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Walter Ricardo Ferrer Santos

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ABSTRACT

Let X be an arbitrary affine variety and K an affine algebraic group acting on it. Then K acts on the algebra P(X) of polynomial functions on X. The objective of this paper is to study the relationship between the existence of a variety structure on the set of orbits of K on X and certain algebraic properties of the action of K on P(X).

INTRODUCTION

This paper is divided into eight sections , with a sharp methodological distinction between the first six and the last two. In the first six sections we study the case in which G is a group and K a closed subgroup of G acting by multiplication. In this case the study of the orbit space G/K is simplified by the use of the intermediate concept of observable subgroup as introduced in [1]. Our representation theoretical methods give elementary and selfcontained proofs of results obtained in [2] and [6]. In the last two sections our methods have a more ideal theoretical and algebraico-geometric emphasis, and some of these results can be used to reinterpret the work of the previous five sections.

We proceed to give a brief description of each section.

1. Observable subgroups - Here we recall some definitions and results from [1]. The systematic study of the semigroup of all rational characters of K that are extendable to G allows us to simplify some of the proofs of that paper.

2. Geometrically reductive subgroups and observability - Here we prove that if K is a geometrically reductive subgroup of G, then K is observable in G.

3. Exactness and observability - We prove that if K is an exact subgroup of G then K is observable in G. For the concept of exact subgroup see [2]. This is the first stage of the proof that K exact in G implies G/K affine. The concept of exact subgroup will be reinterpreted in 7 with regard to the concept of linearly reductive group.

4. Existence of Affine Quotients - We prove that if K is geometrically reductive (or exact in G) then G/K is affine. From 2 and 3 we know that G/K is quasiaffine.

Using this, we cover G/K with affine patches $(G/K)_f$ where f is a K-invariant polynomial function. All that remains is to put all these patches together in a way that guarantees that G/K is affine. This is achieved in Lemma 4.2.

5. Affine Quotients and injectivity - We prove some results that imply a converse of the results of 4, i.e., if the orbit space G/K is affine then K is exact in G. This was proved in [2], our results are more general and refer to the case in wich X is an arbitrary variety and K is a group acting on X in such a way that X/K exists and is affine.

6. More on geometrically reductive groups - Here, using the concept of exact subgroup, we present a proof of the following: If G is geometrically reductive and K is a subgroup of G such that G/K is affine, then K is geometrically reductive. This is a particular case of transitivity results presented in 8. In the framework of section 8 the above statement looses its otherwise rather mysterious character.

We will describe sections 7 and 8 together. The definitions of these two sections have their origin in the following considerations. Let X be an affine variety and K an affine algebraic group acting on it. If K is geometrically (linearly) reductive $P(X)^K$ is finitely generated and (up to a closed subset of X) there is a "quotient" X/K. (See [9] and [10]). The definitions of geometrically (linearly) reductive group have to do with the action of K on any K-module algebra R. This is somewhat unnatural, one should be able to prove these results about orbits spaces using only properties of the action of K on P(X). This leads to the introduction of the concept of geometrically (linearly) reductive action of K on X; then we prove the results about orbit spaces in this case. When X is a group and K a closed subgroup acting by multiplication, the concept of linearly reductive action coincides with the concept of exact subgroup.

This paper was written while the author was a graduate student at U.C. Berkeley working under the direction of G. Hochschild. He suggested the possibility of obtaining new proofs of the results of [2] using the concept of observable subgroup. The author would like to thank Prof. Hochschild for his valuable suggestions and for allowing him to use his private notes on Invariant Theory.

1. Observable Subgroups

The concept of observable subgroup of an affine algebraic group was introduced in [1]. In this section we recall the main definitions and results.

Let G be an affine algebraic group defined over an algebraically closed field F, and let K be a closed subgroup of G. We denote by P(G) and P(K) the Hopf algebras of polynomial functions on G and K respectively. The restriction map π from P(G) to P(K) is a surjective Hopf algebra map. Now suppose that the algebraic group G acts on a vector space M by linear automorphisms. We say that M is a rational G-module if the following two conditions are satisfied:

a) For every $m \varepsilon M$, the space V_m generated by $\{x.m/x \varepsilon G\}$ is finite-dimensional.

b) For every $f \varepsilon V_m^*$, the functions $f/m: G \to F$, given by $(f/m)(g) = f(gm)$, are in $P(G)$.

The group G acts on P(G) from the left by $(x,f)(y)=f(yx)$ and from the right by $(f.x)(y)=f(xy)$ for $f \varepsilon P(G)$ $x,y \varepsilon G$.
The G-module P(G) is a rational G-module when endowed with either one of these actions.

Defn.1.1. Let G be an affine algebraic group and K a closed subgroup. We say that a rational character $\gamma: K \to F$ is extendable to G if there is a non zero element f of P(G) such that $x.f=\gamma(x)\, f$ for every $x \varepsilon K$.
It is easy to see that if there is such an f then there is another such, $\tilde{f}$ say, satisfying the additional requirement $\pi(\tilde{f})=\gamma$. In fact, we have $f(x) \neq 0$ for some element x of G, and we may take $\tilde{f}=f(x)^{-1}f.x$. It is also clear that the character γ is extendable to G if and only if there is an injective K-module map from $F\gamma$ to P(G), as $F\gamma$ is simple as a K-module we deduce that γ is extendable to G if and only if there is a finite-dimensional rational G-module M and an injective K-module map from $F\gamma$ to M. A standard argument that goes back to Chevalley and that is based on certain exterior algebra techniques gives us the following (cf.[1] or [7] Ch XII): For every finite dimensional rational K-module N, there is a finite-dimensional rational G-module M, a character ρ on K extendable to G, and an injective vector space homomorphism

$t:N\to M$ such that for every $x\varepsilon K$ and $n\varepsilon N$ $\rho(x)\ t(x.n) = x.t(n)$. In particular if γ is a rational character of K and we take $N=F\gamma$ we deduce the following:

Theorem 1.2. For every rational character γ of K there is a rational character ρ of K that is extendable to G and such that $\gamma\rho$ is extendable to G.

Defn.1.3. A subgroup K of G is said to be observable in G if given any finite-dimensional rational K-module N, there is a finite-dimensional rational G-module M and an injective K-module map $t:N\to M$.

From now on, we will drop the word rational unless there is danger of confusion.

It follows that K is observable in G if and only if every character of K is extendable to G. Evidently, this condition is necessary. In order to prove the sufficiency, consider a finite-dimensional K-module N, and construct M, t and ρ as above . If we call ρ^* the reciprocal character to ρ, there exists a non zero element u^* of P(G) such that $x.u^* = \rho^*(x)\ u^*$ for every x in K. Let us denote by $\langle Gu^*\rangle$ the subspace of P(G) generated by the translates of u^*, and consider the map $\tilde{t}:N\to M \otimes \langle Gu^*\rangle$ given by $\tilde{t}(n)=t(n)\otimes u^*$. Clearly $\tilde{t}$ is injective. Moreover

$\tilde{t}(x.n)=t(x.n)\otimes u^* = \rho^*(x)x.t(n)\otimes u^* = x.t(n)\otimes\rho^*(x)u^* = x.\tilde{t}(n)$

for every $x\varepsilon K$.

Defn.1.4. We shall denote the set of rational characters of K that are extendable to G by $E_G(K)$. The multiplicative group of all rational characters of K will be denoted by X(K), and X(K) coincides with the subgroup generated by $E_G(K)$.

