quotient is X. Since X is a twisted Hilbert space of order m the induction hypothesis provides a constant C such that $a_n(X) \leq C \log_2^{m-1} n$ for all $n \in \mathbb{N}$. Thus, for $k \in \mathbb{N}$, one has

$$a_{2k}(Z) \leq a_2(H)a_k(Z)+a_2(H)a_k(X)+a_2(Z)a_k(X) = a_k(Z)+(1+a_2(Z))a_k(X) \leq a_k(Z)+(1+a_2(Z))C \log_2^{m-1} k$$

and so

$$a_{2^{k+1}}(Z) \leq a_{2^k}(Z) + (1 + a_2(Z))C \log_2^{m-1} 2^k = a_{2^k}(Z) + (1 + a_2(Z))Ck^{m-1}.$$

Also,

$$a_{2^k}(Z) \leq a_{2^{k-1}}(Z) + (1 + a_2(Z))C(k-1)^{m-1},$$

and, iterating k times, we obtain

$$a_{2^{k+1}}(Z) \leq a_2(Z) + (1 + a_2(Z))C \sum_{1\leq i\leq k} i^{m-1} \leq C(Z)(n+1)^m,$$

where $C(Z)$ is a constant that depends only on Z and we have used Faulhaber's formula [1, p. 108] to dominate $\sum_{1\leq i\leq k} i^{m-1}$. Using that a_n is nondecreasing, there is some constant C' such that $a_n(Z) \leq C' \log_2^m n$ for all n. □

The following computation completes the proof of Theorem 4.2.

Lemma 4.4. *For each $m \geq 0$ there is $c_m > 0$ so that $c_m \log_2^m n \leq a_n(\mathscr{Z}_{m+1})$.*

Proof. Put $s_N = \sum_{1\leq i\leq N} e_i$. Since for $\varepsilon_i = \pm 1$ one has $\|(0,\dots,0,\sum_{i\leq N} \varepsilon_i e_i)\| = \|(0,\dots,0,s_N)\|$ in $\mathscr{Z}_{m+1}$ and $\|s_N\| = \sqrt{N}$ in ℓ_2, the inequality

$$\|(0,\dots,0,s_N)\|_{\mathscr{Z}_{m+1}} \geq c_m \sqrt{N} \log_2^m N,$$

immediately yields the lower estimate for $a_n(\mathscr{Z}_{m+1})$.

We will use the following consequence of the binomial theorem:

$$\frac{1}{n!} + \frac{(-1)}{(n-1)!} + \cdots + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!} = \sum_{0\leq i\leq n} \frac{(-1)^i}{i!}\frac{1}{(n-i)!} = \frac{1}{n!}(1+(-1))^n = 0. \tag{4.2}$$

For the rest of the proof we will use the following notations: given $x \in \Sigma$ and scalars $(\alpha_1,\dots,\alpha_k)$ we write $(\alpha_1,\dots,\alpha_k)x = (\alpha_1 x,\dots,\alpha_k x)$. Also, we set $L = \log(1/N)$. We also take advantage of the fact that each $\mathscr{Z}_{n+1}$ can be written as a twisted sum of $\mathscr{Z}_n$ and ℓ_2, using the map defined by (4.1): taking $x = s_N$ there, we have

$$\Omega^{1,n}(s_N) = \left(\frac{2^n}{n!}\log^n\left(N^{-1/2}\right), \dots, \frac{2^2}{2!}\log^2\left(N^{-1/2}\right), 2\log\left(N^{-1/2}\right)\right) s_N = \left(\frac{L^n}{n!}, \frac{L^{n-1}}{(n-1)!}, \dots, \frac{L^2}{2!}, L\right) s_N.$$

For each n we fix a constant k_n such that $\|(y,x)\|_{\mathscr{Z}_{n+1}} \geq k_n\left(\|(y - \Omega^{1,n}(x)\|_{\mathscr{Z}_n} + \|x\|_2\right)$. Actually one can take $k_n = \frac{1}{3}$ for all n. After this preparation:

$$\begin{aligned}\|(0,\dots,0,s_N)\|_{\mathscr{Z}_{m+1}} &\geq k_m\left\|\Omega^{1,m}(s_N)\right\|_{\mathscr{Z}_m} = k_m\left\|\left(\frac{L^m}{m!}, \frac{L^{m-1}}{(m-1)!}, \dots, \frac{L^2}{2!}, L\right)s_N\right\|_{\mathscr{Z}_m}\\ &\geq k_m k_{m-1}\underbrace{\left\|\left(\frac{L^m}{m!}, \frac{L^{m-1}}{(m-1)!}, \dots, \frac{L^2}{2!}\right)s_N - \Omega^{1,m-1}(Ls_N)\right\|_{\mathscr{Z}_{m-1}}}_{(\star)}.\end{aligned}$$

Now,

$$\begin{aligned}(\star) &= \left\|\left(L^m\left[\tfrac{1}{m!} - \tfrac{1}{(m-1)!}\right], L^{m-1}\left[\tfrac{1}{(m-1)!} - \tfrac{1}{(m-2)!}\right], \dots, L^3\left[\tfrac{1}{3!} - \tfrac{1}{2!}\right], L^2\overbrace{\left[\tfrac{1}{2!} - \tfrac{1}{1!}\right]}^{-1/2!}\right)s_N\right\|_{\mathscr{Z}_{m-1}}\\ &\geq k_{m-2}\left\|\left(L^m\left[\tfrac{1}{m!} - \tfrac{1}{(m-1)!} + \tfrac{1}{2!(m-2)!}\right], L^{m-1}\left[\tfrac{1}{(m-1)!} - \tfrac{1}{(m-2)!} + \tfrac{1}{2!(m-3)!}\right], \dots, L^3\underbrace{\left[\tfrac{1}{3!} - \tfrac{1}{2!} + \tfrac{1}{2!1!}\right]}_{1/3!}\right)s_N\right\|_{\mathscr{Z}_{m-2}}.\end{aligned}$$

Continuing in this way, after ℓ iterations, we see that $\|(0,\dots,0,s_N)\|_{\mathscr{Z}_{m+1}}/(k_m\cdots k_{m-\ell})$ is at least

$$\begin{aligned}&\Bigg\|\Bigg(L^m\left[\tfrac{1}{m!} - \tfrac{1}{(m-1)!} + \tfrac{1}{2!(m-2)!} + \cdots + \tfrac{(-1)^\ell}{\ell!(m-\ell)!}\right], L^{m-1}\left[\tfrac{1}{(m-1)!} - \tfrac{1}{(m-2)!} + \tfrac{1}{2!(m-3)!} + \cdots + \tfrac{(-1)^\ell}{\ell!(m-1-\ell)!}\right],\\ &\qquad \dots, L^{\ell+2}\left[\tfrac{1}{(\ell+2)!} - \tfrac{1}{(\ell+1)!} + \tfrac{1}{2!\ell!} + \cdots + \tfrac{(-1)^\ell}{\ell!2!}\right], L^{\ell+1}\underbrace{\left[\tfrac{1}{(\ell+1)!} - \tfrac{1}{\ell!} + \tfrac{1}{2!(\ell-1)!} + \cdots + \tfrac{(-1)^\ell}{\ell!1!}\right]}_{(-1)^\ell/(\ell+1)!}\Bigg)s_N\Bigg\|_{\mathscr{Z}_{m-\ell}}.\end{aligned}$$

And letting $\ell = m-1$ we conclude that

$$\frac{\|(0,\dots,0,s_N)\|_{\mathscr{Z}_{m+1}}}{(k_m\cdots k_1)} \geq \left\|\left(L^m\underbrace{\left[\tfrac{1}{m!} - \tfrac{1}{(m-1)!} + \tfrac{1}{2!(m-2)!} + \cdots + \tfrac{(-1)^{m-1}}{(m-1)!1!}\right]}_{(-1)^{m-1}/m!}s_N\right)\right\|_{\mathscr{Z}_1} = \frac{|L|^m}{m!}N^{1/2}. \quad \square$$

A more general, less tortuous argument will be given in the final paragraph of Section 5, just after Corollary 5.7. Keep in mind in what follows that $\mathscr{Z}_n \simeq \mathscr{Z}_n \oplus \mathscr{Z}_n$. Indeed, if $\mathbb{N} = A \cup B$ is a partition of $\mathbb{N}$ into two infinite subsets, and we set $\mathscr{Z}_n(A) = \{(x_n,\dots,x_1) \in \mathscr{Z}_n : \operatorname{supp} x_i \subset A \text{ for all } 1 \leq i \leq n\}$ and similarly for $\mathscr{Z}_n(B)$, then $\mathscr{Z}_n(A)$ and $\mathscr{Z}_n(B)$ are isometric to $\mathscr{Z}_n$, and $\mathscr{Z}_n = \mathscr{Z}_n(A) \oplus \mathscr{Z}_n(B)$ for all $n \geq 1$ exactly for the same reason as for $n = 1$, namely $\ell_2 = \ell_2(A) \oplus \ell_2(B)$, with the two direct summands isometric to ℓ_2.

Corollary 4.5. *$\mathscr{Z}_{n+k}$ is not isomorphic to (a subspace of) $\mathscr{Z}_n \oplus \mathscr{Z}_k$.*

Proof. Assuming $n \leq k$, $\mathscr{Z}_n \oplus \mathscr{Z}_k$ is a subspace of $\mathscr{Z}_k \oplus \mathscr{Z}_k \simeq \mathscr{Z}_k$, while $\mathscr{Z}_{n+k}$ is not. $\square$

Corollary 4.6. *Let $0 \leq k, j \leq n$. $\mathscr{Z}_{n-k} \oplus \mathscr{Z}_{n+k} \simeq \mathscr{Z}_{n-j} \oplus \mathscr{Z}_{n+j}$ if and only if $k = j$.*

Proof. Assume otherwise, and assume $j < k$. Then $\mathscr{Z}_{n+k}$ would be a subspace of $\mathscr{Z}_{n-j} \oplus \mathscr{Z}_{n+j}$, which is in turn a subspace of $\mathscr{Z}_{n+j}$, and that is impossible. □

In a forthcoming paper [11] it is proved that $\mathscr{Z}_m$ does not contain complemented copies of $\mathscr{Z}_n$ for $n < m$, which implies that $\mathscr{Z}_j \oplus \mathscr{Z}_k \simeq \mathscr{Z}_n \oplus \mathscr{Z}_m$ if and only if $\{j, k\} = \{n, m\}$.

5. Duality issues

This section studies the conjugate spaces of the Rochberg spaces associated to an admissible space and the corresponding (dual) exact sequences. The material presented here is closely related to [13, 30, 17, 5] and has loose connections with [3, 12, 25]. We deal only with spaces of analytic functions arising from admissible interpolation families. Let $\mathcal{X}$ be such a family, with spaces $(X_u)_{u\in\partial\mathbb{U}}$, containing space Σ, intersection space Δ and containing function $k : \partial\mathbb{U} \longrightarrow (0, \infty)$. We also fix a conformal map $\varphi : \mathbb{D} \longrightarrow \mathbb{U}$, as in Section 2.2. Let $\mathscr{F} = \mathscr{F}(\mathcal{X})$ and $\mathscr{G} = \mathscr{G}(\mathcal{X})$ be as in Section 2.2 and let us keep the traditional notation $X_z^{(n)}$ for $\mathscr{F}_z^{(n)}$, where $z \in \mathbb{U}$. When $n = 1$ we just write X_z. It is an easy consequence from $\mathscr{G}$ being dense in $\mathscr{F}$ that Δ^n is dense in $X_z^{(n)}$ for all $z \in \mathbb{U}$ and all n. Besides, it follows from Lemma 2.6 that for each $x \in \Delta^n$ and every $\varepsilon > 0$ there is $g \in \mathscr{G}$ such that $x = \tau_{(n,0]}g(z)$ and $\|g\|_{\mathscr{F}} \leq (1 + \varepsilon)\|x\|_{X_z^{(n)}}$. This simplification will play a role in the identification of the dual of $X_z^{(n)}$.

5.1. Derivation of duals of interpolation spaces.

Adapting the techniques from [13] we may find the dual of the intermediate spaces X_z the following way: let $\mathscr{F}^\star$ be the space of functions $h : \mathbb{U} \longrightarrow \Delta^\star$ (the algebraic dual of Δ) such that

- $z \longmapsto \langle h(\varphi(z)), x\rangle$ is a function in N^+ for every $x \in \Delta$;
- there is $C > 0$ such that, for each $x \in \Delta$ one has $\lim_{w\to z} |\langle h(\varphi(w)), x\rangle| \leq C\|x\|_{\varphi(z)}$ for almost every $z \in \mathbb{T}$, where the limit is nontangential.

The space $\mathscr{F}^\star$ will be normed taking $\|h\|_{\mathscr{F}^\star}$ as the infimum of the numbers C satisfying the preceding condition. The question of whether $\mathscr{F}$ is complete is irrelevant for the subsequent discussion. For each $z \in \mathbb{U}$ there is an isometry between X_z^* and the "intermediate" space

$$(\mathscr{F}^\star)_z = \{\xi \in \Delta^\star : \xi = h(z) \text{ for some } h \in \mathscr{F}^\star\},$$

with the natural quotient norm. More precisely, $\xi \in \Delta^\star$ belongs to $(\mathscr{F}^\star)_z$ if and only if the functional $x \in \Delta \longmapsto \langle \xi, x\rangle \in \mathbb{C}$ is bounded in the norm of X_z in which case the norm of the obvious extension in X_z^* agrees with the norm of ξ in $(\mathscr{F}^\star)_z$. We take this fact, proved in [13, Theorem 3.1] when Δ is the whole intersection space, as the starting point of this section. From now on we identify X_z^* with that subset of $\Delta^\star$, that is, we use $\Delta^\star$ as a "containing space" for the family X_u^*, with $u \in \mathbb{U}$. In this way the space $\mathscr{F}^\star$ can be used to construct the derived spaces of the family X_z^* using the ideas of Section 3.

First, we need a substitute for the derivatives: given $h \in \mathscr{F}^\star$ and $n \geq 0$ we define $h^{(n)} : \mathbb{U} \longrightarrow \Delta^\star$ by the formula

$$\langle h^{(n)}(z), x\rangle = \frac{d^n}{dz^n} \langle h(z), x\rangle \qquad (x \in \Delta).$$

The meaning of the expressions such as $\tau_A(h), \tau_A(h)(z)$, and the like should be obvious in this context. Now, set

$$(\mathscr{F}^\star)_z^{(n)} = \left\{(\xi_{n-1}, \ldots, \xi_0) \in (\Delta^\star)^n : \text{ there is } h \in \mathscr{F}^\star \text{ such that } \xi_i = \frac{h^{(i)}(z)}{i!} \text{ for } 0 \leq i < n\right\},$$

with the quotient norm. At this juncture most structural properties of the spaces $(\mathscr{F}^\bigstar)_z^{(n)}$ remain obscure: for instance if they are complete, or Hausdorff, or if $(\mathscr{F}^\bigstar)_z^{(n)}$ contains $(\mathscr{F}^\bigstar)_z^{(k)}$ when $k < n$. All these thrilling questions will we settled in the next section.

5.2. **Duality of the twisted sums.** The first part of the following result was proved by Rochberg for finite-dimensional spaces in [30, Theorem 4.1].

Proposition 5.1. *For each $z \in \mathbb{U}$ and each $n \geq 1$, there is a linear homeomorphism $T_n : (\mathscr{F}^\bigstar)_z^{(n)} \longrightarrow (X_z^{(n)})^*$ given by*

$$T_n(\xi_{n-1}, \ldots, \xi_0)(x_{n-1}, \ldots, x_0) = \sum_{j=0}^{n-1} \langle \xi_j, x_{n-j-1} \rangle \tag{5.1}$$

for $(\xi_{n-1}, \ldots, \xi_0) \in (\mathscr{F}^\bigstar)_z^{(n)}$ and $x_j \in \Delta$. In particular, $(\mathscr{F}^\bigstar)_z^{(n)}$ is a Banach space. Moreover,

$$\left\| T_n : (\mathscr{F}^\bigstar)_z^{(n)} \longrightarrow (X_z^{(n)})^* \right\| \leq \frac{1}{\operatorname{dist}(z, \partial\mathbb{U})^{n-1}}\,. \tag{5.2}$$

As the reader may guess, the lion's share of the proof is the boundedness of the pairing (5.1). We shall need a number of intermediate steps, some new notations and a bit of function theory.

Given integers n and k, we consider the maps $\jmath_{n,n+k} : (\Delta^\star)^n \longrightarrow (\Delta^\star)^{n+k}$ and $\varpi_{n+k,k} : (\Delta^\star)^{n+k} \longrightarrow (\Delta^\star)^k$ defined by

$$\jmath_{n,n+k}((\xi_{n-1}, \ldots, \xi_0)) = (\xi_{n-1}, \ldots, \xi_0, \underbrace{0, \ldots, 0}_{k \text{ times}}) \quad \text{and} \quad \varpi_{n+k,k}(\xi_{n+k-1}, \ldots, \xi_k, \xi_{k-1}, \ldots, \xi_0) = (\xi_{k-1}, \ldots, \xi_0)$$

We label them this way to distinguish them from the maps $\imath_{n,n+k} : \Sigma^n \longrightarrow \Sigma^{n+k}$ and $\pi_{n+k,k} : \Sigma^{n+k} \longrightarrow \Sigma^k$ appearing in (3.3), although they are formally the same maps.

