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OPTIMAL COVARIANCE ADJUSTMENT IN GROWTH CURVE MODELS

by

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Optimal covariance adjustment in growth curve models

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Abstract

We consider the selection of covariables for the covariance adjustment of parameter estimators in growth curve models. The procedure consists of obtaining the best subset of linear combinations of higher order polynomials for minimizing the variance of the adjusted estimator of a particular linear combination of the lower order polynomial coefficients. The coefficients of the required linear combinations are expressed in terms of the eigenvectors of appropriate matrices and the gain in precision due to covariance adjustment is measured by the magnitude of the corresponding eigenvalues. The results are numerically illustrated with data previously analyzed in the literature.

Key words: Covariance adjustment, Growth curve model, Selection of covariables.

1 Introduction

A useful model for the analysis of polynomial growth curves was introduced by Potthoff and Roy (1964) and may be described by

$$\mathbf{Y} \sim N(\mathbf{X}\boldsymbol{\tau}\mathbf{A}; \boldsymbol{\Sigma} \otimes \mathbf{I}_N), \tag{1.1}$$

where $\mathbf{Y}$ is a $p \times N$ matrix of observed data, the columns of which constitute a random sample from a p-variate normal distribution with covariance matrix $\boldsymbol{\Sigma}$, $\boldsymbol{\tau}$ is a $m \times g$ matrix of unknown parameters, $\mathbf{X}$ is an $p \times m$ within sample units design matrix of rank m and $\mathbf{A}$ is a $g \times N$ across sample units design matrix of rank g.

This model is particularly attractive because it may be adjusted by standard weighted least squares procedures available in a great number of commercial statistical software packages. This might be one of the reasons why it has been the focus of research of many authors in the past three decades (see Kshirsagar and Smith (1995), for example). In particular, a great deal of attention has been devoted to obtaining more precise estimates of the parameter matrix $\boldsymbol{\tau}$, mainly by using higher order polynomials for covariance adjustment as proposed by Rao (1965).

Essentially, the covariance adjustment procedure is based on the transformation

$$\mathbf{Y}_1 = \mathbf{Q}_1'\mathbf{Y}, \quad \mathbf{Y}_2 = \mathbf{Q}_2'\mathbf{Y} \tag{1.2}$$

where $\mathbf{Q}_1$ and $\mathbf{Q}_2$ denote $p \times m$ and $p \times (p-m)$ matrices, respectively, such that $\mathbf{Q}_1'\mathbf{X} = \mathbf{I}_m$ and $\mathbf{Q}_2'\mathbf{X} = \mathbf{0}$. Since from (1.1) and (1.2), we have $E(\mathbf{Y}_1) = \boldsymbol{\tau}\mathbf{A}$ and $E(\mathbf{Y}_2) = \mathbf{0}$, the m rows of $\mathbf{Y}_1$ generate the estimation space and the $(p-m)$ rows of $\mathbf{Y}_2$ generate the error space.

If the columns of $\mathbf{Y}_1$ are correlated with those of $\mathbf{Y}_2$ it is possible to reduce the variance of the estimators of $\boldsymbol{\tau}$ by fitting conditional models of the form

$$\mathbf{Y}_1|\mathbf{Y}_2 \sim N(\boldsymbol{\tau}\mathbf{A} + \boldsymbol{\eta}\mathbf{Y}_2; \boldsymbol{\Sigma}_{1|2} \otimes \mathbf{I}_N), \tag{1.3}$$

where

$$\boldsymbol{\eta} = \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1},$$

$$\boldsymbol{\Sigma}_{1|2} = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21},$$

and

$$\Sigma_{ij} = \mathbf{Q}_i' \Sigma \mathbf{Q}_j, \quad i, j = 1, 2.$$

In this context, Rao (1965, 1966) discusses the possibility of using less than $(p - m)$ covariables (i.e., rows of $\mathbf{Y}_2$) in the conditional model (1.3). He shows that selecting only the rows of $\mathbf{Y}_2$ which are highly correlated with those of $\mathbf{Y}_1$ as covariates in (1.3) might lead to more precise estimators of the regression parameters $\boldsymbol{\tau}$. Furthermore, he identifies specific structures of the within sample units covariance matrix Σ for which this reduction holds.