Proof. Let γ_1 and γ_2 be extendable characters of K. Consider f_1 and f_2 their extensions to G, and take x_1 and x_2 elements of G such that $f_1(x_1)\neq 0$ and $f_2(x_2)\neq 0$. Consider the function g of P(G) defined as $g=(f_1.x_1x_2^{-1})f_2$. Then $g(x_2) = f_1(x_1)\ f_2(x_2) \neq 0$ and if $x\,\varepsilon\,K$ $x.g=x.(f_1.x_1x_2^{-1})x.f_2 = \gamma_1(x)\ \gamma_2(x)\ g$.

Thus $\gamma_1\ \gamma_2 \varepsilon E_G(K)$. Finally Theo.1.2. says that every character of K can be written as the quotient of two extendable ones.

Q.E.D.

Corollary 1.6. The following four conditions are equivalent:

a) The subgroup K is observable in G.

b) $E_G(K) = X(K)$.

c) For every element of $E_G(K)$ there is a $q>0$ such that $\rho^{*q}\varepsilon E_G(K)$.

d) For every element ρ of $E_G(K)$, the reciprocal ρ^* also belongs to $E_G(K)$.

Proof. a)$\Rightarrow$b) has already been proved. The implication b)$\Rightarrow$c) is obvious. To prove that c)$\Rightarrow$d) we take $\rho\varepsilon E_G(K)$ and $q>0$ such that $\rho^{*q}\varepsilon E_G(K)$. Then $\rho^{q-1}\ \rho^{*q}\varepsilon E_G(K)$, thus $\rho^*\varepsilon E_G(K)$. Finally condition d) says that $E_G(K)$ is a subgroup of X(K). By Lemma 1.5., this implies b).

Q.E.D.

The following result will be extremely useful to us in the sequel. The proof is in [1].

Theorem 1.7. Let G be an affine algebraic group and K a closed subgroup. Then, K is observable in G if and only if the homogeneous space G/K is quasi-affine.

2. Geometrically reductive subgroups and observability

Throughout, $S^q(V)$ will stand for the homogeneous component of degree q of the symmetric algebra built on V, and F will denote the base field.

Defn.2.1. An affine algebraic group K is said to be geometrically reductive if for every rational K-module V and every non-zero K-module map $\lambda: V \to F$, there is a $q>0$ and an x in $S^q(V)^K$ such that $S^q(\lambda)(x) = 1$, where $S^q(\lambda)$ is the map $S^q(V) \to F$ obtained from λ in the canonical fashion.

Defn.2.2. An affine algebraic group K is said to be linearly reductive if for every rational K-module V and every non-zero K-module map $\lambda: V \to F$, there is an x in V^K such that $\lambda(x)=1$.

Clearly condition 2.2 is verified if and only if the map λ splits as a K-module map, and analogously 2.1 is verified if and only if the map $S^q(\lambda)$ splits as a K-module map.

It is known from [5] and [10], that an affine algebraic group is reductive, in the sense that its unipotent radical is trivial, if and only if it is geometrically reductive.

It is also known after Nagata that, if char F=0, then the concepts defined in 2.1 and 2.2 coincide, and if char $F=p>0$, there are very few linearly reductive groups.

The following two results are standard, see [4] or [11], we will state and prove them here in order to have available references. The author was unable to find a proof of the implication a)$\Rightarrow$b) of Theorem 2.4. in the literature.

Theorem 2.3. The following four conditions are equivalent:

a) If R_1 and R_2 are K-module algebras, and $\phi:R_1\to R_2$ is a surjective K-module algebra map, then $\phi(R_1^K)=R_2^K$.

b) If $\lambda:M\to N$ is a surjective map of K-modules, then the restricted map $\lambda/M^K:\ M^K\to N^K$ is surjective.

c) The group K is linearly reductive

d) Every rational K-module is semisimple.

Proof. a)$\Rightarrow$b). Given $\lambda:M\to N$ consider S(M) and S(N), the symmetric algebras built on M and N respectively . If we apply the conclusion of a) to the map $S(\lambda):S(M)\to S(N)$ we deduce the conclusion of b). The implication b)$\Rightarrow$c) is evident. We omit the proof of c)$\Rightarrow$a), which is identical with the proof of the implication b)$\Rightarrow$a) of the next theorem. As to the equivalence of d) and b) we refer the reader to [4] because we won't need the proof in the rest.

Q.E.D.

Theorem 2.4. The following two conditions are equivalent:

a) If R_1 and R_2 are K-module algebras, and $\phi: R_1 \to R_2$ is a surjective K-module algebra map, then for every $r_2 \in R_2^K$, there is a $q>0$ and an $r_1 \in R_1^K$ such that $\phi(r_1) = r_2^q$.

b) The group K is geometrically reductive.

Proof. a)$\Rightarrow$b). Let λ, V and F be as in Defn . 2.1. and consider the map $S(\lambda)$: $S(V) \to S(F)$. The map $S(\lambda)$ is a surjective K-module algebra map. Thus as $1 \in S(F)^K$ there is a t in $S(V)^K$ and a $q>0$ such that $S(\lambda)(t)=1.1....1$ (where the dot indicates the product in S(V), and there are q factors 1). If we look at the part of degree q of t and call it t_q, we have that $t_q \in S^q(V)^K$ and $S(\lambda)(t_q)=1......1$. In the notation of Defn.2.1. $S^q(\lambda)(t_q)=\mu S(\lambda)(t_q)=1$ (where μ indicates multiplication). Thus, we have that t_q verifies the required properties.

b)$\Rightarrow$a). If $r_2=0$ the result is trivial. If $r_2 \neq 0$ consider $s \in R_1$ such that $\phi(s)=r_2$. Define M as the K-module generated by $\{xs/x \in K\}$ and M' as the K-module generated by $\{xs-s/x \in K\}$.Then $M=Fs+M'$ and the sum is direct, because $\phi(M')=(0)$ and $\phi(s) \neq 0$. Define the map $\lambda: M \to F$ by writing $m=\lambda(m)s+m'$, with $m' \in M'$. Then it is clear that λ is a surjective K-module map, and that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{\lambda} & F \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle u_{r_2}} \\ R_1 & \xrightarrow{\phi} & R_2 \end{array}$$

, where i is the inclusion map and $u_{r_2}(a) = ar_2$. Consider the q-th symmetric power of M, where q is as in b). We have the commitative diagram

$$\begin{array}{ccccc}
S^q(M) & \xrightarrow{S^q(\lambda)} & S^q(F) & \xrightarrow{\mu} & F \\
\downarrow S^q(i) & & \downarrow S^q(u_{r_2}) & & \downarrow u^q_{r_2} \\
S^q(R_1) & \xrightarrow{S^q(\phi)} & S^q(R_2) & \xrightarrow{\mu} & R_2 \\
& \searrow^{\mu} & & \nearrow_{\phi} & \\
& & R_1 & &
\end{array}$$

where μ is the multiplication map. Take $x \varepsilon S^q(M)^K$ such that $\mu S^q(\lambda)(x)=1$, and put $r_1=\mu S^q(i)(x)$. As μ and $S^q(i)$ are K-module maps, r_1 is K-invariant. Now, $\phi(r_1)=\mu S^q(\phi) S^q(i)(x)=$ $=u_{r_2^q}\ \mu S^q(\lambda)(x)=r_2^q(1)=r_2^q$.

Q.E.D.

We want to prove that if K is a geometrically reductive subgroup of G, then K is observable in G. First , we establish a Lemma that allows us to go from the irreducible case to the general one.

<u>Lemma 2.5</u>. Let K be a geometrically reductive group and K_1 a normal closed subgroup of finite index. Then K_1 is geometrically reductive.