Lemma 5.2. *For every $n, k \geq 1$ and every $z \in \mathbb{U}$, the map $\jmath_{n,n+k}$ is bounded from $(\mathscr{F}^\bigstar)_z^{(n)}$ to $(\mathscr{F}^\bigstar)_z^{(n+k)}$ and $\varpi_{n+k,k}$ is an isometric quotient map from $(\mathscr{F}^\bigstar)_z^{(n+k)}$ to $(\mathscr{F}^\bigstar)_z^{(k)}$.*

Proof. By [5, Lemma 1] there is a polynomial P of degree at most $n+k-1$ such that $(P \circ \varphi^{-1})^{(i)}(z) = i!\delta_{ik}$ (Kronecker delta) for $0 \leq i < n+k$. Pick $\xi = (\xi_{n-1}, \cdots, \xi_0)$ in $(\mathscr{F}^\bigstar)^{(n)}$ and $h \in \mathscr{F}^\bigstar$ such that $\tau_{(n,0]}h(z) = \xi$. Consider the function $H = (P \circ \varphi^{-1})\dot{h}$. Then $H \in \mathscr{F}^\bigstar$,

$$\tau_{(n+k-1,0]}H(z) = (\xi_{n-1}, \ldots, \xi_0, 0, \ldots, 0) = \jmath_{n,n+k}(\xi),$$

and $\|H\|_{\mathscr{F}^\bigstar} \leq (\sum_i |a_i|)\,\|h\|_{\mathscr{F}^\bigstar}$, where a_i are the coefficients of P, so that $\|\jmath_{n,n+k} : (\mathscr{F}^\bigstar)_z^{(n)} \longrightarrow (\mathscr{F}^\bigstar)_z^{(n+k)}\| \leq \sum |a_i|$. The second part is trivial. □

Lemma 5.3. *Let $h \in \mathscr{F}^\bigstar$ and $g \in \mathscr{G}$. Then the function $f : \mathbb{U} \longrightarrow \mathbb{C}$ given by $f(z) = \langle h(z), g(z) \rangle$ is bounded, analytic on $\mathbb{U}$ and one has*

$$f^{(n)}(z) = n! \sum_{j=0}^{n} \left\langle \frac{h^{(j)}(z)}{j!}, \frac{g^{(n-j)}(z)}{(n-j)!} \right\rangle \qquad \text{and} \qquad \frac{\left|f^{(n)}(z)\right|}{n!} \leq \frac{\|h\|_{\mathscr{F}^\bigstar}\|g\|_{\mathscr{F}}}{\operatorname{dist}(z, \partial\mathbb{U})^n}. \tag{5.3}$$

Proof. We begin by noticing that by our assumptions, and the very definition of $\mathscr{G}$, the composition $f \circ \varphi$ is in N^+, and therefore has almost everywhere nontangential limits on $\mathbb{T}$. If we denote by F the boundary values of $f \circ \varphi$, we have $|F(z)| \leq \|h\|_{\mathscr{F}^\bigstar}\|g\|_{\mathscr{F}}$ for almost every $z \in \mathbb{T}$, so that $z \in \mathbb{T} \longmapsto F(z)$ is in $L_\infty(\mathbb{T})$. This implies that $f \circ \varphi \in H^\infty$, and therefore f is bounded on $\mathbb{U}$.

We will establish (5.3) by induction on $n \geq 0$. The initial step ($n = 0$) is the definition of f. Suppose (5.3) is valid for a given $n \geq 0$, rewrite it as $f^{(n)}(z) = \sum_{j=0}^{n} \binom{n}{j} \langle h^{(j)}(z), g^{(n-j)}(z)\rangle$, and let us check the induction step:

$$\begin{aligned}
\frac{d}{dz} f^{(n)}(z) &= \sum_{j=0}^{n} \binom{n}{j} \frac{d}{dz} \langle h^{(j)}(z), g^{(n-j)}(z)\rangle \\
&= \sum_{j=0}^{n} \binom{n}{j} \Big(\langle h^{(j+1)}(z), g^{(n-j)}(z)\rangle + \langle h^{(j+1)}(z), g^{(n-j+1)}(z)\rangle \Big) \\
&= \sum_{j=1}^{n+1} \binom{n}{j-1} \langle h^{(j)}(z), g^{(n+1-j)}(z)\rangle + \sum_{j=0}^{n} \binom{n}{j} \langle h^{(j)}(z), g^{(n+1-j)}(z)\rangle \\
&= \binom{n}{0} \langle h^{(0)}(z), g^{(n+1)}(z)\rangle + \sum_{j=1}^{n} \binom{n+1}{j} \langle h^{(j)}(z), g^{(n+1-j)}(z)\rangle + \binom{n}{n} \langle h^{(n+1)}(z), g^{(0)}(z)\rangle \\
&= \sum_{j=0}^{n+1} \binom{n+1}{j} \langle h^{(j)}(z), g^{(n+1-j)}(z)\rangle
\end{aligned}$$

The estimate follows from the bound $|f(u)| \leq \|h\|_{\mathscr{F}^\star} \|g\|_{\mathscr{F}}$ for all $u \in \mathbb{U}$ and Cauchy's estimates for the derivatives of a bounded function, taking into account that for every $r < \operatorname{dist}(z, \partial\mathbb{U})$ the disc of radius r centered at z lies inside $\mathbb{U}$. □

Proof of Proposition 5.1. We begin by showing that, for each $z \in \mathbb{U}$, the map T_n is bounded from $(\mathscr{F}^\star)_z^{(n)}$ to $(X_z^{(n)})^*$. Put $x = (x_{n-1}, \cdots, x_0)$ and $\xi = (\xi_{n-1}, \cdots, \xi_0)$. Take $g \in \mathscr{G}$ such that $\tau_{(n,0]} g(z) = x$ and a corresponding $h \in \mathscr{F}^\star$ for ξ. Let $f(u) = \langle h(u), g(u)\rangle$. By Lemma 5.3, f is bounded and analytic on $\mathbb{U}$, with $|f(u)| \leq \|h\|_{\mathscr{F}^\star} \|g\|_{\mathscr{G}}$ for all $u \in \mathbb{U}$ and

$$|T_n(\xi)(x)| = \frac{\left| f^{(n-1)}(z) \right|}{(n-1)!} \leq \frac{\|h\|_{\mathscr{F}^\star} \|g\|_{\mathscr{G}}}{\operatorname{dist}(z, \partial\mathbb{U})^{n-1}}.$$

Since h and g were arbitrary, we obtain that $T_n(\xi)$ extends to a continuous functional on $X_z^{(n)}$ that we call again $T_n(\xi)$, and that T_n is a bounded map, with $\left\| T_n : (\mathscr{F}^\star)_z^{(n)} \longrightarrow (X_z^{(n)})^* \right\| \leq \operatorname{dist}(z, \partial\mathbb{U})^{1-n}$.

The remainder of the proof is easier. First, for $n, k \geq 1$ and $z \in \mathbb{U}$, the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\mathscr{F}^\star)_z^{(n)} & \xrightarrow{\jmath_{n,n+k}} & (\mathscr{F}^\star)_z^{(n+k)} & \xrightarrow{\varpi_{n+k,k}} & (\mathscr{F}^\star)_z^{(k)} & \longrightarrow & 0 \\
 & & \Big\downarrow T_n & & \Big\downarrow T_{n+k} & & \Big\downarrow T_k & & \\
0 & \longrightarrow & (X_z^{(n)})^* & \xrightarrow{{\pi_{n+k,k}}^*} & (X_z^{(n+k)})^* & \xrightarrow{{\imath_{k,n+k}}^*} & (X_z^{(k)})^* & \longrightarrow & 0
\end{array} \tag{5.4}$$

At this stage of the proof we cannot guarantee the exactness of the upper row of the preceding diagram: we have not proved that the image of $\jmath_{n,n+k}$ fills the kernel of $\varpi_{n+k,k}$. However, we know that T_1 is an isomorphism (it is in fact an isometry, by the result of Coifman, Cwikel, Rochberg, Sagher and Weiss mentioned before) and then a diagram chasing argument quickly shows that T_m is an isomorphism for all $m \geq 1$. Indeed let us assume that T_n and T_k are isomorphisms and let us check that then so is T_{n+k}. It is clear that T_{n+k} is injective. We show that it is also onto and open. Pick an arbitrary $x^* \in (X_z^{(n+k)})^*$ and let $\xi \in (\mathscr{F}^\star)_z^{(n+k)}$ be such that $\varpi_{n+k,k}(\xi) = T_k^{-1}(\imath_{n,n+k}{}^*(x^*))$, with

$$\|\xi\|_{(\mathscr{F}^\star)_z^{(n+k)}} \leq C\|T_k^{-1}(\imath_{n,n+k}{}^*(x^*))\|_{(\mathscr{F}^\star)_z^{(k)}}.$$

for a constant C independent of the choices. Now, $x^* - T_{n+k}(\xi)$ belongs to $\ker \imath_{n,n+k}{}^*$ and since the lower row is exact there is $y^* \in X_z^{(n)}$ such that $\pi_{n+k,k}{}^*(y^*) = x^* - T_{n+k}(\xi)$. Letting $\eta = \jmath_{n,n+k}(T_n^{-1}(y^*))$ it is clear that $x^* = T_{n+k}(\xi + \eta)$. Besides,

$$\begin{aligned}\|\xi\| &\leq C\|T_k^{-1}\| \, \|\imath_{k,n+k}\| \, \|x^*\|,\\ \|\eta\| &\leq \|\jmath_{n,n+k}\| \, \|T_n^{-1}\| \, \|(\pi_{n+k,k}{}^*)^{-1}\| \, (1 + \|T_{n+k}\|) \, \|x^*\|.\end{aligned}$$

□

One thus has the following result, in which we adhere to the notation and conventions we we agreed at the beginning of this section.

Theorem 5.4. *If $n, k \geq 1$ and $z \in \mathbb{U}$ then* (5.4) *is a commutative diagram in which the vertical arrows are linear homeomorphisms and the rows are exact.*

5.3. **A useful "norming" subspace to work with couples.** In this section we take advantage of a result by Cwikel [17, Theorem 3.1] to obtain a quite useful subspace of the dual space of the derived spaces of a couple.

Let (X_0, X_1) be a compatible couple with sum Σ and intersection Δ, which is equipped with the norm $x \in X_0 \cap X_1 \longmapsto \max(\|x\|_0, \|x\|_1)$. We assume that (X_0, X_1) is "regular", i.e., Δ is dense in each X_i. Then each X_i^* embeds into Δ^* (not $\Delta^\star$) in such a way that $X_0^* \cap X_1^* = \Sigma^*$. Let $A(\mathbb{S})$ be the algebra of analytic functions on the strip that admit continuous bounded extensions to the closure.

There is a natural bilinear pairing $B : \mathscr{C}_0(X_0, X_1) \times \mathscr{C}(X_0^*, X_1^*) \longrightarrow A(\mathbb{S})$ defined by

$$B(h, g)(z) = \langle h(z), g(z)\rangle$$

for $g \in \mathscr{C}_{00}(X_0, X_1), h \in \mathscr{C}(X_0^*, X_1^*)$; recall that such a g takes values in Δ and that $\mathscr{C}_{00}(X_0, X_1)$ is dense in $\mathscr{C}_0(X_0, X_1)$ so that the previous B can be extended to the completion), where the brackets refer to the duality between Δ^* and Δ. Now, *mutatis mutandis*, the arguments of the preceding section yield:

Proposition 5.5. *For each $z \in \mathbb{S}$ and each $n \geq 1$, let $T_n : \mathscr{C}(X_0^*, X_1^*)_z^{(n)} \longrightarrow (\mathscr{C}_0(X_0, X_1)_z^{(n)})^*$ be given by*

$$T_n(\xi_{n-1}, \ldots, \xi_0)(x_{n-1}, \ldots, x_0) = \sum_{j=0}^{n-1} \langle \xi_j, x_{n-j-1}\rangle \tag{5.5}$$

for $(\xi_{n-1}, \ldots, \xi_0) \in \mathscr{C}(X_0^*, X_1^*)_z^{(n)}$ and $x_j \in \Delta$ for $0 \leq j < n$. The operator T_n is bounded, with

$$\left\| T_n : \mathscr{C}(X_0^*, X_1^*)_z^{(n)} \longrightarrow (\mathscr{C}_0(X_0, X_1)_z^{(n)})^* \right\| \leq \frac{1}{\operatorname{dist}(z, \partial\mathbb{S})^{n-1}}. \tag{5.6}$$

Moreover, T_n "renorms" $\mathscr{C}_0(X_0, X_1)_z^{(n)}$ in the following sense: there exist constants $c, C > 0$ that depend on z and n, but not ξ or x, such that

$$c\,\|x\|_{\mathscr{C}_0(X_0,X_1)_z^{(n)}} \leq \sup\left\{ \left|T_n(\xi)(x)\right| : \xi \in \mathscr{C}(X_0^*, X_1^*)_z^{(n)}, \|\xi\|_{\mathscr{C}(X_0^*,X_1^*)_z^{(n)}} \leq 1 \right\} \leq C\,\|x\|_{\mathscr{C}_0(X_0,X_1)_z^{(n)}}. \tag{5.7}$$

In particular, if X_ζ is reflexive for some $0 < \Re(\zeta) < 1$, which is always the case if one of the spaces of the couple is reflexive, then T_n is an isomorphism for every n and z. The same happens if X_0 or X_1 is an Asplund space (equivalently, the dual has the Radon-Nikodým property).

Sketch of the Proof. The proof of the first part runs parallel to that of Proposition 5.1 and is left to the reader. The "moreover" part follows from Cwikel's result mentioned earlier (namely, that when $n = 1$ the inequalities in (5.7) are actually equalities with $c = C = 1$) by an easy induction argument. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{C}(X_0^*, X_1^*)_z^{(n)} & \longrightarrow & \mathscr{C}(X_0^*, X_1^*)_z^{(n+k)} & \longrightarrow & \mathscr{C}(X_0^*, X_1^*)_z^{(k)} & \longrightarrow & 0 \\
 & & \downarrow T_n & & \downarrow T_{n+k} & & \downarrow T_k & & \\
0 & \longrightarrow & (\mathscr{C}(X_0, X_1)_z^{(n)})^* & \xrightarrow{\pi_{n+k,n}{}^*} & (\mathscr{C}(X_0, X_1)_z^{(n+k)})^* & \xrightarrow{\imath_{k,n+k}{}^*} & (\mathscr{C}(X_0, X_1)_z^{(k)})^* & \longrightarrow & 0
\end{array}$$

and recall our convention about unlabelled arrows. Assuming that T_n and T_k are "renormings", one quickly obtains chasing the diagram T_{n+k} renorms $\mathscr{C}(X_0, X_1)_z^{(n+k)}$. The last assertion in the statement follows from being T_1 a surjective isometry [24, Theorem 4.4], as it was explained during the proof of Proposition 5.1, and thus, by Diagram 5.4, the same occurs to all T_n. □

Corollary 5.6. *For every $z \in \mathbb{S}$ and every $n \geq 1$ the dual of $\mathscr{C}(\ell_\infty, \ell_1)_z^{(n)}$ is isomorphic to $\mathscr{C}(\ell_\infty, \ell_1)_{1-z}^{(n)}$.*

If we specialize to $z = \frac{1}{2}$ we obtain that each of the spaces $\mathscr{Z}_n$ is isomorphic to its dual (Thus, for instance, Theorem 4.2 can be dualized replacing "embeds in" by "is a quotient of", and so on). However the pairing witnessing it is not

$$\langle (y_{n-1}, \ldots, y_0), (x_{n-1}, \ldots, x_0) \rangle = \sum_{i+j=n-1} \langle y_i, x_j \rangle$$

because this pairing induces an isomorphism between $\mathscr{C}(\ell_\infty, \ell_1)_{1/2}^{(n)} = \mathscr{Z}_n$ and the dual of $\mathscr{C}(\ell_1, \ell_\infty)_{1/2}^{(n)}$ which is isometric, but not equal, to $\mathscr{Z}_n$. In general, an isometry between $\mathscr{C}(X_0, X_1)_{1/2}^{(n)}$ and $\mathscr{C}(X_1, X_0)_{1/2}^{(n)}$ can be obtained as follows: pick $x = (x_{n-1}, \ldots, x_0)$ in $\mathscr{C}(X_0, X_1)_{1/2}^{(n)}$ and then $f \in \mathscr{C}(X_0, X_1)$ such that $x = \tau_{(n,0]} f(\frac{1}{2})$ and $\|f\| \leq \|x\| + \varepsilon$. Clearly $g(z) = f(1 - z)$ has the same norm in $\mathscr{C}(X_1, X_0)$ as f in $\mathscr{C}(X_0, X_1)$. Hence $\tau_{(n,0]} g(\frac{1}{2})$ belongs to $\mathscr{C}(X_1, X_0)_{1/2}^{(n)}$ and has the same norm as x. Clearly

$$\tau_{(n,0]} g(\tfrac{1}{2}) = ((-1)^{n-1} x_{n-1}, \ldots, -x_1, x_0).$$

The inexorable conclusion is that the pairing that defines the isomorphism between $\mathscr{Z}_n$ and its dual is

$$\langle (y_{n-1}, \ldots, y_0), (x_{n-1}, \ldots, x_0) \rangle = \sum_{i+j=n-1} (-1)^i \langle y_i, x_j \rangle$$

If we denote by $u_n : \mathscr{Z}_n \longrightarrow \mathscr{Z}_n^*$ the corresponding (noncanonical) isomorphism then the family $(u_n)_{n \geq 1}$ is "almost" compatible with the natural exact sequences:

Corollary 5.7. . *With the same notations as before, for every $k, n \geq 1$ the following diagram is commutative*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{Z}_n & \xrightarrow{\imath_{n,n+k}} & \mathscr{Z}_{n+k} & \xrightarrow{\pi_{n+k,k}} & \mathscr{Z}_k & \longrightarrow & 0 \\
 & & \downarrow (-1)^k u_n & & \downarrow u_{n+k} & & \downarrow u_k & & \\
0 & \longrightarrow & \mathscr{Z}_n^* & \xrightarrow{\pi_{n+k,k}^*} & \mathscr{Z}_{k+n}^* & \xrightarrow{\imath_{n,n+k}^*} & \mathscr{Z}_k^* & \longrightarrow & 0
\end{array}$$