Grizzle and Allen (1969) considered an empirical procedure based on the examination of the within sample units sample covariance matrix of the transformed variables to implement such a covariance adjustment procedure. Rao (1967), Lee and Geisser (1972), Kenward (1985) and Verbyla (1986) provided further insight on covariance adjustment under specific structures for the covariance matrix Σ. Chinchilli and Carter (1984), Fujikoshi and Rao (1991) and Satoh et al. (1997) developed tests for the hypothesis of redundancy of a given subset of rows of $\mathbf{Y}_2$ in this context. The main problem with this approach is the criterion for choosing how many and which rows of $\mathbf{Y}_2$ to include in the covariance adjustment procedure.

We propose a systematic procedure for this purpose. The idea is to identify uncorrelated (noncanonical) linear combinations of the rows of $\mathbf{Y}_2$ which minimize the variance of estimators of linear combinations of the components of $\boldsymbol{\tau}$. The results are expressed in terms of eigenvalues and eigenvectors of appropriate matrices which may be explicitly obtained. Section 2 is devoted to the presentation of the covariance adjustment strategy; first we consider the problem of estimating a single linear combination of the parameters and then generalize the results to two or more of them. In Section 3 we illustrate the method with numerical examples from the statistical literature and discuss possible extensions.

2 A strategy for inclusion of covariables from the error space

Under model (1.3), the estimator of $\boldsymbol{\tau}$ adjusted by all the $(p - m)$ potential covariables is given by

$$\begin{aligned}\hat{\boldsymbol{\tau}}_{\mathrm{I}} &= \hat{\boldsymbol{\tau}}_0 - \mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{Y}_2\mathbf{A}'(\mathbf{AA}')^{-1} \\ &= (\mathbf{X}'\mathbf{S}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{S}^{-1}\mathbf{YA}'(\mathbf{AA}')^{-1}, \end{aligned} \tag{2.1}$$

where

$$\hat{\boldsymbol{\tau}}_0 = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{YA}'(\mathbf{AA}')^{-1}$$

is the ordinary least squares estimator of $\boldsymbol{\tau}$,

$$\mathbf{S} = \mathbf{Y}(\mathbf{I}_N - \mathbf{A}'(\mathbf{AA}')^{-1}\mathbf{A})\mathbf{Y}'$$

and

$$\mathbf{S}_{ij} = \mathbf{Y}_i(\mathbf{I}_N - \mathbf{A}'(\mathbf{AA}')^{-1}\mathbf{A})\mathbf{Y}_j' \quad i,j = 1,2.$$

Rao (1967) and Grizzle and Allen (1969) show that $\hat{\boldsymbol{\tau}}_{\mathrm{I}}$ is unbiased and that its unconditional covariance matrix is

$$Var(\hat{\boldsymbol{\tau}}_{\mathrm{I}}) = (\mathbf{AA}')^{-1} \otimes \frac{N-g-1}{N-g-(p-m)-1}\boldsymbol{\Sigma}_{1|2}. \tag{2.2}$$

From (2.2) we conclude that the utility of covariance adjustment is essentially affected by a balance between the loss in precision due to the removal of variables from the error space (reflected in the term $(N-g-1)/(N-g-(p-m)-1)$) and the gain due to their inclusion as covariables in the estimation space (which shows up in the term $\boldsymbol{\Sigma}_{1|2}$).

Grizzle and Allen (1969) consider all subsets of the columns of $\mathbf{Y}_2$ as potential covariables; their choice falls on the subset for which the generalized variance of the regression parameters is minimized. Since, in general, the potential candidates are correlated, the procedure might not be efficient; error space degrees of freedom might be wasted without the corresponding gain in precision because redundant information might be transferred from the error to the estimation space. To avoid this problem, we propose to use uncorrelated linear combinations of the rows of $\mathbf{Y}_2$ as the potential candidates for covariables.

In this direction first let $\mathbf{b}_1, \ldots, \mathbf{b}_k$, $0 \le k \le p-m$ denote $(p-m)$-dimensional vectors satisfying

$$\mathbf{b}_i'\boldsymbol{\Sigma}_{22}\mathbf{b}_j = 0, \quad i \neq j. \tag{2.3}$$

Also let $\mathbf{B}$ denote a matrix with $\mathbf{b}_k$, $k = 0, \ldots p - m$ as its columns.