<u>Proof</u>. Let V be a rational K_1-module and consider the F-space $F_{K_1}(K,V)$ of all functions $f:K\to V$ satisfying $f(xy)=x.f(y)$ $\forall x \varepsilon K_1$ $y \varepsilon K$. We make K act on $F_{K_1}(K,V)$ by $(z.f)(y)=f(yz)$. Now, for all y and z in K and all x in K_1, we have, $(z.f)(xy)=f(xyz)=x.f(yz)=x.(z.f)(y)$, so that if f belongs to $F_{K_1}(K,V)$ so does z.f. If V is a finite dimensional K_1-module, then $F_{K_1}(K,V)$ is a finite-dimensional K-module.It follows easily that $F_{K_1}(K,V)$ is a rational K-module.Consider

the map E_V: $F_{K_1}(K,V) \to V$ given by $E_V(f)=f(1)$. E_V is a K_1-module map, because $E_V(x.f)=(x.f)(1)=f(x)=f(x1)=x.f(1)$.

Now, let V be the trivial K_1-module F. Evidently $F_{K_1}(K,F)$ can be identifield with the K-module $F(K/K_1,F)$ of all maps from K/K_1 to F with the K-action given by $(z.f)(xK_1)=f(xzK_1)$, and the map E_F is given by $E_F(f)=f(1K_1)$. Let $\gamma: V \to F$ be a surjective morphism of K_1-modules, and consider the map γ^*: $F_{K_1}(K,V) \to F_{K_1}(K,F)$ given by $\gamma^*(f)=\gamma f$. The following diagram is commutative,

$$\begin{array}{ccc} F_{K_1}(K,V) & \xrightarrow{\gamma^*} & F_{K_1}(K,F) = F(K/K_1,F) \\ \Big\downarrow E_V & & \Big\downarrow E_F \\ V & \xrightarrow{\gamma} & F \end{array}$$

Let us make a coset decomposition $K=K_1x_1 \cup K_1x_2 \cup \ldots \cup K_1x_r$, with $x_1=1$, and choose $v_0 \in V$ such that $\gamma(v_0)=1$. Define $f_0: K \to V$ by $f_0(kx_i)=k.v_0$ for every element k of K_1.

It is clear from the very definition that $f_0 \in F_{K_1}(K,V)$.

Moreover, the function $\gamma^*(f_0)$ on K/K_1 is easily seen to be simply the constant function with value 1, which we shall denote by I. Now, consider the K-module M generated by f_0 in $F_{K_1}(K,V)$. The following diagram is commutative and the maps that land in F are surjective,

$$\begin{array}{ccc} M & \xrightarrow{\tilde{\gamma}^*} & F \\ \Big\downarrow E_V & \nearrow_{\gamma} & \\ V & & \end{array}$$

, where $\tilde{\gamma}^*$ denotes the composite of γ^* with the identification of FI with F. Observe that $\tilde{\gamma}^*$

is a K-map.

The diagram

$$\begin{array}{ccc} S^q(M) & \xrightarrow{S^q(\tilde{\gamma}^*)} & F \\ \downarrow{\scriptstyle S^q(E_V)} & \nearrow{\scriptstyle S^q(\gamma)} & \\ S^q(V) & & \end{array}$$

, is commutative for every $q>0$. If we choose q such that $S^q(\tilde{\gamma}^*)$ splits as a K-module map and call t the splitting map, the diagram shows that the K_1-module map $S^q(E_V)t$ splits $S^q(\gamma)$, because $S^q(\gamma)\ S^q(E_V)t = S^q(\tilde{\gamma}^*)t = id_F$.

Q.E.D.

<u>Theorem 2.6</u>. Let G be an affine algebraic group and K a closed subgroup. If K is geometrically reductive then K is observable in G.

<u>Proof</u>. Let G_1 be the connected component of the identity in G.Using [1], Theorem 6, we know that K is observable in G if and only if $K \cap G_1$ is observable in G_1. By Lemma 2.5. it is enough to prove Theorem 2.6. for the case in which G is connected. Let $\pi: P(G) \to P(K)$ be the restriction map and let ρ be a character in $E_G(K)$.There is a non zero element u of P(G) such that $\pi(u)=\rho$ and $x.u=\rho(x)u$ for every element x of K. Choose an element $\tilde{u}$ from P(G) such that $\pi(\tilde{u})=\rho^*$. Let $\langle K\tilde{u}\rangle$ denote the sub K-module of P(G) generated by $\tilde{u}$, and consider the composite F-linear map α: $Fu \otimes \langle K\tilde{u}\rangle \xrightarrow{\pi \otimes \pi} F\rho \otimes F\rho^* \xrightarrow{j} F$, where $j(\rho\otimes\rho^*)=1$. It is clear that α is a K-module map when F is endowed with the trivial K-module structure. As K is geometrically reduc-

tive and α is surjective there is a $q>0$ and an element t in $S^q(Fu \otimes \langle K\tilde{u}\rangle)^K$ such that $S^q(\pi\otimes\pi)(t) = (\rho\otimes\rho^*)^q$. Let us write $t=\sum_i (u\otimes f_{i1})(u\otimes f_{i2})\ldots(u\otimes f_{iq})$, where we use juxtaposition to indicate the product in the symmetric algebra, and the elements f_{ij} are in $\langle K\tilde{u}\rangle$. Set $v=\sum_i f_{i1}\ldots f_{iq}$, and $f = u^q v$. From the fact that t is K-fixed, it follows that f is K-fixed. The following computation proves that v is a ρ^{*q} - semiinvariant:

$x.f = (x.u)^q(x.v) = \rho^q(x)u^q(x.v) = u^q v$, thus, $x.v = \rho^{*q}(x)v$. We have $v(1)=\pi(v)(1) = \sum_i \pi(f_{i1})(1)\ldots\pi(f_{iq})(1)$. On the other hand we have that $\pi(f_{ij})=\lambda_{ij}\rho^*$, from $S^q(\pi\otimes\pi)(t)=(\lambda\otimes\lambda^*)^q$, we find that $\sum_i \lambda_{i1}\ldots\lambda_{iq}=1$, which shows that $v(1)=1$. Thus v is a non-zero ρ^{*q}-semiinvariant, showing that $\rho^{*q}\varepsilon E_G(K)$. Our result follows from Corollary 1.6.

Q.E.D.

3. Exactness and observability

Let G be an affine algebraic group over an algebraically closed field F, and let K be a closed subgroup. We can define induced representations in the category of rational modules as follows. Let N be an arbitrary (rational) K-module and let us endow $P(G)\otimes N$ with a left K-module structure in the usual (diagonal) way. The group G acts on the right on $P(G)\otimes N$ by $(f\otimes n).x = (f.x)\otimes n$ for $x\varepsilon G$. We endow $P(G)\otimes N$ with the left G-module structure associated to the right G-module structure given above. As the diagonal K action and the G action commute $(P(G)\otimes N)^K$ is a left G-submodule of the G-module $P(G)\otimes N$. We call the module $(P(G)\otimes N)^K$ the G-module induced by the K-module N.

Defn.3.1. We say that the subgroup K of G is exact if the induced representation functor is exact. As the induced representation functor is always left exact the meaningful part of the definition above is the following: if α: $M \to N$ is a surjective K-module map, then the restricted map $(id \otimes \alpha)$: $(P(G) \otimes M)^K \to (P(G) \otimes N)^K$ is surjective. It is immediate that K is exact in G if and only if, for every rational K-module M, one has $H^1(K, P(G) \otimes M) = 0$, where H^1 indicates the first rational cohomology group. It is also known that the condition $H^1(K, P(G) \otimes M) = 0$ for every M is equivalent to the assertion that P(G) is injective as a K-module. See [2] for the definition of exact subgroup and [2] or [3] for the results mentioned above.