The continuity of the operators T_n of Proposition 5.5 provides lower bounds for the norm of an element of the form $(0, \ldots, 0, x)$ in $X_c^{(n)} = \mathscr{C}(X_0, X_1)_c^{(n)}$. Note that for $0 < c < 1$ we have $\text{dist}(c, \partial\mathbb{S}) =$

$\min(c, 1-c)$. Now, if $h \in \mathscr{C}(X_0^*, X_1^*)$, and $x \in \Delta$, then

$$\left|\langle \tau_{n-1} h(c), x\rangle\right| = \left|(T_n \tau_{(n,0]} h(c))(0, \dots, 0, x)\right| \leq \frac{\|h\|_{\mathscr{C}(X_0^*, X_1^*)} \, \|(0, \dots, 0, x)\|_{X_c^{(n)}}}{\min(c, 1-c)^{n-1}}. \tag{5.8}$$

Let us consider again the case where $X_0 = \ell_\infty$ and $X_1 = \ell_1$ and estimate the norm of $(0, \dots, 0, s_N)$ in the space $\mathscr{Z}_c^{(n)} = \mathscr{C}(\ell_\infty, \ell_1)_c^{(n)}$, for $0 < c < 1$. Note that $\mathscr{Z}_c^{(1)} = \ell_p$ with $p = 1/c$ and so $\|s_N\|_{\mathscr{Z}_c^{(1)}} = N^{1/p} = N^c$. If we interpret $\mathscr{C}(c_0, \ell_1)$ as a subset of $\mathscr{C}(\ell_\infty, \ell_1)$ in the obvious way, $\mathscr{C}(\ell_\infty, \ell_1)_c = \mathscr{C}(c_0, \ell_1)_c$, with the same norm: just think of the finitely supported sequences. It follows from Lemma 3.3 that $\mathscr{C}(\ell_\infty, \ell_1)_c^{(n)} = \mathscr{C}(c_0, \ell_1)_c^{(n)}$ for all $n \geq 1$ and $0 < c < 1$, still with the same norm. Since (c_0, ℓ_1) is a regular couple we can go to Proposition 5.5 and then compute the extremals in $\mathscr{C}(\ell_1, \ell_\infty)$. Note that $\mathscr{C}(\ell_1, \ell_\infty)_c = \ell_q$, where q is the conjugate exponent of p and that if x is positive and normalized in ℓ_q, then the function $z \mapsto x^{q(1-z)}$ is normalized in $\mathscr{C}(\ell_1, \ell_\infty)$ and assumes the value x at $z = c$. It follows that for any $x \in \ell_q$ the function

$$h(z) = x\left(\frac{|x|}{\|x\|_q}\right)^{-q(z-c)} = x \sum_{n \geq 0} \frac{(-q)^n}{n!}\left(\log^n \frac{|x|}{\|x\|_q}\right)(z-c)^n$$

is an extremal for x in $\mathscr{C}(\ell_1, \ell_\infty)$, with

$$\tau_n h(c) = \frac{(-q)^n}{n!}\, x \, \log^n \frac{|x|}{\|x\|_q}.$$

Letting $x = s_N$ in ℓ_p and taking h as the corresponding extremal for s_N in $\mathscr{C}(\ell_1, \ell_\infty)$ so that $h(c) = s_N$, with $\|h\|_{\mathscr{C}(\ell_1, \ell_\infty)} = N^{1/q}$ and applying (5.8) one obtains

$$\frac{N \log^{n-1} N}{(n-1)!} \leq \frac{N^{1/q} \|(0, \dots, 0, s_N)\|_{X_c^{(n)}}}{\min(c, 1-c)^{n-1}},$$

hence (compare with the proof of Lemma 4.4)

$$\|(0, \dots, 0, s_N)\|_{X_c^{(n)}} \geq \frac{\min(c, 1-c)^{n-1}}{(n-1)!} N^{1/p} \log^{n-1} N.$$

6. Analytic families of Rochberg spaces and interpolation

This section develops the central topic of the paper and it is where acceptable spaces are required and admissible spaces do not suffice. The domain $\mathbb{U}$ on which an acceptable space of analytic functions is based plays an important role here. The simplest domains are: the unit strip $\mathbb{S}$, where classical interpolation for couples occurs, and the unit disk $\mathbb{D}$, where classical interpolation for families occur. Thus, to motivate the problem let us consider first:

6.1. **The case of couples.** The following reiteration-like result is so natural that we can hardly believe it has not been explicitly stated elsewhere. The case $n = 1$ is the classical reiteration theorem for the complex method (cf. [6, Paragraph 12.3]).

Proposition 6.1. *Let (X_0, X_1) be a regular couple of Banach spaces on the strip $\mathbb{S}$, with sum Σ, intersection Δ and $0 < a < b < 1$. For every $n \geq 1$ the Rochberg spaces $X_a^{(n)}$ and $X_b^{(n)}$ form a compatible couple on the strip $\mathbb{S}_{a,b}$ as subspaces of Σ^n and, for every $a < c < b$, the formal inclusion $X_c^{(n)} \longrightarrow [X_a^{(n)}, X_b^{(n)}]_c$ is an isomorphic embedding. If, in addition, Δ^n is dense in $X_a^{(n)} \cap X_b^{(n)}$, which is always the case when X_1 contains X_0, then $[X_a^{(n)}, X_b^{(n)}]_c = X_c^{(n)}$, with equivalent norms.*

Proof. We first remark that in the case of couples we may assume that the norm of Σ is majorized by those of X_0 and X_1. Thus, integrating on large rectangular contours and using Cauchy integral formulæ one gets, for $0 < \theta < 1$, that

$$\|\delta_\theta^{(n)} : \mathscr{C}_0(X_0, X_1) \longrightarrow \Sigma\| \leq \frac{n!}{\min(|\theta|, |1-\theta|)^n}.$$

Thus, if $x = (x_{n-1}, \ldots, x_0)$ belongs to $X_\theta^{(n)}$, and $f \in \mathscr{C}(X_0, X_1)$ is such that $x = \tau_{(n,0]} f(\theta)$, then

$$\max_{0 \leq i < n} \|x_i\|_\Sigma \leq \frac{\|f\|_{\mathscr{C}}}{\min(|\theta|, |1-\theta|)^n},$$

hence Σ^n contains both $X_a^{(n)}$ and $X_b^{(n)}$, the inclusions are continuous and $(X_a^{(n)}, X_b^{(n)})$ is a compatible couple ready for interpolation on the strip $\mathbb{S}_{a,b}$. From now on, we write $Y_a = X_a^{(n)}$ and $Y_b = X_b^{(n)}$. Notice that at the moment we do not know whether $Y_c = X_c^{(n)}$, which which is precisely what we have to prove. We end this preparation noticing that, according to our general notations,

$$X_\eta^{(n)} = \mathscr{C}(X_0, X_1)_\eta^{(n)} = \mathscr{C}_0(X_0, X_1)_\eta^{(n)} \qquad \text{and} \qquad Y_c = [Y_a, Y_b]_c = \mathscr{C}(Y_a, Y_b)_c = \mathscr{C}_0(Y_a, Y_b)_c.$$

Let us see that $X_c^{(n)} \subset Y_c$ with contractive inclusion, which is the easy part. Given $f \in \mathscr{C}(X_0, X_1)$ we define an analytic function $R(f) : \mathbb{S}_{a,b} \longrightarrow \Sigma^n$ by $Rf(z) = \tau_{(n,0]} f(z)$. We claim that R defines a bounded operator from $\mathscr{C}_0(X_0, X_1)$ to $\mathscr{C}(Y_a, Y_b)$. Clearly, if f is a simple function with values in Δ then $Rf \in \mathscr{C}(Y_a, Y_b)$ and $\|Rf\|_{\mathscr{C}(Y_a,Y_b)} \leq \|f\|_{\mathscr{C}(X_0,X_1)}$. For arbitrary $f \in \mathscr{C}_0(X_0, X_1)$ the claim follows from an obvious density argument. We therefore have a commutative square

$$\begin{array}{ccc} \mathscr{C}_0(X_0, X_1) & \xrightarrow{\ R\ } & \mathscr{C}(Y_a, Y_b) \\ {\scriptstyle \delta_c \circ \tau_{(n,0]}} \big\downarrow & & \big\downarrow {\scriptstyle \delta_c} \\ X_c^{(n)} & \xrightarrow{\text{identity}} & Y_c \end{array}$$

witnessing that the formal identity is a bounded operator from $X_c^{(n)}$ to Y_c, with norm at most 1. To complete the proof of the first part we must show that there is a constant C such that $\|x\|_{X_c^{(n)}} \leq C\|x\|_{Y_c}$ for $x \in \Delta^n$. We need here the duality results of the preceding section. Since T_n renorms $X_c^{(n)}$, it suffices to show that there is a constant K such that

$$|T_n \xi(x)| \leq K \|\xi\|_{\mathscr{C}(X_0^*, X_1^*)_c^{(n)}} \|x\|_{Y_c}$$

for $x \in \Delta^n, \xi \in \mathscr{C}(X_0^*, X_1^*)_c^{(n)}$. Pick $\varepsilon > 0$ and a function $g : \mathbb{S}_{a,b} \longrightarrow \Sigma^n$ such that $g(c) = x$ with $\|g\| \leq (1+\varepsilon)\|x\|_{Y_c}$. Now, pick $h \in \mathscr{C}(X_0^*, X_1^*)$ such that $\tau_{(n,0]} h(c) = \xi$, with $\|h\| \leq (1+\varepsilon)\|\xi\|$. Since $X_0^* \cap X_1^* = \Sigma^*$, slightly perturbing ξ if necessary, we may assume that h has the form (2.1), with vectors in Σ^*. Then the components of $\tau_{(n,0]} h$ are Σ^*-bounded on $\mathbb{S}_{a,b}$ and since g is Σ^n-bounded the function

$$f(z) = T_n(\tau_{(n,0]} h(z))(g(z))$$

is bounded and analytic on $\mathbb{S}_{a,b}$, and $f(c) = T_n \xi(x)$. But, for $z \in \partial \mathbb{S}_{a,b}$ one has

$$|f(z)| \leq \left\| T_n : \mathscr{C}(X_0^*, X_1^*)_z^{(n)} \longrightarrow (X_z^{(n)})^* \right\| \left\| \tau_{(n,0]} h(z) \right\|_{\mathscr{C}(X_0^*, X_1^*)_z^{(n)}} \left\| g(z) \right\|_{X_z^{(n)}} \leq \frac{(1+\varepsilon)^2 \left\|\xi\right\|_{\mathscr{C}(X_0^*, X_1^*)_c^{(n)}} \|x\|_{Y_c}}{\min(a, 1-b)^{n-1}}$$

since for $z \in \partial \mathbb{S}_{a,b}$ the space $X_z^{(n)}$ agrees with Y_a when $\Re(z) = a$ and with Y_b when $\Re(z) = b$. The result follows from the maximum principle. The second part is clear: if Δ^n is dense in $X_a^{(n)} \cap X_b^{(n)}$, then it is dense in $[X_a^{(n)}, X_b^{(n)}]_c$ too. □

One may wonder if the irritating hypothesis about the density of Δ^n in $X_a^{(n)} \cap X_b^{(n)}$ is really necessary to get the identity $[X_a^{(n)}, X_b^{(n)}]_c = X_c^{(n)}$. (Even more when we do have [16] for $n = 1$.)

The reader may observe that the preceding proposition does not automatically guarantees that the spaces $X_c^{(n)}$ for $a < c < b$ — or, better, $X_z^{(n)}$ for $a < \Re(z) < b$ — form an analytic family unless $X_0 \subset X_1$ (or vice versa) in which case they coincide with the analytic family attached to the Calderón space $\mathscr{C}(X_a^{(n)}, X_b^{(n)})$, up to equivalence of the norms. In general, the question of which admissible space could have been, and could now be, used to obtain the "derived spaces" of the family $X_z^{(n)}$ an the "higher order Rochberg spaces" admits several answers. The most obvious is to choose:

$$\mathscr{D} = \left\{ g \in \mathscr{C}(X_a^{(n)}, X_b^{(n)}) : g(z) \in X_z^{(n)} \text{ for } a \leq \Re(z) \leq b \right\},$$

with the norm inherited from $\mathscr{C}(X_a^{(n)}, X_b^{(n)})$. One has:

Corollary 6.2. *With the same notations as above, $\mathscr{D}$ is an admissible space of analytic functions on the strip $\mathbb{S}_{a,b}$ and for each $z \in \mathbb{S}_{a,b}$, one has $\mathscr{D}_z = X_z^{(n)}$, with equivalent norms. Besides, if $x \in X_z^{(n)}$ and $f \in \mathscr{C}(X_0, X_1)$ is such that $x = \tau_{(n,0]} f(z)$ and $\|f\|_{\mathscr{C}(X_0,X_1)} \approx \|x\|_{X_c^{(n)}}$, then, if F is the restriction of $\tau_{(n,0]} f$ to $\mathbb{S}_{a,b}$, one has $F(z) = x$, and $\|F\|_{\mathscr{D}} = \|f\|_{\mathscr{C}(X_0,X_1)} \leq C\|x\|_{\mathscr{D}_z}$, where C is a constant depending on z, but not on x.*

Proof. To prove that $\mathscr{D}$ is admissible it suffices to check that if $\varphi : \mathbb{S}_{a,b} \longrightarrow \mathbb{D}$ is a conformal equivalence, $g : \mathbb{S}_{a,b} \longrightarrow \Sigma^n$ is analytic and $\varphi g \in \mathscr{D}$, then $g \in \mathscr{D}$. Of course $g \in \mathscr{C}(X_a^{(n)}, X_b^{(n)})$. Let us see that $g(z) \in X_z^{(n)}$ for all $z \in \mathbb{S}_{a,b}$. This is obvious if $\varphi(z) \neq 0$. Put $\zeta = \varphi^{-1}(0)$ and notice that the reasoning about R contained in the proof of Theorem 6.1 shows that the restriction of g to the line $\Re(z) = \Re(\zeta)$ is a continuous map with values in $[X_a^{(n)}, X_b^{(n)}]_{\Re\zeta}$. As $g(z)$ belongs to $X_z^{(n)} = X_{\Re\zeta}^{(n)}$ for every $z \neq \zeta$ in the line $\Re(z) = \Re(\zeta)$ and this space is closed in $[X_a^{(n)}, X_b^{(n)}]_{\Re\zeta}$, we conclude that $g(\zeta) \in X_{\Re\zeta}^{(n)}$ and so $\mathscr{D}$ is admissible. The "besides" part is clear after Proposition 6.1. □

Thus, starting with a compatible couple (X_0, X_1) sitting on $\mathbb{S}$ one obtains the family $X_c = \mathscr{C}(X_0, X_1)_c$ and the corresponding Rochberg spaces $X_c^{(n)}$ for $0 < c < 1$. These spaces can be twisted in two ways: one is forming the space $X_c^{(2n)}$ which leads to the self-extension

$$0 \longrightarrow X_c^{(n)} \longrightarrow X_c^{(2n)} \longrightarrow X_c^{(n)} \longrightarrow 0 \tag{6.1}$$

described in Section 3. But the preceding corollary also opens up the possibility of considering $X_c^{(n)}$ as one of the spaces of the analytic family induced by $\mathscr{D}$ which leads to the self-extension

$$0 \longrightarrow X_c^{(n)} \longrightarrow \mathscr{D}_c^{(2)} \longrightarrow X_c^{(n)} \longrightarrow 0 \tag{6.2}$$

These extensions are different. Indeed, the differential associated to (6.1) is obtained as follows: given $x = (x_{n-1}, \ldots, x_0)$ in $X_c^{(n)}$ we select $f \in \mathscr{C}(X_0, X_1)$ such that $x = \tau_{(n,0]} f(c)$, with $\|f\|_{\mathscr{C}(X_0,X_1)} \approx \|x\|_{X_c^{(n)}}$ and set

$$\Omega^{n,n}(x) = \tau_{(2n,n]} f(c).$$

As for (6.2) we can use the restriction F of $\tau_{(n,0]} f$ to $\mathbb{S}_{a,b}$ as an extremal for x in $\mathscr{D}$, so that the corresponding derivation is

$$\Phi^{1,1}(x) = F'(c) = \left(\frac{f^{(n)}(c)}{(n-1)!}, \frac{f^{(n-1)}(c)}{(n-2)!}, \ldots, f'(c) \right) = \Bigg(\underbrace{n\frac{f^{(n)}(c)}{n!}}_{\text{nonlinear}}, \underbrace{(n-1)x_{n-1}, \ldots, x_1}_{\text{linear part}} \Bigg).$$

This seems to indicate that, in a sense, (6.1) "twists" $X_c^{(n)}$ more than (6.2) does. This point will be discussed in depth in Section 7, in the broader context of acceptable spaces.

6.2. **The issue of families.** We have encountered insurmountable difficulties to generalize Proposition 6.1 to admissible families. Let us explain this point in detail because, in the end, these difficulties forced us to introduce the notion of acceptability, which will be much more relevant from now on than it has been so far.

Let $\mathbb{U}$ be a domain and let $\mathbb{V}$ be a subdomain with compact closure contained in $\mathbb{U}$. We fix conformal equivalences $\varphi : \mathbb{D} \longrightarrow \mathbb{U}$ and $\phi : \mathbb{D} \longrightarrow \mathbb{V}$ having the extension properties required in Section 2.2 and we denote again by φ and ϕ their extensions to $\mathbb{T}$. These are well-defined up to a null set.