Let $\mathbf{C}'\boldsymbol{\tau}\mathbf{U}$, where $\mathbf{C}$ and $\mathbf{U}$ are fixed constant matrices, denote the linear combinations of interest, e.g., features of the growth curves under investigation that may be expressed in terms of linear combinations of the elements of $\boldsymbol{\tau}$.

Suppose firstly that the interest centres on the estimation of a single linear combination, i.e., with $\mathbf{C} = \mathbf{c}$, a $m \times 1$ vector and $\mathbf{U} = \mathbf{u}$, a $g \times 1$ vector. Under the conditional reduction of model (1.1) to model (1.3) obtained by using the columns of $\mathbf{B}'\mathbf{Y}_2$ as the set of covariables, the adjusted estimator of $\mathbf{c}'\boldsymbol{\tau}\mathbf{u}$ is

$$\mathbf{c}'\hat{\boldsymbol{\tau}}_{\mathbf{B}}\mathbf{u} \tag{2.4}$$

where

$$\hat{\boldsymbol{\tau}}_{\mathbf{B}} = \hat{\boldsymbol{\tau}}_0 - \mathbf{S}_{12}\mathbf{B}(\mathbf{B}'\mathbf{S}_{22}\mathbf{B})^{-1}\mathbf{B}'\mathbf{Y}_2\mathbf{A}'(\mathbf{A}\mathbf{A}')^{-1}.$$

The unconditional variance of $\mathbf{c}'\hat{\boldsymbol{\tau}}_{\mathbf{B}}\mathbf{u}$ is

$$Var(\mathbf{c}'\hat{\boldsymbol{\tau}}_{\mathbf{B}}\mathbf{u}) = \begin{cases} \mathbf{u}'(\mathbf{A}\mathbf{A}')^{-1}\mathbf{u}\,\mathbf{c}'\boldsymbol{\Sigma}_{\mathbf{B}}\mathbf{c}\,\frac{N-g-1}{N-g-k-1} & \text{if } \mathbf{B} \neq \mathbf{0} \\ \mathbf{u}'(\mathbf{A}\mathbf{A}')^{-1}\mathbf{u}\,\mathbf{c}'\boldsymbol{\Sigma}_{11}\mathbf{c} & \text{if } \mathbf{B} = \mathbf{0} \end{cases} \tag{2.5}$$

where

$$\boldsymbol{\Sigma}_{\mathbf{B}} = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\mathbf{B}(\mathbf{B}'\boldsymbol{\Sigma}_{22}\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}_{21}.$$

Note that when $\mathbf{B}$ is a null matrix, the procedure generates the ordinary least squares estimator (in such a case, $k = 0$). Note that the conditional model (1.3) is invariant under any choice of the basis generating the conditioning space, i.e., the conditional distribution of $\mathbf{Y}_1|\mathbf{B}'\mathbf{Y}_2$ is invariant relative to the choice of the $\mathbf{B}$ matrix, for all $\mathbf{B}$ of rank $(p - m)$.

We now wish to choose the number of covariables, $1 \leq k \leq p - m$ and the $\mathbf{B}$ matrix, $\mathbf{B} \neq \mathbf{0}$ in such a way that (2.5) is minimized. For fixed k, this corresponds to obtaining $\mathbf{B}$ such that

$$\mathbf{c}'\boldsymbol{\Sigma}_{12}\mathbf{B}(\mathbf{B}'\boldsymbol{\Sigma}_{22}\mathbf{B})^{-1}\mathbf{B}'\boldsymbol{\Sigma}_{21}\mathbf{c} \tag{2.6}$$

is maximum; note that the procedure does not depend on $\mathbf{u}$. Keeping the restrictions (2.3) in mind, and using standard results in Matrix Algebra,(2.6) reduces to

$$\sum_{i=1}^{k} \mathbf{c}'\boldsymbol{\Sigma}_{12}\mathbf{b}_i(\mathbf{b}_i'\boldsymbol{\Sigma}_{22}\mathbf{b}_i)^{-1}\mathbf{b}_i'\boldsymbol{\Sigma}_{21}\mathbf{c} = \sum_{i=1}^{k} \frac{\mathbf{c}'\boldsymbol{\Sigma}_{12}\mathbf{b}_i'\mathbf{b}_i\boldsymbol{\Sigma}_{21}\mathbf{c}}{\mathbf{b}_i'\boldsymbol{\Sigma}_{22}\mathbf{b}_i} = \sum_{i=1}^{k} \frac{\mathbf{b}_i'\boldsymbol{\Sigma}_{21}\mathbf{c}\mathbf{c}'\boldsymbol{\Sigma}_{12}\mathbf{b}_i}{\mathbf{b}_i'\boldsymbol{\Sigma}_{22}\mathbf{b}_i}. \quad (2.7)$$