If R is a K-module algebra we define the abelian category of (R,K)-modules as follows: the objects of the category are F-spaces M that are at the same time K- modules and R-modules, such that the actions are related by $x(rm) = (xr)(xm)$ for $x \in K$, $r \in R$, $m \in M$; the morphisms in this category are defined in the obvious way. We shall denote by M(R,K) the category of (R,K)-modules.

Lemma 3.2. Let G be an affine algebraic group and K a closed subgroup. Let F denote the fixed point functor from M(P(G),K) to the category of F-spaces. Then K is exact in G if and only if F is exact.

Proof. If M is a K-module then $P(G)\otimes M$ is a $(P(G),K)$-module. Therefore, the condition that F be exact implies that K is exact in G. Conversely, let M and N be $(P(G),K)$-modules and let $\alpha: M\to N$ be a surjective map in the corresponding category. If μ_M denotes the map from $P(G)\otimes M$ to M given by the action of $P(G)$ on M and μ_N indicates the corresponding map for N, then μ_M and μ_N are K-module maps, and the following diagram is commutative

$$\begin{array}{ccc} P(G)\otimes M & \xrightarrow{\mu_M} & M \\ \downarrow{\scriptstyle id\otimes\alpha} & & \downarrow{\scriptstyle \alpha} \\ P(G)\otimes N & \xrightarrow{\mu_N} & N \end{array}$$

. The maps s_M: $M\to P(G)\otimes M$ $s_M(m)=1\otimes m$ and s_N are also K-module maps that split μ_M and μ_N and also fit into a commutative diagram as follows:

$$\begin{array}{ccc} P(G)\otimes M & \xleftarrow{s_M} & M \\ \downarrow{\scriptstyle id\otimes\alpha} & & \downarrow{\scriptstyle \alpha} \\ P(G)\otimes N & \xleftarrow{s_N} & N \end{array}$$

. Taking K-fixed parts we get the following pair of commutative diagrams:

$$\begin{array}{ccc} (P(G)\otimes M)^K & \rightleftarrows & M^K \\ \downarrow{\scriptstyle id\otimes\alpha} & & \downarrow{\scriptstyle \alpha} \\ (P(G)\otimes N)^K & \rightleftarrows & N^K \end{array}$$

, where the horizontal maps are the restrictions of μ and s. From the diagram follows immediately that if $(id\otimes\alpha)(P(G)\otimes M)^K)=(P(G)\otimes N)^K$, then $\alpha(M^K)=N^K$.

Q.E.D.

The concept of exact subgroup of an affine algebraic group was introduced in [2]. Later in this paper we will reinterpret this concept and connect it with a certain generalization of the notion of linearly reductive group. In [2] it was proved that K is exact in G if and only if the homogeneous space G/K is affine. Their proof that K exact in G implies that G/K is affine goes along the following lines. Using the fact that every reductive group is geometrically reductive (see [5]), they prove that G/K is affine if and only if G/K_u is affine, where K_u is the unipotent radical of K. Next they prove that if U is a unipotent subgroup of G, then U is exact if and only if G/U is affine. (See [2], Theorem 3.1, Theorem 4.3 and Lemma 4.1). In this section and the next we present a proof that K exact in G implies G/K affine that does not use the results of [5] and is representation theoretical in spirit.

First, we need to know how to pass from the case in which the group G is irreducible to the general one. This is acomplished using Lemma 3.3.

Lemma 3.3. Let K be an exact subgroup of G and K_1 a normal connected subgroup of K of finite index. Then K_1 is exact in G.

Proof. The induction functor is transitive. This means that if K and L are closed subgroups of G such that $K \subset L \subset G$, and if we denote by $V|^L$ the L-module induced by the

K-module V we have $V|^L|^G = V|^G$. Thus, in order to prove that K_1 is exact in G it is enough to prove that K_1 is exact in K. Consider the decomposition of K into a finite number of cosets module K_1, as follows $K=K_1x_1 \cup K_1x_2 \ldots \cup K_1x_r$, with $x_1=1$. Let $P \doteq \{f \in P(K)/f/K_1=0\}$ and put $Q=\bigcap_{i\neq 1} P.x_i^{-1}$.

The set of zeros of Q can be computed as:

$Z(Q) \doteq \{x \in K/(f.x_i^{-1})(x)=0 \ \forall \ f \in P \ i=2\ldots r\}=$

$\doteq \{x \in K/f(x_i^{-1}x)=0 \ \forall \ f \in P \ i=2\ldots r\}=$

$\doteq \{x \in K/x_i^{-1} \ x \in K_1 \text{ for some } i=2,\ldots,r\} = \bigcup_{i\neq 1} x_i \ K_1 = \bigcup_{i\neq 1} K_1 \ x_i.$

Thus Q is an ideal that is K_1 invariant and $P+Q=P(K)$. This shows that the restriction map $\pi:P(K)\to P(K_1)$ splits as a K_1-module algebra map. Using a result due to Hochschild whose proof (in a more general context) can be found in [3], we deduce that P(K) is injective as a K_1-module, which implies that K_1 is exact in K. If we don't want to use that result, we proceed in the way indicated above to decompose P(K) as a direct sum of algebras $A_1 \oplus \ldots \oplus A_n$ such that A_1 is isomorphic to $P(K_1)$, every A_i is stable under translations by K_1, every translation effected by an element of K permutes the A_i's, and K acts transitively on the set of A_i's from the right as well as from the left. Thus, if we write $1=f_1+\ldots+f_n$ with $f_i \in A_i$, then the Hopf algebra $Ff_1+\ldots+Ff_n$ can be identified with the algebra of polynomial functions on K/K_1. It is clear from the construction above that $P(K) \cong$ $\cong P(K_1) \otimes P(K)^{K_1}$. Hence, as $P(K_1) \otimes V$ with the diagonal K_1-module structure is isomorphic to $P(K_1) \otimes V$ with the trivial

K_1-structure on V and the usual on $P(K_1)$, we deduce that $(P(K_1)\otimes V)^{K_1} \cong P(K_1)^{K_1} \otimes V \cong V$.
Thus, $(P(K)\otimes V)^{K_1} = P(K)^{K_1}\otimes(P(K_1)\otimes V)^{K_1} = P(K)^{K_1}\otimes V$, shows that K_1 is exact in K.

Q.E.D.

Note: In the situation above, $P(K_1)$ is a direct K_1-module summand of P(K).

Theorem 3.4. Let G be an affine algebraic group and let K be a closed subgroup of G. Then K exact in G implies that K is observable in G.

Proof. Let G_1 denote the connected component of the identity in G, and set $K'=K\cap G_1$. The fact that K is exact in G implies that K'is exact in G_1. Indeed, from Lemma 3.3 we have that P(G) is injective as a K'-module. As we noticed before, $P(G_1)$ is a direct G_1-module summand of P(G). A fortiori, $P(G_1)$ is a direct K'-module summand of P(G). Therefore, $P(G_1)$ is injective as a K'-module. By the same considerations we made at the beginning of the proof of Theorem 2.6., it is enough to prove our Theorem in the case where G is connected. Consider a character ρ of K that is extendable to G, and let u be an extension of ρ that restricted to K coincides with ρ. Take $\tilde{u}\in P(G)$ such that $\pi(\tilde{u}) = \rho^*$ (π denotes the restriction map from G to K), and consider the K-module $\langle K\tilde{u}\rangle$ generated by $\tilde{u}$ in P(G). Let α denote the map $\pi\otimes\pi$: $Fu \otimes \langle K\tilde{u}\rangle \to F\rho \otimes F\rho^*$.