Suppose we are given an admissible interpolation family on $\mathbb{U}$, say $\mathcal{X} = (X_z)_{z\in\partial\mathbb{U}}$, with ambient space Σ, intersection Δ and containing function k. Fixing $n \geq 1$ we can consider the family of Rochberg spaces $X_z^{(n)}$ with z varying in $\mathbb{U}$ (note that there are no Rochberg spaces on the original boundary $\partial\mathbb{U}$!), which includes $\partial\mathbb{V}$. In this way we obtain another family, parametrized by $\partial\mathbb{V}$, namely $\mathcal{Y} = (Y_n)_{v\in\partial\mathbb{V}}$, where $Y_v = X_v^{(n)}$ for $v \in \partial\mathbb{V}$. We would like to make $\mathcal{Y}$ an interpolation family. To this end we can choose Σ^n as the ambient space and Δ^n as the intersection space of $\mathcal{Y}$ so that the compactness of $\overline{\mathbb{V}}$ resolves the "containing function" issue:

Lemma 6.3. *Under the above hypotheses there is a constant C such that if $(x_{n-1}, \dots, x_0)$ belongs to $X_v^{(n)}$ for some $v \in \overline{\mathbb{V}}$, then $\sum_{0\leq i<n} \|x_i\|_\Sigma \leq C\|(x_{n-1}, \dots, x_0)\|_{X_v^{(n)}}$.*

Proof. Let $k : \partial\mathbb{U} \longrightarrow (0, \infty)$ be the containing function of $\mathcal{X}$ and $K : \mathbb{D} \longrightarrow \mathbb{C}$ be the outer function associated to $k \circ \varphi$. Then, for every $u \in \mathbb{U}$, every $n \geq 0$ and every $R < \operatorname{dist}(u, \partial\mathbb{U})$, one has

$$\| \delta_u^{(n)} : \mathscr{F}(\mathcal{X}) \longrightarrow \Sigma \| \leq \frac{n!\, M(u,R)}{R^n}, \qquad \text{where} \qquad M(u,R) = \max_{|u-z|\leq R} |K(\varphi^{-1}(z))|.$$

This is straightforward from Cauchy's estimates. Let $r = \frac{1}{2}\operatorname{dist}(\mathbb{V}, \partial\mathbb{U})$. Then $\overline{\mathbb{V}}_r = \overline{\mathbb{V}} + \overline{\mathbb{D}}_r = \{v + z : v \in \overline{\mathbb{V}}, |z| \leq r\}$ is a compact subset of $\mathbb{U}$ containing $\overline{\mathbb{V}}$, where $K \circ \varphi^{-1}$ has to be bounded, say by M. Thus, for every $v \in \overline{\mathbb{V}}$, in particular for $v \in \partial\mathbb{V}$ one has $\|\delta_v^{(n)} : \mathscr{F}(\mathcal{X}) \longrightarrow \Sigma\| \leq n!M/r^n$.

Now, pick $v \in \partial\mathbb{V}$ and $x = (x_{n-1}, \dots, x_0)$ in $X_v^{(n)}$. If $f \in \mathscr{F}(\mathcal{X})$ is such that $\tau_{(n,0]} f(v) = x$, we have

$$\sum_{0\leq i<n} \|x_i\|_\Sigma \leq M \left(\sum_{0\leq i<n} r^{-i} \right) \|f\|_{\mathscr{F}},$$

as required. □

This shows that Σ^n, with the sum norm, is a containing space for the family $\mathcal{Y}$, with containing function (actually constant) $M\left(\sum_{0\leq i<n} r^{-i}\right)$. Up to here the good news. The bad news is that we have been unable to establish the measurability of the function $v \in \partial\mathbb{V} \longmapsto \|x\|_{X_v^{(n)}}$ for fixed $x \in \Delta^n$, that is, we cannot guarantee that $\mathcal{Y}$ is an interpolation family. In the case of couples this was automatic as these functions are constant on each vertical line! Worse yet, even if one could establish measurability in some cases (e.g., if the extremals are unique) or if one could dispose of this issue (replacing $N_{\mathbb{V}}^+$ by $A_{\mathbb{V}}$, or something like that), it is unclear whether the hypothesized interpolation family would be admissible. All we know is the following result, which obviates these difficulties adding to the hypothesis a statement that we would have liked to put into the thesis, namely that the family of derived spaces is admissible.

Proposition 6.4. *With the above notations, if $\mathcal{Y}$ is an admissible interpolation family with intersection space Δ^n, then, for every $z \in \mathbb{V}$, one has $Y_z = X_z^{(n)}$ with equivalence of norms.*

Proof. Let us prove first that, for each $v \in \mathbb{V}$, one has $X_v^{(n)} \subset Y_v$, and the inclusion is contractive. Pick $x \in \Delta^n$ and then $g \in \mathscr{G}(\mathcal{X})$ such that $x = \tau_{(n,0]}g(v)$. Let $f : \mathbb{V} \longrightarrow \Sigma^n$ be the restriction of $\tau_{(n,0]}g$ to $\mathbb{V}$. Then $f \in \mathscr{G}(\mathcal{Y})$: indeed, if we write $g = \sum g_j a_j$, with $g_j \in N_{\mathbb{U}}^+$ and $a_j \in \Delta$, then the successive derivatives of each g_j are all bounded on $\mathbb{V}$ and so they belong to $N_{\mathbb{V}}^+$. Besides, we have $\|f(z)\|_{X_z^{(n)}} \leq \|g\|_{\mathscr{F}}$ for every $z \in \partial\mathbb{V}$, so we have $f \in \mathscr{G}(\mathcal{Y})$, with $x = \delta_v f$, and

$$\|x\|_{Y_v} \leq \|f\|_{\mathscr{G}(\mathcal{Y})} \leq \|g\|_{\mathscr{F}(\mathcal{X})}.$$

Since g is arbitrary and Δ^n is dense in $X_v^{(n)}$ we are done.

We now prove the reversed containment and obtain the corresponding bound. This part uses duality in a critical way. First, since $T_n : \mathscr{W}(\mathcal{X})_v^{(n)} \longrightarrow (X_v^{(n)})^*$ is an isomorphism, it suffices to see that there is a constant K such that, if $x \in \Delta^n$ and $\xi \in (\Delta^\star)^n$, $\|\xi\|_{\mathscr{W}(\mathcal{X})_v^{(n)}} < 1$, then

$$|T_n\xi(x)| \leq K\|x\|_{Y_v}.$$

So, take $g \in \mathscr{G}(\mathcal{Y})$ such that $g(v) = x$, with $\|g\|_{\mathscr{F}(\mathcal{Y})} \leq (1+\varepsilon)\|x\|_{Y_v}$ and $h \in \mathscr{W}(\mathcal{X})$ so that $\xi = \tau_{(n,0]}h(v)$, with $\|h\|_{\mathscr{W}(\mathcal{X})} \leq 1$.

By [13, Proposition 2.5] we can assume that the coefficient functions of g are bounded on $\mathbb{V}$. Therefore, using the conformal map $\phi : \mathbb{D} \longrightarrow \mathbb{V}$ we may consider the function $f : \mathbb{D} \longrightarrow \mathbb{C}$ defined by

$$f(z) = T_n(\tau_{(n,0]}h(\phi(z)))(g(\phi(z))).$$

Then f is analytic, bounded on $\mathbb{D}$ and $f(\phi^{-1}(v)) = T_n\xi(x)$. Moreover, for almost every $z \in \mathbb{T}$, one has

$$|f(z)| \leq \left\|T_n : \mathscr{W}_{\phi(z)}^{(n)} \longrightarrow (X_{\phi(z)}^{(n)})^*\right\| \left\|\tau_{(n,0]}h(\phi(z))\right\|_{\mathscr{W}_{\phi(z)}^{(n)}} \left\|g(\phi(z))\right\|_{X_{\phi(z)}^{(n)}} \leq \frac{\|g(\phi(z))\|_{Y_{\phi(z)}}}{\operatorname{dist}(\partial\mathbb{V}, \partial\mathbb{U})^{n-1}} \leq \frac{(1+\varepsilon)\|x\|_{Y_v}}{\operatorname{dist}(\partial\mathbb{V}, \partial\mathbb{U})^{n-1}},$$

and the result follows from the maximum principle. □

6.3. **The case of analytic families on the disc.** This and the next sections do what we wanted to do in the previous section at the cost of working in the general setting of acceptable spaces. Precisely, what we will show is that if $\mathscr{F}$ is an acceptable space of analytic functions on a domain $\mathbb{U}$ then the family of Rochberg spaces $\mathscr{F}_z^{(n)}$, for z varying in $\mathbb{U}$ and $n \geq 2$ fixed, is the analytic family associated to another acceptable space that is naturally attached to $\mathscr{F}$. This result has no counterpart for admissible spaces. It actually was our original motivation to introduce the notion of an acceptable space and what fully justifies our approach. We will treat in this section the case where the domain is the disc, taking advantage of the fact that the underlying algebra A^∞ admits differentiation. The adjustments required to work on general domains are carried out in the next section.

Let $\mathscr{F}$ be an acceptable space on the disc and let $\mathscr{H} = \mathscr{H}(\mathbb{D}, \Sigma)$ be the space of all holomorphic functions from $\mathbb{D}$ to Σ, the ambient space of $\mathscr{F}$. We inductively define a sequence of Banach spaces $\mathscr{F}^{(n)}$, formally subspaces of the product $\mathscr{H}^n$ as follows:

- $\mathscr{F}^{(1)} = \mathscr{F}$.
- Once $\mathscr{F}^{(n)}$ is defined we consider the linear map $\tau_{[n,1]} : \mathscr{F} \longrightarrow \mathscr{H}^n$ and set

$$\mathscr{F}^{(n+1)} = \mathscr{F}^{(n)} \oplus_{\tau_{[n,1]}} \mathscr{F} = \left\{(f_n, \ldots, f_1, f) \in \mathscr{H}^{n+1} : f \in \mathscr{F} \text{ and } (f_n, \ldots, f_1) - \tau_{[n,1]}(f) \in \mathscr{F}^{(n)}\right\},$$

endowed with the norm $\|(f_n, \ldots, f_1, f)\|_{\mathscr{F}^{(n+1)}} = \left\|(f_n, \ldots, f_1) - \tau_{[n,1]}(f)\right\|_{\mathscr{F}^{(n)}} + \|f\|_{\mathscr{F}}$.

Observe that $\mathscr{F}^{(2)}$ consist of those pairs (g, f) such that both f and $g - f'$ are in $\mathscr{F}$, with norm $\|(g, f)\| = \|g - f'\| + \|f\|$. To compute $\mathscr{F}^{(3)}$, pick (f_2, f_1, f_0). Of course f_0 has to be in $\mathscr{F}$, while $(f_2 - f_0''/2, f_1 - f_0')$ must be in $\mathscr{F}^{(2)}$, that is, both $f_1 - f_0'$ and $f_2 - f_0''/2 - (f_1 - f_0')'$ must be in $\mathscr{F}$, so in the end the norm of (f_2, f_1, f_0) in $\mathscr{F}^{(3)}$ is $\|f_2 - f_1' + f_0'' - f_0''/2\| + \|f_1 - f_0'\| + \|f_0\|$. Instead of spoiling all the fun presenting the 4D case, let us see an explicit formula that works in general. The form of the coefficients that appear in the following result can somehow be considered a lucky strike:

Lemma 6.5. *Fix $n \geq 1$ and let $f_i \in \mathscr{H}$ for $0 \leq i < n$. Then $(f_{n-1}, \dots, f_0)$ belongs to $\mathscr{F}^{(n)}$ if and only if for each $0 \leq i < n$ the sum*

$$f_i + \sum_{1\leq k\leq i} \frac{(-1)^k}{k!} f_{i-k}^{(k)}$$

falls into $\mathscr{F}$, where the sum over the empty set is treated as zero. Moreover, for such an array $(f_{n-1}, \dots, f_0)$ one has

$$\|(f_{n-1}, \dots, f_0)\|_{\mathscr{F}^{(n)}} = \|f_0\|_{\mathscr{F}} + \sum_{0<i<n} \left\| f_i + \sum_{1\leq k\leq i} \frac{(-1)^k}{k!} f_{i-k}^{(k)} \right\|_{\mathscr{F}}.$$

Proof. The proof goes by induction on n. The initial step $n = 1$ is trivial, so let us assume that the lemma holds for n and let us check the corresponding statement for $n + 1$. Pick $n + 1$ functions $f_i \in \mathscr{H}$ for $0 \leq i \leq n$. By the very definition, $(f_n, \dots, f_0) \in \mathscr{F}^{(n+1)}$ if and only if $f_0 \in \mathscr{F}$ and $(f_n, \dots, f_1) - \tau_{[n,1]} f_0$ belongs to $\mathscr{F}^{(n)}$. Write

$$(f_n, \dots, f_1) - \tau_{[n,1]} f_0 = \left(f_n - \frac{f_0^{(n)}}{n!}, \dots, f_1 - f_0' \right) = (g_{n-1}, \dots, g_0).$$

Then the induction hypothesis says that $(g_{n-1}, \dots, g_0) \in \mathscr{F}^{(n)}$ if and only if for each $0 \leq i \leq n - 1$ the following sum belongs to $\mathscr{F}$:

$$g_i + \sum_{0<k\leq i} \frac{(-1)^k}{k!} g_{i-k}^{(k)} = f_{i+1} - \frac{f_0^{(i+1)}}{(i+1)!} + \sum_{0<k\leq i} \frac{(-1)^k}{k!} \left(f_{i+1-k}^{(k)} - \frac{f_0^{(i+1-k+k)}}{(i+1)!} \right) = f_{i+1} + \sum_{0<k\leq i+1} \frac{(-1)^k}{k!} f_{i+1-k}^{(k)}$$

because

$$\frac{-1}{(i+1)!} + \sum_{0<k\leq i} \frac{(-1)^k}{k!} \frac{-1}{(i+1)!} = \frac{(-1)^{i+1}}{(i+1)!};$$

(see Equation 4.2). Probably it is not necessary to say anything more. □

Note that the Lemma implies, among other things, that

$$\mathscr{F}^{(n+1)} = \mathscr{F} \oplus_\Lambda \mathscr{F}^{(n)}, \quad \text{with} \quad \Lambda(f_{n-1}, \dots, f_1, f_0) = -\sum_{1\leq k\leq n} \frac{(-1)^k}{k!} f_{n-k}^{(k)} = -\sum_{0\leq k\leq n-1} \frac{(-1)^{n-k}}{(n-k)!} f_k^{(n-k)},$$

which is a linear map $\mathscr{F}^{(n)} \longrightarrow \mathscr{H}$, hence a (trivial) quasilinear map from $\mathscr{F}^{(n)}$ to $\mathscr{F}$. It follows that each $\mathscr{F}^{(n)}$ is isometric, but not equal, to the direct sum $\mathscr{F} \oplus \cdots \oplus \mathscr{F}$ — there are n summands.

We also have the following representation of $\mathscr{F}^{(n)}$ that will be very useful in Section 7:

Corollary 6.6. *With the same notations as before $F \in \mathscr{H}^n$ belongs to $\mathscr{F}^{(n)}$ if and only it has the form*

$$F = \left(\frac{f_0^{(n-1)}}{(n-1)!} + \frac{f_1^{(n-2)}}{(n-2)!} + \cdots + f_{n-1}, \dots, \frac{f_0''}{2!} + f_1' + f_2, f_0' + f_1, f_0 \right)$$

with $f_i \in \mathscr{F}$ for $0 \leq i < n$, in which case $\|F\|_{\mathscr{F}^{(n)}}$ is equivalent to $\sum_{0\leq i<n} \|f_i\|_{\mathscr{F}}$.

Let us then prove what has brought us here:

Proposition 6.7. *If $\mathscr{F}$ is an acceptable space of analytic functions on the disc, then so is $\mathscr{F}^{(n)}$ for every $n \geq 1$. Moreover:*

- *If $f \in \mathscr{F}$, then $\tau_{(n,0]}f \in \mathscr{F}^{(n)}$ and $\|\tau_{(n,0]}f\|_{\mathscr{F}^{(n)}} = \|f\|_{\mathscr{F}}$.*
- *The analytic family associated to $\mathscr{F}^{(n)}$ are the Rochberg spaces $(\mathscr{F}_z^{(n)})_{z\in\mathbb{D}}$, up to equivalence of norms.*

Proof. We first observe that each n-tuple $(g_n, \ldots, g_1)$ in $\prod_{i=1}^n \mathscr{H}(\mathbb{D}, \Sigma)$ can be seen as an analytic function from $\mathbb{D}$ to Σ^n just letting $(g_n, \ldots, g_1)(z) = (g_n(z), \ldots, g_1(z))$, where Σ^n can be equipped with the direct sum norm, so certainly $\mathscr{F}^{(n)}$ is a space of analytic functions.

The result is trivial when $n = 1$ and will be established by induction on n. So, let us assume it true for $1, \ldots, n$ and prove it for $n + 1$. To check completeness, just observe that $\mathscr{F}^{(n+1)}$ is a twisted sum of $\mathscr{F}$ by $\mathscr{F}^{(n)}$ and that those spaces are complete by the induction hypothesis. A classical 3-space result [10, Lemma 1.5.b] then asserts that a twisted sum of complete spaces is complete. In truth, all this is unnecessary since Corollary 6.6 tells us that each $\mathscr{F}^{(n)}$ is linearly homeomorphic (actually isometric) to the direct sum of n copies of $\mathscr{F}$.

In order to prove that the evaluations $\delta_z : \mathscr{F}^{(n+1)} \longrightarrow \Sigma^{n+1}$ are bounded we can assume that $\delta_z : \mathscr{F}^{(n)} \longrightarrow \Sigma^n$ are bounded. By Lemma 2.4, the successive derivatives $\delta_z^{(k)} : \mathscr{F} \longrightarrow \Sigma$ are all bounded. Pick $(f_n, \ldots, f_1, f_0) \in \mathscr{F}^{(n+1)}$ and consider the decomposition

$$(f_n, \ldots, f_1, f_0) = (f_n, \ldots, f_1, f_0) - \tau_{[n,0]}f_0 + \tau_{[n,0]}f_0$$

We have

$$\|(f_n, \ldots, f_0)\|_{\mathscr{F}^{(n+1)}} = \|(f_n, \ldots, f_1) - \tau_{[n,1]}f_0\|_{\mathscr{F}^{(n)}} + \|f_0\|_{\mathscr{F}}.$$

Also,

$$\begin{aligned}\|\delta_z(f_n, \ldots, f_0)\|_{\Sigma^{n+1}} &\leq \|\delta_z((f_n, \ldots, f_1) - \tau_{[n,1]}f_0)\|_{\Sigma^n} + \sum_{0\leq k\leq n} \left\| \frac{f_0^{(k)}(z)}{k!} \right\|_{\Sigma} \\ &\leq \left\|\delta_z : \mathscr{F}^{(n)} \longrightarrow \Sigma^n\right\| \|(f_n, \ldots, f_1) - \tau_{[n,1]}f_0\|_{\mathscr{F}^{(n)}} + \sum_{0\leq k\leq n} \left\| \frac{\delta^{(k)}}{k!} : \mathscr{F} \longrightarrow \Sigma \right\| \|f\|_{\mathscr{F}},\end{aligned}$$

which is enough.