Note that the matrix $\boldsymbol{\Sigma}_{21}\mathbf{c}\mathbf{c}'\boldsymbol{\Sigma}_{12}$ is symmetric, positive semidefinite with rank 1 and that the matrix $\boldsymbol{\Sigma}_{22}$ is positive definite. Then, recalling that $\mathbf{c}$ is fixed, and using well known results in Matrix Algebra (see Johnson and Wichern (1992), for example) it follows that the maximum of (2.6) is attained with $k = 1$ and $\mathbf{b}_1 = \mathbf{p}_1$, where $\mathbf{p}_1$ is the eigenvector associated with the single nonzero root of the determinantal equation

$$|\boldsymbol{\Sigma}_{21}\mathbf{c}\mathbf{c}'\boldsymbol{\Sigma}_{12} - \lambda\boldsymbol{\Sigma}_{22}| = 0. \quad (2.8)$$

In fact, the solution is given by $\lambda = \mathbf{c}'\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\mathbf{c}$ and the corresponding eigenvector by $\mathbf{p}_1 = \alpha\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\mathbf{c}$ for any nonzero real constant α. Under the restrictions (2.3), it follows that $\alpha = (\mathbf{c}'\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}\mathbf{c})^{-1/2}$. Since this minimizes (2.5) for $\mathbf{B} \neq \mathbf{0}$, we must compare the result with that obtained with no covariance adjustment, i.e., for $\mathbf{B} = \mathbf{0}$.

For practical applications, the parameters $\boldsymbol{\Sigma}_{ij}$ must be replaced by appropriate estimates which implies that we must rely on approximate results. Thus, the required linear combination of the error space variables will be given by

$$\mathbf{b}_1 = (\mathbf{c}'\mathbf{S}_{12}\mathbf{S}_{22}^{-1}\mathbf{S}_{21}\mathbf{c})^{-1/2}\mathbf{S}_{22}^{-1}\mathbf{S}_{21}\mathbf{c}. \quad (2.9)$$

Without loss of generality, suppose now that $\mathbf{C} = \mathbf{I}_m$ and $\mathbf{U} = \mathbf{I}_g$, i.e., that we are interested in estimating the original parameters. As opposed to the previous case, where the interest was centred on the minimization of the variance of the estimator of a single contrast, here, several criteria are available. Among them, a sensible alternative is to choose $\mathbf{B}$ to minimize the trace of the covariance matrix of $\hat{\boldsymbol{\tau}}_B$, i.e.,

$$trVar(\hat{\boldsymbol{\tau}}_B) = \begin{cases} (\sum_{j=1}^{g} \frac{1}{N_j})\frac{N-g-1}{N-g-k-1} tr\boldsymbol{\Sigma}_B & if \ \ \mathbf{B} \neq \mathbf{0} \\ (\sum_{j=1}^{g} \frac{1}{N_j}) tr\boldsymbol{\Sigma}_{11} & if \ \ \mathbf{B} = \mathbf{0} \end{cases} \quad (2.10)$$

where N_j denotes the sample size associated to the jth treatment ($j = 1, \ldots g$). For fixed k, this corresponds to obtaining $\mathbf{B}$ that maximizes

$$
\begin{aligned}
tr\Sigma_{12}\mathbf{B}(\mathbf{B}'\Sigma_{22}\mathbf{B})^{-1}\mathbf{B}'\Sigma_{21} &= tr(\mathbf{B}'\Sigma_{22}\mathbf{B})^{-1}\mathbf{B}'\Sigma_{21}\Sigma_{12}\mathbf{B} \\
&= \sum_{i=1}^{k} \frac{\mathbf{b}_i'\Sigma_{21}\Sigma_{12}\mathbf{b}_i}{\mathbf{b}_i'\Sigma_{22}\mathbf{b}_i}. \qquad (2.11)
\end{aligned}
$$