Clearly, α is a surjective K-module map. Using the exactness of K in G, we deduce that the map $(id \otimes \alpha):(P(G)\otimes Fu\otimes\langle K\tilde{u}\rangle)^K \to (P(G)\otimes F\rho\otimes\rho^*)^K$ is surjective. The element $1\otimes\rho\otimes\rho^*$ is K-invariant. Therefore, there is an element $t = \Sigma f_i \otimes u \otimes g_i \in (P(G) \otimes Fu \otimes \langle K\tilde{u}\rangle)^K$ such that $\Sigma f_i \otimes \rho \otimes \pi(g_i) = 1 \otimes \rho \otimes \rho^*$. Now, $\pi(g_i)=\lambda_i\rho^*$, with $\Sigma\lambda_i f_i = 1$. Now consider the element $f=u\Sigma f_i g_i$ of $P(G)$. As t is K-invariant, f is in $P(G)^K$. Moreover, $\Sigma f_i(1)g_i(1) = \Sigma f_i(1)\pi(g_i)(1) = \Sigma f_i(1)\lambda_i = 1$. Consequently, if we put $v=\Sigma f_i g_i$, we have

$uv=f = x.f = x.u\ x.v = \rho(x)u\ (x.v)$, whence $x.v=\rho^*(x)v$.

This shows that the character ρ^* is extendable to G. Thus K is observable in G by Corollary 1.6.

Q.E.D.

4. Existence of Affine Quotients

We prove that, in the cases studied above, where K is geometrically reductive or K is exact in G, the orbit space G/K is not only quasi-affine but actually affine. This is a consequence of the following algebraic lemma.

Lemma 4.1. Let K be an observable subgroup of G. In addition, suppose that, for every proper ideal I of $P(G)^K$, the ideal $IP(G)$ is a proper ideal of $P(G)$. Then G/K is affine.

Proof. As G/K is quasi-affine, there is a non-zero element f in $P(G)^K$ such that the corresponding principal open set $(G/K)_f$ is affine. Consider the ideal I of

$P(G)^K$ that is generated by the elements f.x with x in G. The ideal IP(G) cannot have any zero in G, because if there is a y_0 in G such that $(f.x)(y_0) = 0$ for every x in G, the function f is zero on $Gy_0=G$. Consequently IP(G)=P(G). By assumption , we must therefore have $I=P(G)^K$. Thus we can find a finite set of points $x_1,\ldots,x_r$ in G such that the ideal generated by the $f.x_i$'s is all of $P(G)^K$. Now $(G/K)_{f.x_i} = x_i^{-1}\ (G/K)_f$, so that $(G/K)_{f.x_i}$ is affine for each i=1,...,r. From this it follows that G/K is affine.

Q.E.D.

At the end of the proof of Lemma 4.1 we made use of the following result: Let X be a quasi-affine variety and $f_1,\ldots,f_n$ elements of P(X) the algebra of everywhere defined regular functions, such that:

a) The functions f_i generate the unit ideal of P(X).

b) The open sets X_{f_i} are affine for every i. Then X is an affine variety. The proof proceeds as follows. First we observe that the algebra of polynomial functions on X_{f_i} is isomorfic with the localization, $P(X)_{f_i}$ of P(X) with respect to the multiplicative set of the powers of f_i. Using that $P(X)_{f_i}$ is finitely generated as an algebra over the base field and condition a) we deduce that P(X) is finitely generated as an algebra. Then the cannonical map from X to the Spect (P(X)) is injective because X is quasi-affine, and is an isomorfism on every open set X_{f_i}.

The next Lemma appeared in a slightly different form in [9]. We need to have a proof avaible for future reference and we include it here.

Lemma 4.2. Let K be a geometrically reductive affine algebraic group, and let R be an arbitrary K-module algebra. If I is an ideal of R^K such that $IR=R$, then $I=R^K$.

Proof. Let $1=f_1r_1+\dots+f_nr_n$ with $f_i \varepsilon I$, $r_i \varepsilon R$. We shall prove by induction on n that the ideal generated by the f_i's in R^K coincides with R^K. If n=1 from the equality $1=f_1r_1$ we deduce that for every $x \varepsilon K$, $1=f_1(x.r_1)$. Thus $x.r_1-r_1 \varepsilon \mathrm{Ann}(f_1)$, and then $r_1+\mathrm{Ann}(f_1) \varepsilon (R/\mathrm{Ann}(f_1))^K$. By Theorem 2.4. there is a $t \varepsilon R^K$ such that $t-r_1^q \varepsilon \mathrm{Ann}(f_1)$. Then $tf_1=r_1^q f_1$, thus, if we raise the equality $1=f_1r_1$ to the q-th power we get $1=f_1^q\, r_1^q = f_1^{q-1}t\, f_1 = f_1(tf_1^{q-1})$. Thus $I=R^K$. For a general n, since Rf_1 is K-stable, we may apply the inductive hypothesis to the cannonical image of I in $(R/Rf_1)^K$. This shows that $1+f_1R = s_2f_2+\dots+s_nf_n+f_1R$ where $s_i+f_1R \;\varepsilon\; (R/f_1R)^K$. By Theorem 2.4., there are elements $\tilde{s}_i$ in R^K such that $\tilde{s}_i-s_i^k \;\varepsilon\; f_1R$, for some $k>0$. There is an r in R such that $1-f_1r = s_2f_2+\dots+s_nf_n$. Raising $1-f_1r$ to a convenient power we deduce that there are elements $\bar{s}_2,\dots,\bar{s}_n \;\varepsilon\; R^K$ and $\bar{r} \varepsilon R$ such that $1=f_1\bar{r} + \bar{s}_2f_2 + \dots + \bar{s}_nf_n$. Thus if $x \varepsilon K$ we have that $1 = f_1(x.\bar{r})+\bar{s}_2f_2+\dots+\bar{s}_nf_n$. This shows that $x.\bar{r} - \bar{r}$ belongs to the annihilator, J say, of f_1 in R. Then the element $\bar{r}+J \varepsilon (R/J)^K$. Applying Theorem 2.4 we deduce the existence of an element $s \varepsilon R^K$ and a $q>0$ such that $\bar{r}^q - s \varepsilon J$. Thus $\bar{r}^q f_1^q = s\, f_1^q$. Raising the equality $1-\bar{s}_2f_2-\ldots-\bar{s}_nf_n = f_1\bar{r}$ to the q-th power we deduce that $1=r_1'f_1+\ldots+r_n'f_n$ with $r_i'\varepsilon R^K$. Thus $I=R^K$.

Q.E.D.

From Lemma 4.2. Lemma 4.1. and the results of Section 2 we deduce the following result.

Theorem 4.3. Let G be an affine algebraic group and K a closed geometrically reductive subgroup of G. Then the quotient space G/K is affine.

A relative criterion for a quotient space to be affine is as follows.

Theorem 4.4. Let G be an affine algebraic group and K a closed subgroup that is exact in G, then the quotient space G/K is affine.

Proof. Let I be an ideal of $P(G)^K$ such that $P(G)I=I$. Consider $a_1,\ldots,a_r$ elements of I such that $a_1f_1+\ldots+a_rf_r=1$ for some $f_i\varepsilon P(G)$. Look at the map $\phi: \bigoplus_{i=1}^{r} P(G) \to P(G)$ given by $\phi(\tilde{f}_1,\ldots,\tilde{f}_r)=\Sigma a_i\tilde{f}_i$. The map ϕ is a surjective $(P(G),K)$-module map. Thus, as K is exact in G, we deduce, using Lemma 3.2., that $\phi(\bigoplus_{i=1}^{r} P(G)^K) = P(G)^K$. In particular $I=P(G)^K$.