Let us check that $\mathscr{F}^{(n+1)}$ is an A^∞-module under pointwise multiplication assuming that $\mathscr{F}^{(n)}$ is. As a preparation we consider the following general situation. Suppose we have a (topological) algebra A and that X and Y are topological left-modules over A. Let H be another A-module, not necessarily carrying a topology, that contains Y as a submodule. Finally, suppose $\Phi : X \longrightarrow H$ is quasilinear from X to Y; see Definition 3.1. It is very easy to see that the "coordinatewise" product $a(h, x) = (ah, ax)$ makes $Y \oplus_\Phi X$ into a topological A-module if and only if for every $a \in A$ and $x \in X$ one has $\Phi(ax) - a\Phi(x) \in Y$ and

$$\|\Phi(ax) - a\Phi(x)\| \longrightarrow 0 \qquad \text{as} \qquad (a, x) \longrightarrow 0 \text{ in } A \times X.$$

(Such quasilinear maps are called "centralizers", but that's is another story, the first episodes of which can be found in Kalton's memoir [22], and the stellar moments in [23].) As the space $\mathscr{F}^{(n+1)}$ is just $\mathscr{F}^{(n)} \oplus_\Phi \mathscr{F}$ when Φ is the quasilinear map (linear in fact) given by $\tau_{[n,1]} : \mathscr{F} \longrightarrow \mathscr{H}^n$ what we need to prove is that if $f \in \mathscr{F}$ and $a \in A^\infty$, then the difference $\tau_{[n,1]}(af) - a\tau_{[n,1]}(f)$ falls into $\mathscr{F}^{(n)}$ and

$$\|\tau_{[n,1]}(af) - a\tau_{[n,1]}(f)\|_{\mathscr{F}^{(n)}} \longrightarrow 0 \qquad \text{as} \qquad (a, f) \longrightarrow 0 \text{ in } A^\infty \times \mathscr{F}. \tag{6.3}$$

Note that if $f \in \mathscr{F}$, then, for each $k \geq 1$, the array $\tau_{[k,0]}(f)$ belongs to $\mathscr{F}^{(k+1)}$, with $\|\tau_{[k,0]}(f)\|_{\mathscr{F}^{(k+1)}} = \|f\|_{\mathscr{F}}$ and so every array of the form

$$\left(\frac{f^{(k)}}{k!}, \dots, f', f, 0, \dots, 0\right),$$

ending with ℓ zeroes, belongs to $\mathscr{F}^{(k+\ell+1)}$ and its norm there agrees with $\|f\|_{\mathscr{F}}$. Fix now $f \in \mathscr{F}, a \in A^\infty$ and let us compute the difference $\tau_{[n,1]}(af) - a\tau_{[n,1]}(f)$. Note, that, by the Leibniz formula

$$\frac{(af)^{(k)}}{k!} = \sum_{0\leq i\leq k} \frac{a^{(k-i)}}{(k-i)!}\frac{f^{(i)}}{i!},$$

so

$$\begin{aligned}\tau_{[n,1]}(af) &= \left(\frac{(af)^{(n)}}{n!}, \dots, (af)'\right)\\ &= \underbrace{a\left(\frac{f^{(n)}}{n!}, \dots, f'\right)}_{a\tau_{[n,1]}f} + a'\left(\frac{f^{(n-1)}}{(n-1)!}, \dots, f,\right) + \frac{a''}{2!}\left(\frac{f^{(n-2)}}{(n-2)!}, \dots, 0\right) + \cdots + \frac{a^{(n)}}{n!}(f, 0, \dots, 0)\end{aligned}$$

Hence

$$\tau_{[n,1]}(af) - a\tau_{[n,1]}(f) = a'\left(\frac{f^{(n-1)}}{(n-1)!}, \dots, f\right) + \frac{a''}{2!}\left(\frac{f^{(n-2)}}{(n-2)!}, \dots, 0\right) + \cdots + \frac{a^{(n)}}{n!}(f, 0, \dots, 0),$$

with each summand in $\mathscr{F}^{(n)}$, and

$$\|\tau_{[n,1]}(af) - a\tau_{[n,1]}(f)\|_{\mathscr{F}^{(n)}} \leq \sum_{1\leq k\leq n} \left\|\frac{a^{(k)}}{k!}\right\|_{L(\mathscr{F}^{(n-k)})} \|f\|_{\mathscr{F}}.$$

To complete the proof that $\mathscr{F}^{(n+1)}$ is acceptable let us assume that $(f_n, \dots, f_0) \in \mathscr{H}(\mathbb{D}, \Sigma^{n+1})$ and $\psi \in \mathrm{Aut}(\mathbb{D})$ are such that $\psi(f_n, \dots, f_0) = (\psi f_n, \dots, \psi f_0)$ falls into $\mathscr{F}^{(n+1)}$. We must check that $(f_n, \dots, f_1, f_0)$ belongs to $\mathscr{F}^{(n+1)}$ and that

$$\|(f_n, \dots, f_1, f_0)\|_{\mathscr{F}^{(n+1)}} \leq K[\psi, n+1]\|\psi(f_n, \dots, f_1, f_0)\|_{\mathscr{F}^{(n+1)}},$$

where $K[\psi, n+1]$ is a constant depending on ψ and the "dimension" n only. The hypothesis means that $\psi f_0 \in \mathscr{F}$ (hence $f_0 \in \mathscr{F}$) and $\psi(f_n, \dots, f_1) - \tau_{[n,1]}(\psi f_0) \in \mathscr{F}^{(n)}$. On the other hand, since $\psi \in A^\infty$ (see the Appendix), we know from the previous step that the difference $\tau_{[n,1]}(\psi f_0) - \psi\tau_{[n,1]}(f_0)$ belongs to $\mathscr{F}^{(n)}$. Thus,

$$\psi(f_n, \dots, f_1) - \psi\tau_{[n,1]}(f_0) \in \mathscr{F}^{(n)}$$

and the induction step yields $(f_n, \dots, f_1) - \tau_{[n,1]}(f_0) \in \mathscr{F}^{(n)}$, hence $(f_n, \dots, f_1, f_0) \in \mathscr{F}^{(n+1)}$.

As for the norm, one has

$$\begin{aligned}\|(f_n, \dots, f_1, f_0)\|_{\mathscr{F}^{(n+1)}} &= \|(f_n, \dots, f_1) - \tau_{[n,1]}(f_0)\|_{\mathscr{F}^{(n)}} + \|f_0\|_{\mathscr{F}}\\ &\leq K[\psi, n]\|\psi(f_n, \dots, f_1) - \psi\tau_{[n,1]}(f_0)\|_{\mathscr{F}^{(n)}} + K[\psi, 1]\|\psi f_0\|_{\mathscr{F}}\\ &\leq K[\psi, n]\left(\|\psi(f_n, \dots, f_1) - \tau_{[n,1]}(\psi f_0)\|_{\mathscr{F}^{(n)}} + \|\tau_{[n,1]}(\psi f_0) - \psi\tau_{[n,1]}(f_0)\|_{\mathscr{F}^{(n)}}\right) + K[\psi, 1]\|\psi f_0\|_{\mathscr{F}}\\ &\leq \max(K[\psi, n], K[\psi, 1])\|\psi(f_n, \dots, f_1, f_0)\|_{\mathscr{F}^{(n+1)}} + K[\psi, n]\sum_{1\leq k\leq n}\left\|\frac{\psi^{(k)}}{k!}\right\|_{L(\mathscr{F}^{(n-k)})}\|f_0\|_{\mathscr{F}},\end{aligned}$$

which is enough as it implies that

$$K[\psi, n+1] \le \max\Big(K[\psi, n], K[\psi, 1]\Big) + K[\psi, n]K[\psi, 1] \sum_{1\le k\le n} \left\| \frac{\psi^{(k)}}{k!} \right\|_{L(\mathscr{F}^{(n-k)})}.$$

Finally, we prove the "moreover" part. For each $k \ge 1$ let $(\mathscr{F}^{(k)})_z$ denote the analytic family induced by $\mathscr{F}^{(k)}$, while we keep the notation $\mathscr{F}_z^{(k)}$ for the k-th Rochberg space induced by $\mathscr{F}$ at z. In particular:

$$(\mathscr{F}^{(n+1)})_z = \{x \in \Sigma^{n+1} : x = F(z) \text{ for some } F \in \mathscr{F}^{(n+1)}\};$$
$$\mathscr{F}_z^{(n+1)} = \{x \in \Sigma^{n+1} : x = \tau_{[n,0]} f(z) \text{ for some } f \in \mathscr{F}\}.$$

Now, if $f \in \mathscr{F}$, then the array $F = \tau_{[n,0]}(f)$ belongs to $\mathscr{F}^{(n+1)}$ by the very definition, and evaluating at z one obtains the Taylor coefficients of f. Besides, $\|\tau_{[n,0]}(f)\|_{\mathscr{F}^{(n+1)}} = \|f\|_{\mathscr{F}}$, hence $(\mathscr{F}^{(n+1)})_z$ contains $\mathscr{F}_z^{(n+1)}$ and the inclusion is contractive. To establish the other containment, one has to check that if $(f_n, \dots, f_0)$ belongs to $\mathscr{F}^{(n+1)}$ then, for each $z \in \mathbb{D}$, there is $f \in \mathscr{F}$ such that

$$f_k(z) = \frac{f^{(k)}(z)}{k!} \qquad (0 \le k \le n)$$

with $\|f\|_{\mathscr{F}} \le M\|(f_n, \dots, f_0)\|_{\mathscr{F}^{(n+1)}}$, where $M = M[z, n+1]$ depends only on z and n, but not on the array. So, fix $z \in \mathbb{D}$ and pick $(f_n, \dots, f_0)$ in $\mathscr{F}^{(n+1)}$. Then since the array $(f_n, \dots, f_1) - \tau_{[n,1]} f_0$ belongs to $\mathscr{F}^{(n)}$ we can assume by the induction hypothesis that there is $g \in \mathscr{F}$ such that

$$g(z) = f_1(z) - f_0'(z), \dots, \frac{g^{(n-1)}(z)}{(n-1)!} = f_n(z) - \frac{f_0^{(n)}(z)}{n!}, \tag{6.4}$$

with $\|g\|_{\mathscr{F}} \le M[z, n] \left\|(f_n, \dots, f_1) - \tau_{[n,1]} f_0\right\|_{\mathscr{F}^{(n)}}$. Take $\psi \in \mathrm{Aut}(\mathbb{D})$ vanishing at z and use [5, Lemma 1] to get a polynomial P of degree at most n so that if $a = P(\psi)$, then $a^{(k)}(z) = \delta_{k1}$ (Kronecker delta) for $0 \le k \le n$. Obviously, $a \in A^\infty$ and so $f = ag + f_0 \in \mathscr{F}$. We have

$$\begin{aligned}\|f\|_{\mathscr{F}} &\le \|a\|_{L(\mathscr{F})}\|g\|_{\mathscr{F}} + \|f_0\|_{\mathscr{F}} \\ &\le \|a\|_{L(\mathscr{F})} M[z, n] \left\|(f_n, \dots, f_1) - \tau_{[n,1]} f_0\right\|_{\mathscr{F}^{(n)}} + \|f_0\|_{\mathscr{F}} \\ &\le \max\left(\|a\|_{L(\mathscr{F})} M[z, n], 1\right) \|(f_n, \dots, f_0)\|_{\mathscr{F}^{(n+1)}}.\end{aligned}$$

Regarding the Taylor coefficients of f, by Leibniz rule and (6.4),

$$\frac{f^{(k)}(z)}{k!} = \frac{f_0^{(k)}(z)}{k!} + \sum_{0\le i\le k} \frac{a^{(i)}(z)}{i!} \frac{g^{(k-i)}(z)}{(k-i)!} = \frac{f_0^{(k)}(z)}{k!} + \frac{g^{(k-1)}(z)}{(k-1)!} = f_k(z). \quad \square$$

6.4. **General domains.** We transplant our results from the disc to general domains. The main obstruction to proceed as we did in Proposition 6.7 is that the grafted algebras $A_{\mathbb{U}}^\infty$ are not closed under differentiation, even if $\mathbb{U}$ is a strip (see the Appendix). Therefore, most of the computations done along the proof just do not make any sense for general domains. The idea is then to use a conformal map between $\mathbb{U}$ and $\mathbb{D}$ to transfer the acceptable space $\mathscr{F}$ from $\mathbb{U}$ to $\mathbb{D}$, then use Proposition 6.7 and then move back to $\mathbb{U}$. This involves the most basic operations in calculus: the Chain and Leibniz rule. The paper [29] contains many deeper "translations" to vector valued analytic functions of much deeper facts about complex analytic functions.

6.4.1. *Chain rule.* Let $\mathscr{F}$ be an acceptable space on $\mathbb{U}$ and suppose $\psi : \mathbb{V} \longrightarrow \mathbb{U}$ is a conformal equivalence. Then we can consider the space

$$\mathscr{G} = \psi^*[\mathscr{F}] = \{g \in \mathscr{H}(\mathbb{V}, \Sigma) : g = f \circ \psi, f \in \mathscr{F}\},$$

with norm $\|g\|_{\mathscr{G}} = \|f\|_{\mathscr{F}}$. It is clear that $\mathscr{G}$ is acceptable, or admissible if $\mathscr{F}$ is. In some sense, $\mathscr{G}$ and $\mathscr{F}$ are "equivalent" objects. This is indeed the case for the "degree zero" theory as shown by the fact that, for each $z \in \mathbb{V}$, one has $\mathscr{G}_z = \mathscr{F}_{\psi(z)}$, with identical norms. We omit the obvious proof.

What about the corresponding Rochberg spaces? They are still isometric but, in general, different. To see this, fix $z \in \mathbb{V}$ and put $u = \psi(z)$. Take $(x_1, x_0) \in \mathscr{F}_u^{(2)}$ and pick $f \in \mathscr{F}$ so that $x_1 = f'(u), x_0 = f(u)$. Then take $g = f \circ \psi$ and evaluate $\tau_{[1,0]}g$ at z:

$$(g'(z), g(z)) = (f'(u)\psi'(z), f(u)) = (\psi'(z)x_1, x_0).$$

This shows at once:

- The map $(x_1, x_0) \longmapsto (\psi'(z)x_1, x_0)$ is a surjective isometry between $\mathscr{F}_u^{(2)}$ and $\mathscr{G}_z^{(2)}$.
- If $\psi'(z) \neq 1$, then $\mathscr{F}_u^{(2)} = \mathscr{G}_z^{(2)}$ as subspaces of Σ^2 if and only if $\mathscr{F}_u^{(2)} = \mathscr{F}_u \times \mathscr{F}_u$.
- It $\lambda = \psi'(z)$, then we have a commutative diagram (recall that $\mathscr{F}_u$ and $\mathscr{G}_z$ are the same space)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{F}_u & \longrightarrow & \mathscr{F}_u^{(2)} & \longrightarrow & \mathscr{F}_u & \longrightarrow & 0 \\ & & \lambda\big\downarrow & & \lambda\times\mathbf{1}\big\downarrow & & \big\| & & \\ 0 & \longrightarrow & \mathscr{G}_z & \longrightarrow & \mathscr{G}_z^{(2)} & \longrightarrow & \mathscr{G}_z & \longrightarrow & 0 \end{array}$$

  in which the middle arrow is an isometry.

In general we can describe nice isometries between $\mathscr{F}_u^{(n)}$ and $\mathscr{G}_z^{(n)}$ as follows. Take $x \in \mathscr{F}_u^{(n)}$ and let $f \in \mathscr{F}$ be a representative, that is, $x = \tau_{(n,0]}f(u)$. Set $g = f \circ \psi$ and put $y = \tau_{(n,0]}g(z)$. It is clear that y depends only on x (if f has a zero of order k at u, then g has a zero of order k at z, and vice versa) and that this correspondence defines a surjective isometry between $\mathscr{F}_u^{(n)}$ and $\mathscr{G}_z^{(n)}$ that we may denote by $L[n, u]$ thus emphasizing the fact that it depends on the base point. To understand the dependence between the input $x = (x_{n-1}, \dots, x_0)$ and the output $y = (y_{n-1}, \dots, y_0)$ we can invoke Faà di Bruno's formula for higher derivatives of composite functions (see [20] for an exposition). Write

$$f(v) = \sum_{m\geq 0} x_m(v-u)^m \qquad \text{and} \qquad \psi(w) = \sum_{m\geq 0} z_m(w-z)^m$$

with positive radii of convergence. Then

$$g(w) = f(\psi(w)) = \sum_{m\geq 0} y_m(w-z)^m, \qquad \text{with} \qquad y_m = \sum_{(b_1,\dots,b_m)} \frac{z_1^{b_1}}{b_1!} \cdots \frac{z_m^{b_m}}{b_m!}\, k!\, x_k,$$

where the sum is taken over all different solutions $(b_1, \dots, b_m)$ of the equation $b_1 + 2b_2 + \cdots + mb_m = m$ in which each b_i is a nonnegative integer and $k = b_1 + b_2 + \cdots + b_m$; in particular $k \leq m$. Hence, each $L[n, u]$ is implemented by an upper triangular matrix with complex coefficients that we will denote $\mathbf{FdB}[n, u, \psi]$, emphasizing the dependence on n, u and ψ; see Section 8.3.