Since the rank of $\Sigma_{21}\Sigma_{12}$ is $\min(m, p-m)$, there are only $\min(m, p-m)$ nonnull eigenvalues θ_i, ($i = 1, \ldots, \min(m, p-m)$), satisfying $\theta_1 > \theta_2 > \ldots > \theta_{\min(m,p-m)}$. Then, the maximum of (2.11) is attained when $\mathbf{b}_1 = \mathbf{p}_1, \ldots, \mathbf{b}_k = \mathbf{p}_k$, $\mathbf{p}_1, \ldots, \mathbf{p}_k$ being the first k eigenvectors of the matrix $\Sigma_{21}\Sigma_{12}$ in the metric of Σ_{22}, i.e., associated with the determinantal equation

$$
|\Sigma_{21}\Sigma_{12} - \theta_i\Sigma_{22}| = 0. \qquad (2.12)
$$

Repeating the procedure for $k = 1, \ldots, \min(m, p-m)$ and comparing the results to that corresponding to $k = 0$ we may choose the set of linear combinations that minimize $trVar(\hat{\boldsymbol{\tau}}_B)$.

A similar approach was employed in a different context by Rao (1964); he maximizes an expression like (2.11) to obtain the principal components of instrumental variables.

3 Examples

To compare the procedure proposed in Section 2 with other variable selection strategies for covariance adjustment, we consider two datasets previously analyzed in the literature.

We first focus on the data presented in Grizzle and Allen (1969), concerning ramus bone heights (in mm) for 20 boys at 8, 8.5, 9 and 9.5 years of age. A plot of the average profiles suggests that a straight line should fit the data. Thus, model (1.1) may be specified by letting $\mathbf{A} = \mathbf{1}_{20}$, $\boldsymbol{\tau} = (\tau_0, \tau_1)'$ and the within sample units design matrix expressed in terms of orthogonal polynomials, i.e.,

$$
\mathbf{X} = \begin{pmatrix} 1/4 & -3/20 \\ 1/4 & -1/20 \\ 1/4 & 1/20 \\ 1/4 & 3/20 \end{pmatrix}. \qquad (3.1)
$$

Under this parametrization, the transformation (1.2) may be carried out by taking $\mathbf{Q}_1 = \mathbf{X}$ and

$$\mathbf{Q}_2 = \begin{pmatrix} 1/4 & -1/20 \\ -1/4 & 3/20 \\ -1/4 & -3/20 \\ 1/4 & 1/20 \end{pmatrix}.$$

Here the terms in the estimation space (spanned by the rows of $\mathbf{Y}_1$) are transformed by the values of orthogonal polynomials of orders 0 and 1 ($m = 2$) and those in the error space (spanned by the rows of $\mathbf{Y}_2$) correspond to the quadratic and cubic orthogonal polynomials ($p - m = 2$). The sample correlations between the rows of $\mathbf{Y}_1$ (labeled Intercept and Linear) and $\mathbf{Y}_2$ (labeled Quadratic and Cubic) are given by

	Intercept	Linear
Quadratic	-0.0828	0.1517
Cubic	-0.0629	-0.5912

and suggest that some gain may be attained by covariance adjustment. Let us first consider the problem of estimating the slope τ_1. For comparison purposes, we present the results obtained under different alternatives: no adjustment, adjustments based on the quadratic and cubic terms separately or simultaneously and adjustment based on linear combinations of them obtained via the strategy outlined in Section 2. The results are summarized in Table 3.1. The entries denoted $\mathbf{c}'\mathbf{S_B}\mathbf{c}$ and Constant correspond to an estimate of $\mathbf{c}'\mathbf{\Sigma_B}\mathbf{c}$ and to $N - g - 1/N - g - k - 1$, respectively, in the expression for the variance of $\hat{\tau}_1$ given in (2.5) with $\mathbf{u} = 1$ and $\mathbf{c} = (0, 1)'$. Additionally, we also illustrate that the introduction of the second linear combination obtained under the same conditions mentioned above does not contribute to the covariance adjustment process, since it does not change the term $\mathbf{c}'\mathbf{S_B}\mathbf{c}$ but increases the term $N - g - 1/N - g - k - 1$.