Q.E.D.

5. Affine Quotients and Injectivity

The main purpose of this section is to prove the converse of Theorem 4.4. We consider a somewhat more general situation with a view to the generalizations of Section 8. In the case where G is a group and K a closed subgroup, the methods used here were developped in [2], §4, to obtain similar results. Using the relationship between the sheaf cohomology of the homogeneous space G/K and the rational cohomology of certain K-modules, Habousch in [6], claims to prove Corollary 5.5 below. However, his proof seems incomplete to the author.

Let F be a fixed algebraically closed field. All our varieties will be defined over F, and their "points" will be undestood to be F-rational points. Let X be an affine variety, and let K be an affine algebraic group acting on X from the right in such a way that the action $X \times K \to X$ is a morphism of affine varieties.

Defn.5.1. We say that an orbit variety for the action of K on X exists if there is a pair (Y,τ) where Y is an algebraic variety and τ a surjective open morphism from X to Y such that:

a) $\tau(x) = \tau(x')$ if and only if $xK = x'K$.

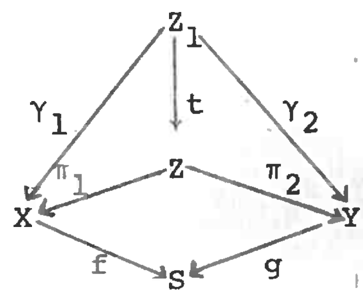
b) If U is an open subset of X the map $\tau^*\colon 0_Y(\tau(U)) \to 0_X(U)$ is injective and its image consists of all elements f of $0_X(U)$ that are constant on sets of the from $xK \cap U$ for $x \in K$. If the orbit variety exists, it is unique up to isomorphisms and we denote it by X/K.

Theorem 5.2. Suppose there is an abstract group L actings transitively on X from the left as a group of variety automorphisms in such a way that the actions of L and K commute. If the orbit variety X/K exists and is affine, then the algebra P(X) is faithfully flat as a ${}^{K}P(X)$-module, where ${}^{K}P(X)$ denotes the K-fixed part of P(X).

Proof. Consider the inclusion ${}^{K}P(X) \subset P(X)$. It is an easy consequence of Noether's normalization theorem that there is an f in ${}^{K}P(X)$ such that P(X) is free as a ${}^{K}P(X)_f$-module. For every element l of L, we have $f.l \in {}^{K}P(X)$, and $P(X)_{f.l}$ is free as a ${}^{K}P(X)_{f.l}$-module. As the action of L on X is, transitive, it follows that the ideal generated by the f.l's, with l ranging over L, coincides with ${}^{K}P(X)$. Hence, there is a finite set $(f_1,\ldots,f_n)$ of elements of ${}^{K}P(X)$ such that $P(X)_{f_i}$ is free as a ${}^{K}P(X)_{f_i}$-module and moreover the ideal generated by the f_i's is all of ${}^{K}P(X)$. By standard commutative algebra, this implies that P(X) is faithfully flat as a ${}^{K}P(X)$-module.

Q.E.D.

We recall the definition of fiber products of algebraic varieties. Given a pair of varieties X and Y and maps $f:X\to S$ $g:Y\to S$ into another variety S, the product of X and Y over S, denoted by $X \times_S Y$ is a triple (Z,π_1,π_2) where Z is an algebraic variety, π_1 and π_2 are morphisms of varieties from Z to X_1 and X_2 respectively that verify $f\pi_1 = g\pi_2$, and such that if (Z_1,γ_1,γ_2) is another triple as above, there is one and only one map $t:Z_1\to Z$ that makes the diagram below commutative.



The map t will be denoted by $\gamma_1 \times_S \gamma_2$. If X,Y and S are affine varieties given by X=Spect(A), Y=Spect(B) and S=Spect(C), then also Z is affine, being given by Spec $(A\otimes_C B)$.

Let X be an affine variety and K a group acting on X as in Defn.5.1. Consider the fiber product $X \times_{X/K} X$ coming from the cannonical morphism $\tau: X \to X/K$. Clearly $X \times_{X/K} X$ can be identified with the subset of X x X given by the condition $\{(x_1,x_2) \in X \times X / x_1K = x_2K\}$. Then if we define the map $\alpha: X \times K \to X \times X$ by $\alpha(x,k)=(x,xk)$, it is clear that the image of α in X x X is $X \times_{X/K} X$. For the rest of this section we will consider only such actions of K on X for which the map α defined above is an isomorphism onto its image. This implies that K acts on X without fixed points and that for every orbit T the map $f: T \times T \to K$ given by $f(x_1,x_2)=k$ if $x_1=x_2k$ is a polynomial map. Under these hypotheses $\alpha: X \times K \to X \times_{X/K} X$ is an isomorphism of affine varieties. It induces an isomorphism $\alpha^*: P(X) \otimes_{K_{P(X)}} P(X) \to P(X) \otimes P(K)$. Moreover if χ denotes the P(K)-comodule structure on P(X) and μ is the multiplication of the algebra P(X), then the map α^* is given by $\alpha^*=(\mu\otimes id)(id\otimes_{K_{P(X)}} \chi)$ or; explicitly, if $t=\Sigma f_i \otimes_{K_{P(X)}} g_i$ and $\chi(g_i)=\Sigma g_{ij} \otimes h_{ij}$, then $\alpha^*(t) = \Sigma f_i g_{ij} \otimes h_{ij}$.

If we let K act on $P(X) \otimes_{K_{P(X)}} P(X)$ trivially on the first factor and with the induced action on the second factor, and on $P(X) \otimes P(K)$ trivially on the first factor and with the induced action on the second; the map α^* is a left K-module map.

<u>Theorem 5.3</u>. Let X be an affine variety and let K be an affine algebraic group acting on X in such a way that the map $\alpha(x,k)=(x,xk)$ is an isomorphism onto its image. Suppose, moreover, that the orbit variety X/K exists and is affine, and that there is a group L acting on the left in a transitive way as a group of automorphisms of X commuting with the K-action. Then P(X) is injective as a K-module.

<u>Proof</u>. It is well known (see [3] or [8]) that P(X) is injective as a K-module if and only if the functor $W \to {}^{K}(P(X) \otimes W)$ is exact where we regard $P(X) \otimes W$ as a K-module with the diagonal left K-action. Let $W_1 \xrightarrow{\gamma} W_2$ be a surjective K-module map. We want to examine the map ${}^{K}(P(X) \otimes W_1) \xrightarrow{id \otimes \gamma} {}^{K}(P(X) \otimes W_2)$. An elementary computation shows that ${}^{K}(P(X) \otimes W_i) \cong W_i$, where the isomorphism is given by $\Sigma f_k \otimes w_k \to \Sigma f_k(1) w_k$, and the K-module structure on $P(X) \otimes W_i$ is the diagonal structure.