Take $n, k \geq 1$ and let $\pi_{n+k,n} : \Sigma^{n+k} \longrightarrow \Sigma^n$ denote the projection onto the last n coordinates. Clearly, $L[n, u] \circ \pi_{n+k,n} = \pi_{n+k,n} \circ L[n + k, u]$, so $L[n + k, u]$ maps the kernel of $\pi_{n+k,n} : \mathscr{F}_u^{(n+k)} \longrightarrow \mathscr{F}_u^{(n)}$ onto that of $\pi_{n+k,n} : \mathscr{G}_u^{(n+k)} \longrightarrow \mathscr{G}_u^{(n)}$ and we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{F}_u^{(k)} & \longrightarrow & \mathscr{F}_u^{(n+k)} & \longrightarrow & \mathscr{F}_u^{(n)} & \longrightarrow & 0 \\ & & {\scriptstyle I}\big\downarrow & & {\scriptstyle L[n+k,u]}\big\downarrow & & \big\downarrow{\scriptstyle L[n,u]} & & \\ 0 & \longrightarrow & \mathscr{G}_z^{(k)} & \longrightarrow & \mathscr{G}_z^{(n+k)} & \longrightarrow & \mathscr{G}_z^{(n)} & \longrightarrow & 0 \end{array} \tag{6.5}$$

in which I is an isomorphism, depending on n, k and u, in general different from $L[k, u]$.

Moral: If you are interested in twisted sums, Banach space properties of the derived spaces and the like you can change variables without causing any harm to your conclusions. If you are rather interested in interpolation spaces, interpolation of operators and the like, you should be careful.

6.4.2. *Leibniz rule.* The preceding considerations suggest the following formal procedure to correct the distorsion introduced by a change of variable. Let $\mathscr{F}$ be an admissible/acceptable space of analytic functions from $\mathbb{U}$ to Σ and suppose $L : \mathbb{U} \longrightarrow \operatorname{Aut}(\Sigma)$ is analytic when $\operatorname{Aut}(\Sigma)$ carries the restriction of the norm topology of $L(\Sigma)$. We can define a weighted version of $\mathscr{F}$, denoted $L_*[\mathscr{F}]$ with a slight abuse of notation, taking those functions $g : \mathbb{U} \longrightarrow \Sigma$ of the form $g(z) = L(z)(f(z))$, for some $f : \mathbb{U} \longrightarrow \Sigma$, with norm $\|g\|_{L_*[\mathscr{F}]} = \|f\|_{\mathscr{F}}$. It is clear that $L_*[\mathscr{F}]$ is admissible/acceptable if and only if $\mathscr{F}$ is. Moreover, for each $z \in \mathbb{U}$, one has $L_*[\mathscr{F}]_z = L(z)[\mathscr{F}_z]$ and that $L(z) : \mathscr{F}_z \longrightarrow L_*[\mathscr{F}]_z$ is a surjective isometry.

The connection between the Rochberg spaces of $\mathscr{F}$ and those of $L_*[\mathscr{F}]$ is as follows. Suppose $(x_{n-1}, \dots, x_0) \in \Sigma^n$ belongs to $\mathscr{F}_z^{(n)}$ and that it agrees with the evaluation of $\tau_{[n-1,0]}(f)$ at z. Then $g(\zeta) = L(\zeta)(f(\zeta))$ belongs to $L_*[\mathscr{F}]$ and since by Leibniz's rule

$$\frac{g^{(k)}(z)}{k!} = \sum_{0 \le i \le k} \frac{L^{(k-i)}(z)}{(k-i)!} \left(\frac{f^{(i)}(z)}{i!} \right)$$

we see that the isometry between $\mathscr{F}_z^{(n)}$ and $L_*[\mathscr{F}]_z^{(n)}$ is implemented by the following operator valued matrix evaluated at z

$$\begin{pmatrix} \frac{L^{(n-1)}}{(n-1)!} & \frac{L^{(n-2)}}{(n-2)!} & \dots & L' & L \\ 0 & \frac{L^{(n-2)}}{(n-2)!} & \dots & L' & L \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & L' & L \\ 0 & 0 & \dots & 0 & L \end{pmatrix}$$

We are ready to state the conclusion of all this:

Theorem 6.8. *Let $\mathscr{F}$ be an acceptable space of analytic functions $\mathbb{U} \longrightarrow \Sigma$. For every $n \ge 2$ there exists an acceptable space $\mathscr{T}$ of analytic functions $\mathbb{U} \longrightarrow \Sigma^n$ with the following properties:*

- *For every $f \in \mathscr{F}$, the array $\tau_{(n,0]}f : \mathbb{U} \longrightarrow \Sigma^n$ belongs to $\mathscr{T}$, and $\|\tau_{(n,0]}f\|_{\mathscr{T}} = \|f\|_{\mathscr{F}}$.*
- *For every $u \in \mathbb{U}$ one has $\mathscr{T}_u = \mathscr{F}_u^{(n)}$, with equivalent norms.*

Proof. Fix a conformal map $\psi : \mathbb{D} \longrightarrow \mathbb{U}$ and let $\mathscr{G} = \psi^*[\mathscr{F}]$. Then $\mathscr{G}$ is acceptable on $\mathbb{D}$ and $\mathscr{F}_u = \mathscr{G}_z$, where $u = \psi(z)$. If $\mathscr{G}^{(n)}$ is the space provided by Proposition 6.7, we have:

- $\mathscr{G}^{(n)}$ is an acceptable space of Σ^n-valued functions on the disc.
- The analytic family induced to $\mathscr{G}^{(n)}$ is $\mathscr{G}_z^{(n)}$, up to equivalence of norms.
- If $g \in \mathscr{G}$, then $\tau_{(n,0]}g$ belongs to $\mathscr{G}^{(n)}$, and $\|\tau_{(n,0]}g\|_{\mathscr{G}^{(n)}} = \|g\|_{\mathscr{G}}$.

Moreover, we know from Section 6.4.1 that there is an analytic mapping $L(n,\cdot):\mathbb{U}\longrightarrow M[n]$, the space of $n\times n$ matrices with complex coefficients, such that, if $u=\psi(z), f\in\mathscr{F}, g=f\circ\psi$, then

$$\tau_{(n,0]}g(z)=L(n,u)\big((\tau_{(n,0]}f)(u)\big).$$

Each $L(n,u)$ is upper triangular and invertible and restricts to a surjective isometry between $\mathscr{F}_u^{(n)}$ and $\mathscr{G}_z^{(n)}$ and so to an isomorphism from $\mathscr{F}_u^{(n)}$ to $(\mathscr{G}^{(n)})_z$. Now, we continue with this n fixed, and define $M:\mathbb{D}\longrightarrow M[n]$ by $M(z)=L(n,\psi(z))^{-1}$. Consider the space

$$M_*[\mathscr{G}^{(n)}]=\left\{H\in\mathscr{H}(\mathbb{D},\Sigma^n):H(w)=M(w)(G(w)),\ \text{with } G\in\mathscr{G}^{(n)}\right\}.$$

It should be obvious by now that $M_*[\mathscr{G}^{(n)}]$ is an acceptable space on the disc and also that $(M_*[\mathscr{G}^{(n)}])_z=\mathscr{F}_u^{(n)}$, with equivalent norms, where $u=\psi(z)$. Finally, set $\mathscr{T}=(\psi^{-1})[M_*[\mathscr{G}^{(n)}]]$ and check the details. □

There is a puzzling fact in that one is much less interested in which are the spaces $\mathscr{T}$ appearing in Theorem 6.8 than in their mere existence. Indeed, $\mathscr{T}$ has been constructed to provide a framework that legitimates the manipulations we will perform next. On the other hand, the formalism developed in this paper for acceptable spaces is rather satisfactory in the sense that produces, under minimal hypotheses, both the Rochberg spaces and the process to derive them. A reader interested in interpolation theory could miss some concrete applications beyond Section 6. The main obstacle to derive "classical" interpolation results from the material in Sections 6.3 and 6.4 is that, while admissible interpolation families lead to admissible spaces of analytic functions in the way explained in Section 2.2, we do not know how to travel the way back, if there is a way back. Precisely, assume that $\mathscr{F}$ is an admissible space on the disc and let us fix $0<r<1$. Under which conditions one can guarantee that the spaces $(\mathscr{F}_z)_{|z|=r}$ form an interpolation family (in the sense of Definition 2.5) so that a new admissible space $\mathscr{X}$ can be eventually formed? And, if so, do the new interpolation spaces $(\mathscr{X}_z)_{|z|<r}$ agree with the old ones $\mathscr{F}_z$?

7. Derivation of Rochberg families

Let $\mathscr{F}$ be an acceptable space on $\mathbb{U}$. Fix $m\geq 2$ and let $\mathscr{T}$ be the space provided by Theorem 6.8 so that $\mathscr{T}_z^{(1)}=\mathscr{F}_z^{(m)}$; the fact that $\mathscr{T}$ depends on the choice of a conformal map does not affect the ensuing considerations. Since $\mathscr{T}$ is acceptable, given any integer $n\geq 2$ one can construct the corresponding Rochberg spaces $\mathscr{T}_z^{(n)}$ and the associated exact sequences (3.6) they naturally form. This section makes the first steps in the study of these objects. While our knowledge on this issue is very limited, the general impression is that one arrives to certain degenerate versions of the Rochberg spaces generated by the original $\mathscr{F}$.

Let us agree on the following notations. For fixed $m\geq 1$, if $\mathscr{T}$ is the space provided by Theorem 6.8 so that $\mathscr{T}_z=\mathscr{F}_z^{(m)}$ for all $z\in\mathbb{U}$. Let us fix $z\in\mathbb{U}$ for the remainder of the section, write $F[m,n]=\mathscr{T}_z^{(n)}$ and rename the exact sequences entwinning the successive Rochberg spaces of $\mathscr{T}$ as

$$0\longrightarrow F[m,n]\xrightarrow{\iota^m_{n,n+k}}F[m,n+k]\xrightarrow{\pi^m_{n+k,k}}F[m,k]\longrightarrow 0 \tag{7.1}$$

We may describe the elements of $F[m,n]$ by means of $m\times n$-matrices with entries in the ambient space Σ as follows. Each function in $\mathscr{T}$ can be written as $F=(f_{m-1},\dots,f_0)$ where $f_j:\mathbb{U}\longrightarrow\Sigma$ are certain analytic functions. Thus, a typical element of $F[m,n]$ arises by evaluation of the following array of functions

$$\begin{pmatrix} \frac{F^{(n-1)}}{(n-1)!} \\ \frac{F^{(n-2)}}{(n-2)!} \\ \vdots \\ F' \\ F \end{pmatrix} = \begin{pmatrix} \frac{f_{m-1}^{(n-1)}}{(n-1)!} & \frac{f_{m-2}^{(n-1)}}{(n-1)!} & \cdots & \frac{f_{1}^{(n-1)}}{(n-1)!} & \frac{f_{0}^{(n-1)}}{(n-1)!} \\ \frac{f_{m-1}^{(n-2)}}{(n-2)!} & \frac{f_{m-2}^{(n-2)}}{(n-2)!} & \cdots & \frac{f_{1}^{(n-2)}}{(n-2)!} & \frac{f_{0}^{(n-2)}}{(n-2)!} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f'_{m-1} & f'_{m-2} & \cdots & f'_1 & f'_0 \\ f_{m-1} & f_{m-2} & \cdots & f_1 & f_0 \end{pmatrix}$$

at z. There is a quite natural operator $E_{m,n} : \mathscr{F}_z^{(m+n-1)} \longrightarrow F[m,n]$. To see which one is, pick $x = (x_{m+n-2}, \dots, x_0)$ in $\mathscr{F}_z^{m+n-1}$. Let $f \in \mathscr{F}$ be an extremal for x so that $x = \tau_{(m+n-1,0]}f(z)$ and put $F(\cdot) = \tau_{(m,0]}f(\cdot)$. Then $F \in \mathscr{T}$ and (the transpose of) $\tau_{(n,0]}F$ is

$$\begin{pmatrix} \frac{F^{(n-1)}}{(n-1)!} \\ \frac{F^{(n-2)}}{(n-2)!} \\ \vdots \\ F' \\ F \end{pmatrix} = \begin{pmatrix} \frac{f^{(m-1+n-1)}}{(m-1)!(n-1)!} & \frac{f^{(m-2+n-1)}}{(m-2)!(n-1)!} & \cdots & \frac{f^{(n)}}{(n-1)!} & \frac{f^{(n-1)}}{(n-1)!} \\ \frac{f^{(m-1+n-2)}}{(m-1)!(n-2)!} & \frac{f^{(m-2+n-2)}}{(m-2)!(n-2)!} & \cdots & \frac{f^{(n-1)}}{(n-2)!} & \frac{f^{(n-2)}}{(n-2)!} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{(m-1)!}f^{(m)} & \frac{1}{(m-2)!}f^{(m-1)} & \cdots & f'' & f' \\ \frac{1}{(m-1)!}f^{(m-1)} & \frac{1}{(m-2)!}f^{(m-2)} & \cdots & f' & f \end{pmatrix}$$

Evaluating at z we obtain

$$E_{m,n}(x) = \begin{pmatrix} \frac{(m-1+n-1)!}{(m-1)!(n-1)!}x_{m-1+n-1} & \frac{(m-2+n-1)!}{(m-2)!(n-1)!}x_{m-2+n-1} & \cdots & nx_n & x_{n-1} \\ \frac{(m-1+n-2)!}{(m-1)!(n-2)!}x_{m-1+n-2} & \frac{(m-2+n-2)!}{(m-2)!(n-2)!}x_{m-2+n-2} & \cdots & (n-1)x_{n-1} & x_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ mx_m & (m-1)x_{m-1} & \cdots & 2x_2 & x_1 \\ x_{m-1} & x_{m-2} & \cdots & x_1 & x_0 \end{pmatrix} = \left(\frac{(i+j)!}{i!j!}x_{i+j}\right)_{\substack{0\le i<n \\ 0\le j<m}}$$

It is clear that each $E_{m,n}$ is injective and continuous. We shall see very soon that $E_{m,n}$ is an embedding with complemented range if m or n is 2. To this end we need the following remark that implicitly concerns the pushout construction (The reader who is curious about this stuff can see it explained in [4, §2.6]). We apologize for the tendentious notation.

Lemma 7.1. *Let Z be a Banach space and let K and Y be closed subspaces of Z, with $K \subset Y$. Assume one has another Banach space* PO *and a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\text{inclusion}} & Z & \xrightarrow{\text{quotient}} & Z/K & \longrightarrow & 0 \\ & & {\scriptstyle\text{inclusion}}\downarrow & & \downarrow{\scriptstyle E} & & \| & & \\ 0 & \longrightarrow & Y & \xrightarrow{J} & \mathrm{PO} & \xrightarrow{Q} & Z/K & \longrightarrow & 0 \end{array}$$

with exact rows. Then E is an embedding with complemented range and $\mathrm{PO}/E[Z]$ *is isomorphic to Y/K. In particular* PO *is isomorphic to $Y/K \oplus Z$.*

Proof. The "three-lemma" (see the version in [4, Lemma 2.1.8]) tell us that E is an embedding and after a short reflection on the meaning of the operator Q one realizes that $\mathrm{PO} = J[Y] + E[Z]$, so that $(y,z) \longmapsto J(y) + E(z)$ is open from $Y \oplus Z$ onto PO. Define $U : Y/K \oplus Z \longrightarrow \mathrm{PO}$ letting $U(y+K,z) = J(y) + E(z-y)$, which is an operator whose inverse can be obtained as follows: given $x \in \mathrm{PO}$ take $y \in Y$ and $z \in Z$ such that $x = J(y) + E(z)$ and set $V(x) = (y+K, z+y)$. Check, check, check. □

The copies of Z and Y/K inside PO that arise by restricting U to each "factor" are obvious: the restriction of U to Z is just E; as for Y/K one has

$$U(y+K,0)=J(y)-E(y) \tag{7.2}$$

which depends only on the class of y in Y/K since J and E agree on K.