The estimated variance of the covariance adjusted estimator of the parameter τ_1 is smaller when a single covariable, given by $\mathbf{b}_1'\mathbf{Y}_2$ with $\mathbf{b}_1 = (-0.036, 0.177)'$ is used. As indicated in Section 2, $\mathbf{b}_1$ is given by (2.9) and corresponds to the eigenvector associated to the single nonull eigenvalue of $\mathbf{S}_{21}\mathbf{c}\mathbf{c}'\mathbf{S}_{12}\mathbf{S}_{22}^{-1}$. The reduction in the term $\mathbf{c}'\mathbf{S_B}\mathbf{c}$ relatively to the nonadjusted estimator is given by the corresponding eigenvalue $\lambda = 0.576$. Clearly, the

Table 3.1: *Covariance adjusted estimation of the ramus bone growth slope*

Covariable(s)	$\hat{\tau}_1$	$20\widehat{Var}(\hat{\tau}_1)$	$\mathbf{c}'\mathbf{S_B}\mathbf{c}$	Constant
None	0.467	0.085	1.613	0.053
Quadratic	0.471	0.093	1.576	0.059
Cubic	0.463	0.062	1.049	0.059
Quadratic and Cubic	0.465	0.069	1.037	0.066
-0.036 Quad + 0.117 Cubic	0.465	0.061	1.037	0.059
-0.036 Quad + 0.117 Cubic and 0.241 Quad + 0.031 Cubic	0.465	0.069	1.037	0.066

introduction of the second linear combination of the variables in the error space ($\mathbf{b}_2'\mathbf{Y}_2$ with $\mathbf{b}_2 = (0.241, 0.031)'$) does not contribute to further reduce the variance of the estimator. Other authors, like Rao (1965, 1966), Grizzle and Allen (1969) or Fujikoshi and Rao (1991), working with the same data, choose the cubic term as the covariable to reduce the variance under investigation. To a certain extent, this concurs with the optimal solution, which places more weight on this term than on the quadratic one.

Now suppose that the interest lies both in the intercept τ_0 and the slope τ_1 and that the objective is to minimize the trace of the covariance matrix of the corresponding vector of estimators. Following the same strategy considered above, we present the results in Table 3.2. Here, the entries denoted $tr\mathbf{S_B}$ and Const correspond to an estimate of $tr\mathbf{\Sigma_B}$ and to $N-g-1/N-g-k-1$, respectively, in the expression for the trace of the covariance matrix of $\hat{\tau}_\mathbf{B}$ given in (2.10).

In this case, under a minimum trace criterion, covariance adjustment procedures do not contribute to obtain better estimates of the parameters, since the decrease in the term $tr\mathbf{S_B}$ obtained by including either one or two linear combinations of the error space variates as covariables does not compensate the corresponding increase in the constant term. The decrease corresponding to the introduction of the first linear combination is equal to the first nonnull eigenvalue of $\mathbf{S}_{21}\mathbf{S}_{12}\mathbf{S}_{22}^{-1}$, i.e., 1.598 and that corresponding to the introduction of both linear combinations is equal to the sum of both nonnull eigenvalues, i.e., $1.914 = 1.598 + 0.416$.

The selected covariables in the second case differ from that considered

Table 3.2: *Covariance adjusted estimation of the ramus bone growth intercept and slope*

Covariable(s)	$\hat{\tau}_0$	$\hat{\tau}_1$	$20trVar(\hat{\tau}_B)$	trS_B	Const
None	50.075	0.467	6.350	120.644	0.053
Quadratic	50.055	0.471	7.047	119.792	0.059
Cubic	50.072	0.463	7.036	119.610	0.059
Quadratic and Cubic	50.050	0.465	7.851	118.631	0.066
0.169 Quad + 0.096 Cub	50.053	0.458	7.003	119.047	0.059
0.169 Quad + 0.096 Cub and 0.177 Quad - 0.074 Cub	50.050	0.465	7.851	118.631	0.066

in the first case because the estimation of an additional parameter is also considered. Also note that either the model including the quadratic and cubic terms or the model including both linear combinations generate the same estimates and the same estimated variances of the parameters because of the invariance property mentioned in Section 2.