Thus, from the surjectivity of γ, we deduce that the map $P(X) \otimes {}^{K}(P(K) \otimes W_1) \xrightarrow{id \otimes id \otimes \gamma} P(X) \otimes {}^{K}(P(K) \otimes W_2)$ is surjective. If we endow P(X) with the trivial left K-module action we deduce that the map

${}^{K}(P(X) \otimes P(K) \otimes W_1) \xrightarrow{id \otimes id \otimes \gamma} {}^{K}(P(X) \otimes P(K) \otimes W_2)$ is surjective. Using the fact that α^* is a K-module map we deduce that the map

${}^{K}(P(X) \otimes_{K_{P(X)}} P(X) \otimes W_1) \xrightarrow{id \otimes id \otimes \gamma} {}^{K}(P(X) \otimes_{K_{P(X)}} P(X) \otimes W_2)$

is surjective provided we endow the tensor products above with the diagonal left K-module structure by making K act trivially on the first tensor factor and with the given actions on the second and third factor. Thus the map

$P(X) \otimes_{K_{P(X)}} {}^{K}(P(X) \otimes W_1) \xrightarrow{id \otimes id \otimes \gamma} P(X) \otimes_{K_{P(X)}} {}^{K}(P(X) \otimes W_2)$

is surjective. Now, as P(X) is faithfully flat as a $K_{P(X)}$-module, we deduce that the map

${}^{K}(P(X) \otimes W_1) \xrightarrow{id \otimes \gamma} {}^{K}(P(X) \otimes W_2)$, is surjective.

Q.E.D.

Now suppose that X is an affine algebraic group G and that K is a closed subgroup of G. All the hypotesis of Theorem 5.3. are satisfield, and we deduce the following result.

<u>Theorem 5.4</u>. Let G be an affine algebraic group and K a closed subgroup of G. If G/K is affine, then P(G) is injective as a K-module, and the induced representation functor is exact.

6. More on Geometrically Reductive Groups

Here we prove that, if G is a geometrically reductive affine algebraic group and K a closed subgroup of G such that the quotient G/K is affine, then K is geometrically reductive.

This result was first proved by Bialinicki-Birula in the case of characteristic zero. Later, Richardson in [12] and Habousch in [6] presented proofs for arbitrary characteristic. Our proof relies on the concept of exactness and is similar to the one in [6]. In section 7, we will reinterpret this theorem as a particular case of a transitivity theorem.

Theorem 6.1. Let G be an affine algebraic group and K a closed subgroup of G such that the quotient G/K is affine. Then, if G is geometrically reductive, so is K.

Proof. Let V be an arbitrary K-module, and let $\gamma: V \to F$ be a surjective K-map. Applying the induced representation functor to γ we get a map $\tilde{\gamma}: (P(G) \otimes V)^K \to (P(G) \otimes F)^K = P(G)^K$ given by $\tilde{\gamma}(\Sigma f_i \otimes v_i) = \Sigma \gamma(v_i) f_i$.

Consider the map $E_V: (P(G) \otimes V)^K \to V$ given by $E_V(\Sigma f_i \otimes v_i) = \Sigma f_i(1) v_i$. The K-fixed part $(P(G) \otimes V)^K$ is taken with respect to the left diagonal action and the G-action on $(P(G) \otimes V)^K$ is : $g.(\Sigma f_i \otimes v_i) = \Sigma f_i . g^{-1} \otimes v_i$. The following calculation shows that E_V is a K-module map.

$\Sigma k^{-1}.f_i \otimes k^{-1}.v_i = \Sigma f_i \otimes v_i$ implies that

$\Sigma f_i(k^{-1}) k^{-1}.v_i = \Sigma f_i(1) v_i$, whence $\Sigma f_i(k^{-1}) v_i = k.E_V(\Sigma f_i \otimes v_i)$.

Now we have $E_V(k.\Sigma f_i \otimes v_i) = E_V(\Sigma f_i . k^{-1} \otimes v_i) = \Sigma f_i(k^{-1}) v_i =$

$= k.E_V(\Sigma f_i \otimes v_i)$.

The diagram

$$\begin{array}{ccc} (P(G)\otimes V)^K & \xrightarrow{\tilde{\gamma}} & P(G)^K \\ \downarrow{\scriptstyle E_V} & & \downarrow{\scriptstyle E} \\ V & \xrightarrow{\gamma} & F \end{array}$$

is commutative where E is the map given by the evaluation at 1. As K is exact in G there is a $t\varepsilon(P(G) \times V)^K$ such that $\tilde{\gamma}(t)=1$, where 1 denotes the constant function of value 1 on G. If we denote by $\langle Gt\rangle$ the sub G-module of $(P(G)\otimes V)^K$ generated by t, the map $\tilde{\gamma}$ can be considered as a G-module map from $\langle Gt\rangle$ to F, and with that interpretation the diagram

$$\begin{array}{ccc} \langle Gt\rangle & \xrightarrow{\tilde{\gamma}} & F \\ \downarrow{\scriptstyle E_V} & \nearrow{\scriptstyle \gamma} & \\ V & & \end{array}$$

, is commutative.

As G is geometrically reductive, there is a $q>0$ and an $x\varepsilon S^q(\langle Gt\rangle)^G$ such that $S^q(\tilde{\gamma})(x)=1$. Let $y=S^q(E_V)(x)$. Then $y\varepsilon S^q(V)^K$, and $S^q(\gamma)(y) = S^q(\gamma)S^q(E_V)(x) = S^q(\tilde{\gamma})(x)=1$.

Q.E.D.

Using Theorem 6.1., we can answer the following question. What groups are universally exact ? In other words, what are the affine algebraic groups K such that, whenever K is embedded in an arbitrary G as a closed subgroup, the quotient G/K is affine ? The answer is "The (geometrically) reductive groups". Indeed, we know from Section 4 that geometrically reductive groups are universally exact. Now let K be a universally exact group. We can imbed K in a GL(n,F) as a closed subgroup . Now K is exact in GL(n,F), and since GL(n,F) is geometrically reductive, it follows that K is geometrically reductive.

It is an open problem to characterize the universally observable groups.

7. Invariant Theory

In this section, we establish results on rings of invariants that we use in Section 8 for studying orbits spaces. We use a degree of generality that allows us to clarify and unify certain results.

Let K be an affine algebraic group defined over an algebraically closed field F and let C be an arbitrary abelian subcategory of the category of all rational K-modules. An F-algebra object in C is an object in C that is at the same time a commutative algebra with identity. We assume throughout that, if R is an algebra object in C and $r \varepsilon R^K$, then the set Rr is an object of C and the maps $R \to Rr \to R$ defined respectively as multiplication by r and set theoretical inclusion are C-maps.

As we saw in Theorems 2.3 and 2.4, the following two (alternative) axioms are highly relevant to the reductivity properties of K.

Axiom A. Let R_1 and R_2 be F-algebra objects of C and let ϕ be a surjective algebra map from R_1 to R_2 that is also a morphism of C. Then, for every $r_2 \varepsilon R_2^K$, there is a $q>0$ such that $r_2^q \varepsilon \phi(R_1^K)$.

Axiom B. Let R_1, R_2 and ϕ be as in Axiom A. Then $\phi(R_1^K) = R_2^K$.

Note that, in the case where C is the category of all rational K-modules, Axiom B holds if and only if K is linearly reductive, and Axiom A holds if and only if K is geometrically reductive (Theorems 2.3 and 2.4). It is also clear that if K is geometrically reductive then Axiom A holds for any category C as above, and similarly for Axiom B. Conversely if K satisfies Axiom A for every category C as above, then K is geometrically reductive (and similarly for Axiom B and linear reductivity). It will became clear in section 8 how to construct an affine algebraic group K that is not geometrically reductive, but satisfies Axiom A on a certain category C of rational K-modules.

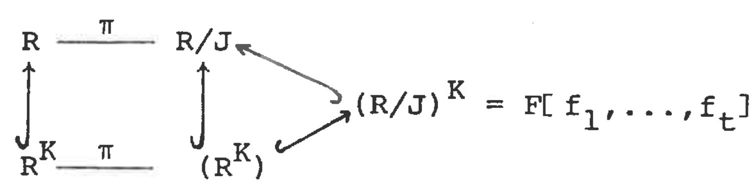
We will prove that, if C is a category as above for which Axiom A holds, and that satisfies some mild additional hypothesis, then, R^K is finitely generated for every finitely generated F-algebra object R of C.