Proposition 7.2. *For each $m\geq 1$ and $z\in\mathbb{U}$ the operator $E_{m,2}:\mathscr{F}_z^{(m+1)}\longrightarrow F[m,2]$ is an embedding with complemented range and the quotient of $F[m,2]$ by $E_{m,2}[\mathscr{F}_z^{(m+1)}]$ is isomorphic to $\mathscr{F}_z^{(m-1)}$.*

Proof. We consider $\mathscr{F}_z$ and $\mathscr{F}_z^{(m)}$ as subspaces of $\mathscr{F}_z^{(m+1)}$ and check that the following diagram is commutative

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{F}_z & \xrightarrow{\imath_{1,m+1}} & \mathscr{F}_z^{(m+1)} & \longrightarrow & \mathscr{F}_z^{(m)} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \imath_{1,m}} & & \downarrow{\scriptstyle E_{m,2}} & & \| & & \\ 0 & \longrightarrow & \mathscr{F}_z^{(m)} & \xrightarrow{m\,\imath_{1,2}^{m}} & F[m,2] & \longrightarrow & \mathscr{F}_z^{(m)} & \longrightarrow & 0 \end{array} \tag{7.3}$$

where we have identified $\mathscr{F}_z^{(m)}$ with $F[m,1]$ in the obvious way. Given $x=(x_m,x_{m-1},\dots,x_0)$ in $\mathscr{F}_z^{(m+1)}$ one has

$$E_{m,2}(x_m,x_{m-1},\dots,x_0)=\begin{pmatrix} mx_m & (m-1)x_{m-1} & \dots & 2x_2 & x_1 \\ x_{m-1} & x_{m-2} & \dots & x_1 & x_0 \end{pmatrix} \tag{7.4}$$

The left square is commutative since for $x\in\mathscr{F}_z$ the two possible compositions lead to

$$\begin{pmatrix} mx & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

The right square is commutative as well: given $(x_m,x_{m-1},\dots,x_0)$ in $\mathscr{F}_z^{(m+1)}$ one has

$$\pi^2_{2,1}E_{m,2}(x_m,x_{m-1},\dots,x_0)=\pi^2_{2,1}\begin{pmatrix} mx_m & (m-1)x_{m-1} & \dots & 2x_2 & x_1 \\ x_{m-1} & x_{m-2} & \dots & x_1 & x_0 \end{pmatrix}=(x_{m-1},\dots,x_0).$$

Applying the preceding lemma concludes the proof. □

The remark after the lemma shows where the copies of $\mathscr{F}_z^{(m+1)}$ and $\mathscr{F}_z^{(m-1)}$ are located in $F[m,2]$. The first one is given by the action of $E_{m,2}$, described by (7.4). The position of the complementary copy of $\mathscr{F}_z^{(m-1)}$ is defined by (7.2): if $(y_{m-2},\dots,y_0)\in\mathscr{F}_z^{(m-1)}$ and $\tilde{y}=(y_{m-1},y_{m-2},\dots,y_0)$ is a "lifting" in $\mathscr{F}_z^{(m)}$ we have

$$U(\tilde{y}+\mathscr{F}_z,0)=m\imath^m_{1,2}\tilde{y}-E_{m,2}(\underbrace{y_{m-1},y_{m-2},\dots,y_0,0}_{\imath_{m,m+1}(\tilde{y})})=\begin{pmatrix} 0 & y_{m-2} & 2y_{m-3} & \dots & (m-2)y_1 & (m-1)y_0 \\ -y_{m-2} & -y_{m-3} & -y_{m-4} & \dots & -y_0 & 0 \end{pmatrix}$$

Reversing the parameters leads to similar conclusions:

Proposition 7.3. *For each $m\geq 1$ and $z\in\mathbb{U}$ the operator $E_{2,m}:\mathscr{F}_z^{(m+1)}\longrightarrow F[2,m]$ is an embedding with complemented range and the quotient of $F[2,m]$ by $E_{2,m}[\mathscr{F}_z^{(m+1)}]$ is isomorphic to $\mathscr{F}_z^{(m-1)}$.*

Proof. We write the proof when $\mathbb{U} = \mathbb{D}$ with base point at the centre of the disc. The general case follows suit. Let us check that $F[2,m]$ fits into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{F}_0 & \longrightarrow & \mathscr{F}_0^{(m+1)} & \longrightarrow & \mathscr{F}_0^{(m)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow E_{2,m} & & \| & & \\ 0 & \longrightarrow & \mathscr{F}_0^{(m)} & \xrightarrow{mJ} & \mathscr{F}[m,2] & \xrightarrow{Q} & \mathscr{F}_0^{(m)} & \longrightarrow & 0 \end{array} \tag{7.5}$$

with exact rows. Recall what we agreed on unlabelled arrows. The other operators are defined as follows:

$$E_{2,m}(x_m,\dots,x_0) = \begin{pmatrix} mx_m & x_{m-1} \\ (m-1)x_{m-1} & x_{m-2} \\ \dots & \dots \\ 2x_2 & x_1 \\ x_1 & x_0 \end{pmatrix};\ J(y_{m-1},\dots,y_0) = \begin{pmatrix} y_{m-1} & 0 \\ y_{m-2} & 0 \\ \dots & \dots \\ y_1 & 0 \\ y_0 & 0 \end{pmatrix};\ Q\begin{pmatrix} u_{m-1} & v_{m-1} \\ u_{m-2} & v_{m-2} \\ \dots & \dots \\ u_1 & v_1 \\ u_0 & v_0 \end{pmatrix} = (v_{m-1},\dots,v_0)$$

While it is clear that the diagram commutes (when one replaces each space by its containing Σ^k) the continuity of J and Q is not completely obvious.

But an analytic function $F : \mathbb{D} \longrightarrow \Sigma^2$ belongs to $\mathscr{F}^{(2)}$ if and only if there are $f_0, f_1 \in \mathscr{F}$ such that $F = (f_0' + f_1, f_0)$ in which case $\|F\|_{\mathscr{F}^{(2)}} = \|f_0\|_{\mathscr{F}} + \|f_1\|_{\mathscr{F}}$; take $n = 2$ in Corollary 6.6.

This implies that if f_0, f_1 are in $\mathscr{F}$ and $f_0 = \sum_{k\geq 0} a_k z^k$ and $f_1 = \sum_{k\geq 0} b_k z^k$ are their respective Taylor expansions, then

$$M = \begin{pmatrix} ma_m + b_{m-1} & a_{m-1} \\ (m-1)a_{m-1} + b_{m-2} & a_{m-2} \\ \dots & \dots \\ 2a_2 + b_1 & a_1 \\ a_1 + b_0 & a_0 \end{pmatrix} \text{ belongs to } F[2,m], \text{ with } \|M\|_{F[2,m]} \leq \|f_0\|_{\mathscr{F}} + \|f_1\|_{\mathscr{F}}. \tag{7.6}$$

and also that all points of $F[2,m]$ have that form. Hence J is bounded (actually contractive) from $\mathscr{F}_0^{(m)}$ to $F[2,m]$: given $x \in \mathscr{F}_0^{(m)}$ take an extremal $f \in \mathscr{F}$ with $x = \tau_{(m,0]}f(0)$, set $F = (f', 0)$ (that is, $f_0 = 0, f_1 = f$) and evaluate F at the origin. Since all elements of $F[2,m]$ can be written as in (7.6) we see that their right columns are in $\mathscr{F}_0^{(m)}$ and that Q is onto, with $\|Q : F[2,m] \longrightarrow \mathscr{F}_0^{(m)}\| \leq 1$.

Clearly J is injective. It remains to check that $\ker Q$ agrees with the image of J. One containment is trivial since $QJ = 0$. As for the other assume $M \in F[2,m]$ is such that $Q(M) = 0$. If we write M as in (7.6), with $f_0 = \sum_{k\geq 0} a_k z^k$ and $f_1 = \sum_{k\geq 0} b_k z^k$ in $\mathscr{F}$ we have that $a_k = 0$ for $0 \leq k < m$, so that

$$M = \begin{pmatrix} ma_m + b_{m-1} & 0 \\ b_{m-2} & 0 \\ \dots & \dots \\ b_1 & 0 \\ b_0 & 0 \end{pmatrix} \quad \text{with} \quad \begin{cases} f_0 = a_m z^m + a_{m+1} z^{m+1} + \cdots \\ f_1 = b_0 + b_1 z + b_2 z^2 + \cdots \end{cases}$$

Since $f_0(0) = 0$ we have that $f(z) = f_1(z) + mf_0(z)/z$ defines a function in $\mathscr{F}$; letting $y = \tau_{(m,0]}f(0)$ it should be obvious that $y \in \mathscr{F}_0^{(m)}$ is such that $J(y) = M$. The proof concludes using Diagram (7.5) and the preceding lemma. □

The just proved proposition describes, in particular, the sucessive Rochberg (derived) spaces of the "analytic family" of the Kalton–Peck spaces; see Section 6.1. It turns out that the spaces $F[m,2]$ and $F[2,m]$ are isomorphic since they are isomorphic to $\mathscr{F}_z^{(m+1)} \oplus \mathscr{F}_z^{(m-1)}$.

It is both tempting and hasty to conjecture that $E_{m,n}$ is always an embedding with complemented range with $F[m,n]/E_{m,n}[\mathscr{F}^{(m+n-1)}]$ isomorphic to $F[m-1,n-1]$ for $m,n \geq 2$. We do not even know whether $E_{3,3}$ is an embedding or if $F[3,3]$ has a subspace isomorphic to $\mathscr{F}_z^{(5)}$.

8. The solution of some problems. Counter-examples

In this section we will solve some problems left unanswered in [5, 7, 8, 30].

8.1. **A totally incomparable family with nonsingular derivation at any point.**

Recall that the Banach spaces X and Y are totally incomparable if no infinite-dimensional of X is isomorphic to a subspace of Y. Recall also that an operator between Banach spaces is said to be strictly singular if it restrictions to an infinite-dimensional subspace is never an isomorphism.

The paper [8] is devoted to different aspects of the stability of the differentials associated to an analytic family $(\mathscr{C}_z)$. One problem not considered, though implicit, there is whether the total incomparability of the spaces $\mathscr{C}_t$ for t real in a neighborhood of $\theta = \Re(z)$ forces the quotient map $\pi_{2,1} : \mathscr{C}_z^{(2)} \longrightarrow \mathscr{C}_z$ to be singular.

The answer is negative. Indeed, if $m \geq 2$, the quotient map $F[m,2] \longrightarrow \mathscr{F}_z^{(m)}$ in Diagram (7.4) is *never* strictly singular because the composition

$$\mathscr{F}_z \longrightarrow \mathscr{F}_z^{(m+1)} \xrightarrow{E_{m,2}} F[m,2] \longrightarrow \mathscr{F}_z^{(m)}$$

agrees with the natural inclusion $\imath_{1,m}$. It therefore suffices to consider a couple (X_0, X_1) of Banach spaces and some $m \geq 2$ for which the spaces $\mathscr{C}(X_0,X_1)_t^{(m)}$ are mutually totally incomparable for $0 < t < 1$. This is easily achieved for all $m \geq 2$ taking $X_0 = \ell_\infty, X_1 = \ell_1$ since in this case, the spaces $\mathscr{C}(X_0,X_1)_t^{(m)}$, begin "iterated" twisted sums of ℓ_p for $p = 1/t$ are ℓ_p-saturated, by a simple 3-space argument; cf. [10, Theorem 3.2.d].

8.2. **Answer to a question of Rochberg.**

In [30, p. 266, last paragraph of Section 6], Rochberg observes that, when $\mathscr{F}$ is the Calderón space associated to a couple of Banach lattices with associated differential Ω then $\Omega^{1,k}(f)$ depends only on f and $\Omega^{1,1}(f)$. He asked if the same is true for arbitrary families. The answer is strongly negative since one can build, for each $k \geq 1$, an admissible family such that $\Omega_z^{1,i} = 0$ for $1 \leq i \leq k$ but $\Omega_z^{1,i}$ is not trivial for $i > k$.

Let us proceed with the counter-example. Fix a function $\omega : \mathbb{D} \longrightarrow \mathbb{S}$ that extends to an analytic function on a neighborhood of $\overline{\mathbb{D}}$ that we denote again ω. We set $p(z) = 1/\Re(\omega(z))$. Let $\mathscr{W}$ be the space of analytic functions $F : \mathbb{D} \longrightarrow \ell_\infty$ that admit a continuous extension $\overline{F} : \overline{\mathbb{D}} \longrightarrow \ell_\infty$ such that $\|F\|_{\mathscr{W}} = \sup_{|z|\leq 1} \|\overline{F}(z)\|_{\ell_{p(z)}} < \infty$. One has:

Lemma 8.1.

(a) *$\mathscr{W}$ is an admissible space.*
(b) *$\mathscr{W}_\zeta = \ell_{p(\zeta)}$ for every $\zeta \in \mathbb{D}$.*
(c) *Given $|\zeta| < 1$ and $f \geq 0$ normalized in $\ell_{p(\zeta)}$, the (restriction to $\mathbb{D}$ of the) function $\overline{\mathbb{D}} \longrightarrow \ell_\infty$ defined by $F(z) = f^{\frac{\omega(\zeta)}{\omega(z)}}$ is normalized in $\mathscr{Z}[\omega]$ and $F(\zeta) = f$.*

Proof. (a) It is clear that for each $z \in \mathbb{D}$ the evaluation δ_z is bounded as a map $\mathscr{W} \longrightarrow \ell_\infty$. Since conformal automorphisms of the open unit disc extend continuously to the boundary (they are Möbius transformations) in order to stablish that $\mathscr{W}$ has the required invariance property, it suffices to check that for each $F \in \mathscr{W}$ one has $\|F\| = \sup_{z\in\mathbb{T}} \|F(z)\|_{\ell_{p(z)}}$, which follows from the maximum principle. The space $\mathscr{W}$ is complete since a uniform limit of analytic / continuous / bounded functions is analytic / continuous / bounded. Part (b) follows from the very definition of the norm of $\mathscr{W}$ and (c), which we prove next: Fix $\zeta \in \mathbb{D}$. Pick then a nonnegative, normalized $f \in \ell_{p(\zeta)}$ and define $F : \overline{\mathbb{D}} \longrightarrow \ell_\infty$ by

$$F(z) = f^{\frac{\omega(\zeta)}{\omega(z)}}$$

with the convention that any power of zero is again zero. It is clear that F is continuous on the closed disc and analytic on the interior. We are thus done because F (or rather its restriction to $\mathbb{D}$) belongs to $\mathscr{W}$ since for every $z \in \overline{\mathbb{D}}$,

$$\left\|F(z)\right\|_{\ell_{p(z)}} = \left\|f^{\frac{\omega(\zeta)}{\omega(z)}}\right\|_{\ell_{p(z)}} = \left\|f^{\frac{\Re\omega(\zeta)}{\Re\omega(z)}}\right\|_{\ell_{p(z)}} = \left\|f^{\frac{p(\zeta)}{p(z)}}\right\|_{\ell_{p(z)}} = \left\|f^{p(\zeta)}\right\|_{\ell_1}^{\frac{1}{p(z)}} = \left\|f\right\|_{\ell_{p(\zeta)}}^{\frac{p(\zeta)}{p(z)}} = 1. \quad \square$$

The answer to Rochberg's question comes now. For each $z \in \mathbb{D}$, let Ω_z be the differential generated by $\mathscr{W}$ at z. Recall that η is a zero of order k of h if $h(\zeta) = h'(\zeta) = \cdots = h^{(k-1)}(\zeta) = 0$ but $h^{(k)}(\zeta) \neq 0$.

Proposition 8.2. *If ζ is a zero of order $k \geq 1$ of ω', then $\Omega_\zeta^{i,j}$ is trivial if and only if $i + j \leq k + 1$.*

Proof. Although a good assimilation of Proposition 3.2 would simplify the proof, we present an argument based on the fact that the Kalton–Peck space Z_p is not isomorphic to a closed subspace of ℓ_p for $1 < p < \infty$; see [25, Theorem 6.1 and Corollary 6.7] or [4, Proposition 3.2.7] for a simplification.

Since ω has no zeros on $\mathbb{D}$ one easily checks that ω' has a zero of order $k \geq 1$ at ζ if and only if the derivative of $1/\omega$ does: just apply the Leibniz rule to $1 = \omega \cdot (1/\omega)$. The hypothesis means that for $|z - \zeta|$ small enough we have

$$\frac{\omega(\zeta)}{\omega(z)} = 1 + \sum_{n=k+1}^{\infty} a_n (z - \zeta)^n$$

with $a_{k+1} \neq 0$, where ζ is considered "fixed" and z "variable". Set $a(z) = \omega(\zeta)/\omega(z) - 1$, so that a has a zero of order $k + 1$ at ζ. Take a positive, normalized $f \in \mathscr{W}_\zeta = \ell_{p(\zeta)}$ and let F be the extremal provided above:

$$F(z) = f^{\frac{\omega(\zeta)}{\omega(z)}} = \exp\left(\frac{\omega(\zeta)}{\omega(z)} \log f\right) = \exp\left((1 + a(z)) \log f\right) = f\, \exp\left(a(z) \log f\right),$$

where f can be treated as a "constant" in ℓ_∞. Differentiating F we obtain $F'(\zeta) = \cdots = F^{(k)}(\zeta) = 0$ which immediately implies that $\Omega_\zeta^{i,j}$ is bounded for $i + i \leq k + 1$; which gives

$$\mathscr{W}_\zeta^{(n)} = \underbrace{\mathscr{W}_\zeta \oplus \cdots \oplus \mathscr{W}_\zeta}_{n \text{ summands}} = \ell_{p(\zeta)}^n \simeq \ell_{p(\zeta)} \tag{8.1}$$

for $1 \leq n \leq k + 1$. On the other hand, $F^{(k+1)}(\zeta) = a^{(k+1)}(\zeta) f \log f$ and thus

$$\Omega_\zeta^{1,k+1}(f) = \left(a_{k+1} f \log |f|, 0, \ldots, 0\right),$$

for all normalized $f \in \ell_{p(\zeta)}$. This map cannot be trivial since projection onto the first factor (which is bounded according to (8.1)) yields (a multiple of) the genuine Kalton–Peck map, which is not. This also shows that $\mathscr{W}_\zeta^{(k+2)}$ is naturally isomorphic to

$$Z_{p(\zeta)} \oplus \underbrace{\ell_{p(\zeta)} \oplus \cdots \oplus \ell_{p(\zeta)}}_{k \text{ summands}} \simeq Z_{p(\zeta)} \oplus \ell_{p(\zeta)}.$$

Hence $\Omega_\zeta^{i,j}$ cannot be trivial when $i + j = k + 2$ since otherwise $\mathscr{W}_\zeta^{(k+2)} = \mathscr{W}_\zeta^{(j)} \oplus_{\Omega_\zeta^{i,j}} \mathscr{W}_\zeta^{(i)}$ would be isomorphic to the direct product — and so to $\ell_{p(\zeta)}^{i+j} \simeq \ell_{p(\zeta)}$, which is not. From this one easily gets that $\Omega_\zeta^{i,j}$ is nontrivial for $i + j \geq k + 2$. □

The most obvious examples where the preceding Proposition applies are obtained taking $\omega(z) = \frac{1}{2} + rz^{k+1}$, with $0 < r < \frac{1}{2}$ and $k \geq 1$. In this case $\omega'(z) = (k+1)rz^k$ has a zero of order k at 0 and thus $\mathscr{W}_0^{(i)} = (\ell_2)^i \simeq \ell_2$ for $1 \leq i \leq k+1$, while $\mathscr{W}_0^{(k+2)} = \mathscr{Z}_2 \oplus \ell_2^k \simeq \mathscr{Z}_2 \oplus \ell_2$ where $\mathscr{Z}_2 \simeq Z_2$ is the Kalton–Peck space according to the notation in Section 4. The distribution of the spaces on $\mathbb{T}$ induced by the configuration ω consists of a "periodic" family of $\ell_{p(\theta)}$ spaces where $\theta \in [0, 2\pi)$, and

$$p(\theta) = \frac{1}{\Re(\frac{1}{2} + re^{i(k+1)\theta})} = \frac{2}{1 + 2r\cos((k+1)\theta)}.$$

In [5, Corollary 6], it is shown that if the first differential $\Omega^{1,1}$ induced by an admissible space $\mathscr{F}$ is not trivial at z then all $\Omega^{n,m}$ are nontrivial at z. Problem 6.1 in [5] asks whether the reciprocal is true. The preceding example shows that the answer is negative.