For the second example, we consider the data on coronary sinus potassium measured in dogs also presented in Grizzle and Allen (1969). The design involved four groups of dogs observed at seven different occasions. Here, a third degree polynomial ($m = 4$) was adopted as an adequate description of the mean response for each group. Therefore, the candidate covariables are linear combinations of the ($p - m = 3$) vectors that span the error space, i.e., those associated to the polynomials of degrees 4, 5 and 6. The results for the selection of the best covariable to minimize the estimated variance of a linear combination of the parameters designed to compare the coefficients of the quadratic terms corresponding to groups 1 and 4 are summarized in Table 3.3 displayed under the same format as Tables 3.1 and 3.2. The linear combination of interest is actually a contrast defined as $\mathbf{c}'\boldsymbol{\tau}\mathbf{u}$ with $\mathbf{c} = (0, 0, 1, 0)'$ and $\mathbf{u} = (1, 0, 0, -1)'$.

In Table 3.4, we present similar results for the case where interest is focused on the entire vector of 16 parameters and we choose to minimize the trace of the corresponding estimated covariance matrix. We neither present estimates of the individual parameters nor of their estimated variances, since

Table 3.3: *Covariance adjusted estimation of the difference between the quadratic coefficients of the first and fourth group for the coronary sinus potassium curve*

Covariable(s)	Estimated contrast	Estimated variance	$c'S_Bc$	Const
None	-0.0127	0.0030	0.0973	0.0313
4th degree (D4)	-0.0390	0.0015	0.0448	0.0333
5th degree (D5)	-0.0134	0.0032	0.0970	0.0333
6th degree (D6)	-0.0112	0.0032	0.0967	0.0333
4th and 5th degree	-0.0388	0.0016	0.0448	0.0356
4th and 6th degree	-0.0381	0.0016	0.0447	0.0356
5th and 6th degree	-0.0118	0.0034	0.0965	0.0356
4th, 5th and 6th degree	-0.0379	0.0017	0.0447	0.0382
- 0.0454 D4 - 0.0016 D5 - 0.0017 D6	-0.0379	0.0015	0.0447	0.0333
- 0.0454 D4 - 0.0016 D5 - 0.0017 D6 and - 0.0031 D4 - 0.0695 D5 + 0.0012 D6	-0.0379	0.0016	0.0447	0.0356
- 0.0454 D4 - 0.0016 D5 -0.0017 D6, - 0.0031 D4 - 0.0695 D5 + 0.0012 D6 and 0.0044 D4 - 0.0032 D5 - 0.0314 D6	-0.0379	0.0017	0.0447	0.0382

our interest lies solely on the reduction of the sum of the latter.

In either case, the use of a single covariable conveniently constructed from the vectors spanning the error space produces estimates with smaller "variances" than their unadjusted counterparts.

The proposed procedure for selecting covariables for covariance adjustment yields optimal results and is easily implemented computationally avoiding either the arbitrary choice implied by the strategy suggested by Grizzle and Allen (1969) or the possibly sub-optimal solution obtained by trying all possible subsets. The maximum number of covariables to be included in the adjustment process is also determined. A possible extension useful for cases where the parameter of interest is multidimensional, involves the optimization of other criteria such as the generalized variance or the Euclidean norm of the covariance matrix of the corresponding estimators as suggested by Rao (1964 and 1973).

Table 3.4: *Covariance adjusted estimation of the coronary sinus potassium cubic polynomial coefficients*

Covariable(s)	$trVar(\hat{\tau}_B)$	trS_B	Const
None	0.3070	9.8227	0.0313
4th degree (D4)	0.3077	9.2305	0.0333
5th degree (D5)	0.3191	9.5726	0.0333
6th degree (D6)	0.3239	9.7184	0.0333
4th and 5th degree	0.3212	9.0132	0.0356
4th and 6th degree	0.3256	9.1390	0.0356
5th and 6th degree	0.3375	9.4728	0.0356
4th, 5th and 6th degree	0.3407	8.9246	0.0382
- 0.0381 D4 + 0.0266 D5 - 0.0085 D6	0.3038	9.1154	0.0333
- 0.0381 D4 + 0.0266 D5 - 0.0085 D6 and 0.0253 D4 + 0.0531 D5 - 0.0137 D6	0.3180	8.9246	0.0356
- 0.0381 D4 + 0.0266 D5 - 0.0085 D6, 0.0253 D4 + 0.0531 D5 - 0.0137 D6 and - 0.0015 D4 + 0.0363 D5 + 0.0269 D6	0.3407	8.9246	0.0382

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