8.3. **A remark on "reiteration" for higher order differentials.** The spaces $\mathscr{W}$, or rather, the admissible interpolation family $\{\ell_{p(z)} : z \in \mathbb{T}\}$ are "toy-examples" of the general construction of Coifman, Cwikel, Rochberg, Sagher and Weiss mentioned in Section 2.2 whose "first degree" version is studied in [7]. The key result we need is the basic reiteration for families of [13, Theorem 5.1]: *Let $\alpha : \mathbb{T} \longrightarrow [0,1]$ be a measurable function such that both its infimum and supremum are attained. Let (X_0, X_1) be a couple of Banach spaces. Then $\mathcal{X} = \{(X_0, X_1)_{\alpha(z)} : z \in \mathbb{T}\}$ is an admissible interpolation family (in the sense of Section 2) and, if $\mathscr{F} = \mathscr{F}(\mathcal{X})$ denotes the corresponding admissible space, then $\mathscr{F}_z = (X_0, X_1)_{\tilde{\alpha}(z)}$, with equality of norms, where $\tilde{\alpha}(z) = \int_{\partial\mathbb{T}} \alpha(\omega) dP_z(\omega)$ is the harmonic extension to $\mathbb{D}$ provided by the Poisson kernel P_z.*

(Here α corresponds to the real part of the restriction of "our" ω.) The crucial fact inside the proof of this theorem is that if β is the harmonic conjugate of $\tilde{\alpha}$ (with $\beta(0) = 0$, say) and $\psi = \tilde{\alpha} + i\beta$ then, given $z \in \mathbb{D}$ and $x \in \mathscr{F}_z = (X_0, X_1)_{\tilde{\alpha}(z)}$ one can obtain an extremal in $\mathscr{F}$ just taking an extremal f for x in $\mathscr{C}(X_0, X_1)$ and letting $f \circ \psi$.

It follows that if Ω_θ denote the differentials associated to $\mathscr{C}(X_0, X_1)$ for $0 < \theta < 1$, then the differentials associated to $\mathscr{F}$ are given by $\Phi_z = \psi'(z)\Omega_{\psi(z)} = \psi'(z)\Omega_{\tilde{\alpha}(z)}$ at the first degree level (this is [7, Theorem 3.20]).

More generally, recall from Section 6.4.1 that for each n there is an upper triangular matrix $\mathbf{FdB}[n, z, \psi]$ such that $\tau_{(n,0]}(f \circ \psi)(z) = \mathbf{FdB}[n, z, \psi]\tau_{(n,0]} f(\psi(z))$. It is clear that these matrices intertwine the successive differentials by the formulæ

$$(\Phi_z^{n,k}(\mathbf{FdB}[n, z, \psi]\, x), \mathbf{FdB}[n, z, \psi]\, x) = \mathbf{FdB}[n+k, z, \psi](\Omega_{\tilde{\alpha}(z)}^{n,k}(x), x),$$

where $x \in \mathscr{C}(X_0, X_1)_{\alpha(z)}^{(n)}$, which matches with Diagram 6.5. Note that $\Omega_{\tilde{\alpha}(z)}^{n,k} = \Omega_{\psi(z)}^{n,k}$.

9. Appendix: A Fréchet algebra of analytic functions

This appendix contains the definition and basic properties of the algebra that supports the notion of an acceptable space. There are a number of reasons, most of them implicit in Section 6, suggesting that one must start with an algebra of analytic functions on the disc which contains Aut($\mathbb{D}$), the conformal automorphisms of the disc, and admits differentiation. The heuristic argumentation could be like this: Pick an admissible space $\mathscr{F}$ of Σ-valued functions. To generate $\mathscr{F}^{(2)}$ one would itch to set the space of Σ^2-valued functions $\{(f', f) : f \in \mathscr{F}\}$; since $\mathscr{F}$ is admissible the product φf is in $\mathscr{F}$ for every $f \in \mathscr{F}$ and every conformal φ as in Definition 2.1. Now the point is that $((\varphi f)', \varphi f)$ does not behave as expected; and this is because $(\varphi f)' = \varphi' f + \varphi f'$. The term $\varphi f'$ is harmless since $\mathscr{F}$ is admissible, but $\varphi' f$ is not, unless we somehow have a product $A \times \mathscr{F} \longrightarrow \mathscr{F}$ by an algebra containing all derivatives of conformal maps.

In the search for A, observe that Banach algebras tend not to admit differentiation. So, instead of struggling to get an artificial one it is perhaps a better move to give up and look into the realm of Fréchet algebras, the natural habitat of derivatives. This is what we will do. A sequence of complex numbers (c_n) is said to be rapidly decreasing if, for every positive real α, one has $|c_n| = O(n^{-\alpha})$. Let us denote by (s) the Fréchet space of rapidly decreasing sequences in its natural topology generated by the system of norms $|(c_n)|_\alpha = \sup_{n\geq 0} |c_n|\, n^\alpha$ for $0 < \alpha < \infty$. Note that (s) contains every geometric progression $(a^n)_{n\geq 0}$ with $a \in \mathbb{D}$.

Let A^∞ denote the linear space of all analytic functions $f : \mathbb{D} \longrightarrow \mathbb{C}$ whose Taylor coefficients at the origin belong to (s), with the obvious Fréchet topology. The following facts about A^∞ are not hard to check:

- A^∞ is a unital Fréchet algebra with the pointwise product (which does not correspond to the coordinatewise product of sequences, but to their convolution).
- Ordinary differentiation is a continuous, linear endomorphism on A^∞.
- $\mathrm{Aut}(\mathbb{D}) \subset A^\infty$.

To prove the third point, recall that all conformal automorphisms of the disc are Möbius transformations and so they have the form

$$\varphi(z) = \lambda \frac{z - a}{\bar{a}z - 1} \qquad (|\lambda| = 1, |a| < 1).$$

Assuming $\lambda = 1$ we have

$$\varphi(z) = \frac{a - z}{1 - \bar{a}z} = (a - z) \sum_{n\geq 0} \bar{a}^n z^n = a + \sum_{n\geq 1} \left(a\bar{a}^n + \bar{a}^{n-1}\right) z^n \qquad \Longrightarrow \qquad \varphi \in A^\infty.$$

A minor drawback of the definition of A^∞ is that everything seems to depend on the behaviour of the functions at the origin. We now characterize those functions which are in A^∞ by means of their boundary values. First of all, note that since the Taylor coefficients of any $f \in A^\infty$ are absolutely summable, f extends continuously to the closed disc and in particular, it belongs to the disc algebra A and even to the positive Wiener algebra W^+ (see definiton below). Let us denote this extension again by f. If f is any function defined on the closed disc, then $f_{\mathbb{T}}$ denotes the "boundary values", that is, the periodic function defined by $f_{\mathbb{T}}(t) = f(e^{it})$ for real t. We denote by Dg the ordinary derivative of $g : \mathbb{R} \longrightarrow \mathbb{C}$ with respect to the real variable t:

$$Dg(t) = \lim_{h\longrightarrow 0} \frac{g(t + h) - g(t)}{h}$$

provided that limit exists. Given a continuous 2π-periodic function $g : \mathbb{R} \longrightarrow \mathbb{C}$, the n-th Fourier coefficient of g is

$$c_n = c_n(g) = \frac{1}{2\pi} \int_0^{2\pi} g(t) e^{-int} dt \qquad (n \in \mathbb{Z}).$$

Note that if g corresponds to the boundary values of some function of the disc algebra, then $c_n(g) = 0$ for each $n < 0$. If, moreover, $f \in A$, then, by Cauchy formulæ,

$$f^{(n)}(0) = \frac{n!}{2\pi i} \oint_{\mathbb{T}} \frac{f(z)}{z^{n+1}}\, dz = \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(e^{it})}{e^{i(n+1)t}}\, de^{it} = \frac{n!}{2\pi} \int_0^{2\pi} \frac{f(e^{it})}{e^{int}}\, dt,$$

so the n-th Taylor coefficient of f at the origin agrees with the the n-th Fourier coefficient of $f_{\mathbb{T}}$.

Differentiability properties of periodic functions are related to the decay of their Fourier coefficients; indeed, a continuous 2π-periodic function $g : \mathbb{R} \longrightarrow \mathbb{C}$ is smooth (that is, it has derivatives of all orders) if and only if the (bilateral) sequence of Fourier coefficients of g belongs to (s); see, for instance, [26, Lemma 3]. All this shows:

Lemma 9.1. *An analytic function $f : \mathbb{D} \longrightarrow \mathbb{C}$ belongs to A^∞ if and only if it has a continuous extension to the boundary which is smooth on $\mathbb{T}$.* □

Corollary 9.2. *If $\psi \in \mathrm{Aut}(\mathbb{D})$, then ψ^* is a (continuous) automorphism of A^∞.*

Proof. Here, $\psi^*(a) = a \circ \psi$. It suffices to prove that ψ^* is correctly defined (that is, it maps A^∞ to itself) since the closed graph theorem implies continuity and the inverse is given by $(\psi^{-1})^*$. But the restriction of ψ is a smooth diffeomorphism of $\mathbb{T}$ and so the boundary values of $\psi^*(a)$ are smooth if and only if so are those of a. □

We now graft our algebra A^∞ into an arbitrary domain $\mathbb{U}$, conformally equivalent to the disc. Suppose $\psi : \mathbb{U} \longrightarrow \mathbb{C}$ is a conformal equivalence. We define

$$\psi^*[A^\infty] = \{g : \mathbb{U} \longrightarrow \mathbb{C} \text{ such that } g = f \circ \psi \text{ for some } f \in A^\infty\},$$

with the obvious (Fréchet) topology. One has.

Lemma 9.3. *$\psi^*[A^\infty]$ is independent of ψ.*

Proof. Suppose $\psi_i : \mathbb{U} \longrightarrow \mathbb{C}$ are conformal equivalences for $i = 1, 2$. Then $\psi = \psi_2 \circ \psi_1^{-1}$ is an automorphism of the disc and so ψ^* is an automorphism of A^∞. It is unnecessary to continue. □

From now on we write $A^\infty_{\mathbb{U}}$ instead of $\psi^*[A^\infty]$. Of course $A^\infty_{\mathbb{D}}$ is just A^∞.

The positive Wiener algebra W^+ is the algebra of all analytic functions on the disc whose Taylor coefficients at the origin are absolutely summable. It is clear that each function in W^+ has a continuous extension to $\overline{\mathbb{D}}$ and, in particular, it is bounded on $\mathbb{D}$. Given $f \in W^+$ we put $\|f\|_{W^+} = \sum_{n\geq 0} |c_n|$, where $f(z) = \sum_{n\geq 0} c_n z^n$ for $z \in \mathbb{D}$. As before, if $\psi : \mathbb{U} \longrightarrow \mathbb{D}$ is a conformal map, we define

$$\psi^*[W^+] = \{g : \mathbb{U} \longrightarrow \mathbb{C} \text{ such that } g = f \circ \psi \text{ for some } f \in W^+\}$$

and we transfer the norm of W^+ to $\psi^*[W^+]$ by stipulating that $\|g\|_{\psi^*[W^+]} = \|f\|_{W^+}$ provided $g = f \circ \psi$.

Note that $g : \mathbb{U} \longrightarrow \mathbb{C}$ belongs to $\psi^*[W^+]$ if and only if there is $(c_n)_{n\geq 0}$ in ℓ_1 such that $g(u) = \sum_{n\geq 0} c_n \psi(u)^n$ for all $u \in \mathbb{U}$ in which case $\|g\|_{\psi^*[W^+]} = \|(c_n)\|_{\ell_1} = \sum_n |c_n|$. One has:

Lemma 9.4. *$\psi^*[W^+]$ contains $A^\infty_{\mathbb{U}}$, and the inclusion is continuous.*

Proof. Since $A^\infty_{\mathbb{U}} = \psi^*[A^\infty]$ it suffices to check that W^+ contains A^∞ and the inclusion is continuous. Which is obvious: every rapidly decreasing sequence $(c_n)_{n\geq 1}$ is absolutely summable, with $\|(c_n)\|_{\ell_1} \leq \frac{\pi}{6}|(c_n)|_2$. □

In spite of our good intentions, and rather unexpectedly, the grafted algebras $A^\infty_{\mathbb{U}}$ are not closed under differentiation, even for very natural choices of $\mathbb{U}$. To convince the skeptical reader let us work out the following example: the function $\varphi(z) = (e^z - 1)/(e^z + 1)$ maps conformally the (horizontal) strip $\mathbb{U} = \{z : \Im(z) \in (-\frac{\pi}{2}, \frac{\pi}{2})\}$ onto $\mathbb{D}$. Obviously $\varphi \in A^\infty_{\mathbb{U}}$. But if we write $w = e^z$, then

$$\varphi'(z) = \frac{2w}{(w+1)^2}$$

and we see that $\varphi'(z)$ has poles at $z = \pm\frac{\pi}{2}i$. In particular φ' is unbounded on $\mathbb{U}$, and therefore it cannot be in $A^\infty_{\mathbb{U}}$ which contains bounded functions only. In the end, this is one of the reasons why the generation of Rochberg families in general domains requires to move back and forth from $\mathbb{U}$ to $\mathbb{D}$ which, in turn, requires the versions of the Chain and Leibniz's rules presented in Section 6.4.

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Authors' answers to the referee's report

We have incorporated most of your suggestions into the text. Here we will only refer to the points where we have not followed the report to the letter. In this case the relevant point of the report is shown in italics, followed by our reply.

(1) *p7 bot; boundedness of evaluation of the derivatives. I am not comfortable with what is said. I don't see why the norm boundedness of point evaluations implies the boundedness of, say, evaluation of the first derivative at that point. I may just be missing something obvious and/or the resolution might be buried in the definition of "Banach space of analytic functions"*

We have added a lemma with this (Lemma 2.4).

(2) ★★★ *p8 par 3; I don't think that at this point you have defined quasilinear map or twisted direct sum (I couldn't find "Definition 1") [....I found "Definition 1". It's on Page 32. There is a Tex mistake; the definitions in Section 2 are numbered 2.1, 2.2, etc., by that scheme Definition 1 should be Definition 7.1 (which is why I couldn't find it.)*

You are right. We have placed these definitions on p. 8, see Definition 3.1. Thank you.

(3) *p9 bot; "...norm of Hardy type...Pisier's..." The meaning is not clear (but I suspect I know what you mean) nor is the justification. You could fill in more details or perhaps the comment should be dropped.*

Here we have adopted an intermediate solution, being very precise in the reference to Pisier's work, so that the interested reader can understand what we are talking about. See the paragraph following Lemma 3.3.

(4) *p 13 bot; "...in the final paragraph of Section 5". If the reader then turns to Section 5, as I did, the result is there, but I was looking for a numbered lemma to compare to Lemma 4.4. Perhaps make that change in Section 5.*

Well, we put a more precise comment on Section 4, but we didn't add any label in the end.

(5) *p14 top: "It is likely..." It would be nice if you could establish that. Perhaps the duality result helps.*

This is actually true, but requires a lot of additional work. We added a reference at the end of Section 4.

(6) *p 17 bot: I am guessing that you want "Δ^* (not $\Delta^\star$)" but [...] I suggest you use the same $\star$ in the superscript on Δ as you used for $\mathscr{F}$. There may also be some other places this is an issue.*

Since $\mathscr{F}$ is a linear space, this would lead us to a certain ambiguity given that $\mathscr{F}^\star$ could be interpreted as the algebraic dual of $\mathscr{F}$. We have left it as it was.

(7) *p23 mid; here the good...*

Ok, but "news" takes a singular verb.

(8) *p 31 Section 7; I think that this section has some very nice ideas but it is not well integrated with the rest of the paper. In addition to the issues I mentioned in the comment* ★★★ *(moving the introduction of quasilinear maps and twisted sums earlier, regularizing the numbering of the Definition, etc,) I suggest a pointer to this work earlier in the introduction; perhaps a sentence such as "We consider the relationship between the exact sequences associated to T and to F in*

Section 7." in the middle of the middle paragraph of p2. The analysis which leads to the formula for $F[m,3]$ *should be more clearly separated from the earlier discussion. ("We continue our analysis..." isn't strong enough,) It looks to me as if that result should be a numbered proposition. (I would be tempted to reorganize the section and put all the homological work in a technical proposition and then have a corollary with the representations of* $F[m,2]$ *and* $F[m,3]$. *Also, consider doing the computations similar to those on the top of page 35 for the case of F[3, 3].*

We have completely rewritten Section 7, partly because there was a bug that we have not managed to fix (in the computation of $F[3,3]$) and partly to integrate it better with the rest of the work (which we think we have achieved, you can judge for yourself).

(9) We also have added a result that we felt was important and which is new even in the context of admissible spaces / families (Proposition 3.2).

(10) Finally, we have used a more consistent notation and terminology: "Compatible couple" Always. No Banach couple or Interpolation couple. No interpolation pair. "Degree" of a Rochberg space, theory, and other similar things. Always. Not "order": polynomials have degree, not order. Zeros have both "order" and "degree" but, while a polynomial can have degree 0 (a "constant"), zeros cannot be of 0-th order because a single zero has order 1.

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