We need some preparation for proving that result. Let R be an F-algebra object in C. If $r\varepsilon R^K$, then the map $R\to Rr$ has kernel $I=\{s\varepsilon R/sr=0\}$ that is (by our assumptions) an object of C. The map $R\to R/I$ is a C-map. In this situation, the following analogous of Lemma 4.2. is valid.

<u>Lemma 7.1</u>. Let C be an abelian category as above that satisfies Axiom A. Let R be an algebra object of C. If I is an ideal of R^K such that $IR=R$ then $I=R^K$. Note that we do not assume that I is an object of C. The proof of this result is identical with that of Lemma 4.2.

<u>Lemma 7.2</u>. Let C be an abelian category of K-modules satisfying Axiom A. Let R be an algebra object of C, and let J be an ideal of R that is also an object of C and such that the inclusion $J \subset R$ is a morphism of C. If $(R/J)^K$ is finitely generated as an F-algebra, then so is $R^K/J \cap R^K$.

<u>Proof</u>. Consider the commutative diagram



where all the non horizontal maps are inclusions. As R and R/J are algebra objects of C and π is a surjective algebra homomorphism that is also a morphism of the category C, Axiom A yields the existence of exponents $q_1,\ldots,q_t$ such that $T=F[f_1^{q_1},\ldots,f_t^{q_t}] \subset \pi(R^K)$. Now let us look at $T \subset \pi(R^K) \subset F[f_1,\ldots,f_t]$. Clearly $F[f_1,\ldots,f_t]$ is finitely generated as a T-module. A fortiori $F[f_1,\ldots,f_t]$ is finitely generated as a $\pi(R^K)$-module. A well known Lemma due to Artin and Tate says that, in this case, $\pi(R^K)$ is finitely generated as an F-algebra. Now the kernel of the map $\pi: R^K \to R/J$ is $J \cap R^K$. Thus $\pi(R^K) \cong R^K/J \cap R^K$.

Q.E.D.

Lemma 7.3. Let C be an abelian category of K-modules satisfying Axiom A. Let R be an algebra object of C and, for every element r of R, let I(r) denote the annhilator of r in R. Suppose that there exists an $r \varepsilon R^K$ such that $(R/Rr)^K$ and $(R/I(r))^K$ are finitely generated F-algebras. Then R^K is finitely generated as an F-algebra.

Proof. Using Lemma 7.2. for J=Rr and J=I(r) respectively we deduce that $R^K/Rr \cap R^K$ and $R^K/I(r) \cap R^K$ are finitely generated as F-algebras. Thus, we may write

$R^K = F[u_1,\dots,u_t] + Rr \cap R^K \quad R^K=F[v_1,\dots,v_s] + I(r) \cap R^K$.

Put $S = F[u_1,\dots,u_t,v_1,\dots,v_s]$. In proving Lemma 7.2. we saw that $(R/I(r))^K$ is finitely generated as an $R^K/I(r) \cap R^K$-module. Accordingly, we choose elements $c_1,\dots,c_l \varepsilon R$ such that the elements $c_i+I(r)$ of $R/I(r)$ are K-fixed and generate $(R/I(r))^K$ as an $R^K/I(r) \cap R^K$-module. Now, if $x \varepsilon K$, then $xc_i - c_i \varepsilon I(r)$, whence $c_i r \varepsilon R^K$. We shall prove that $F[u_1,\dots,u_t,v_1,\dots,v_s,c_1r,\dots,c_lr]$ coincides with R^K. Let $r_0 \varepsilon R^K$. There is an element s of S such that $r_0-s \varepsilon (Rr) \cap R^K$, i.e., $r_0-s=ar \varepsilon R^K$ with $a \varepsilon R$. Now $(xa-a)r=x(ar) - ar = 0$. Thus, xa-a is in I(r), so that $a+I(r) \varepsilon (R/I(r))^K$. By the definition of the c_i's, there is an element b in $Sc_1+\dots+Sc_l$ such that $a-b \varepsilon I(r)$. Now $ar=br \varepsilon S[c_1r,\dots,c_lr]$. Thus $r_0-s \varepsilon S[c_1r,\dots,c_lr]$, whence $r_0 \varepsilon F[u_1,\dots,u_t,v_1,\dots,v_s,c_1r,\dots,c_lr]$.

Q.E.D.

Theorem 7.4. Let C be an abelian category of K-modules satisfying Axiom A. Let R be an algebra object of C that is finitely generated as an F-algebra and graded in such a way that the action of K preserves the grading. Then R^K is finitely generated as an F-algebra.

Proof. Let F denote the family of all those homogeneous ideals of R which are also objects of C and have the property that $I \hookrightarrow R$ is a morphism of C and $(R/I)^K$ is not finitely generated as an F-algebra. We wish to prove that $F=\phi$. We do this by deriving a contradiction from the assumption $F \neq \phi$. As R is noetherian, we have a maximal element I_m in F. Then $R_m = R/I_m$ is again an algebra object of C that is finitely generated as an F-algebra, is graded and such that the action of K preserves the grading. The ring R_m^K is not finitely generated, but $(R_m/J)^K$ is finitely generated for every non zero homogeneous ideal J that is in C and such that $J \hookrightarrow R_m$ is a map in C. Replacing R with R_m, we achieve that R^K is not finitely generated but $(R/J)^K$ is finitely generated for every ideal J as above. Let r be an homogeneous non zero element of R^K. If the annihilator I(r) of r in R is not zero, then Lemma 7.3. gives a contradiction. Thus, we may assume that r is not a zero divisor. Consider J=Rr. This is a non zero homogeneous ideal that is also an object in C, and the inclusion $J \hookrightarrow R$ is a map in C. Thus, $(R/J)^K$ is finitely generated. By Lemma 7.2., this implies that $R^K/J \cap R^K$, i.e. , $R^K/R^K r$ is finitely generated. There are homogeneous elements

$v_1,\ldots,v_k$ in R^K whose images in R^K/R^Kr generate R^K/R^Kr as an F-algebra. We prove by induction on the degree that $R^K=F[v_1,\ldots,v_k,r]$. Let $s\varepsilon R^K$ and put e=deg(s). By definition of the v_i's, we have $s=p(v_1,\ldots,v_k)+tr$ where p is a polynomial and $t\varepsilon R^K$. If e'=deg(r) we have $s=p(v_1,\ldots,v_k)_e+t_{e-e'}r$. By induction we have that $t_{e-e'}\varepsilon F[v_1,\ldots,v_k,r]$, then our conclusion follows.

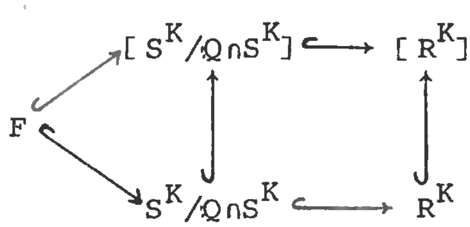
Q.E.D.

To prove the theorem in the case of an arbitrary algebra object of C (not necessarily graded) we have to add multiplicative conditions to our category C.

Defn. 7.5. An abelian subcategory of the category of rational K-modules is said to be multiplicative if, for every algebra object R of C, the algebra $R\otimes_F R$ is again an element of C and the multiplication map $\mu: R\otimes_F R\to R$ is morphism of the category C.

Theorem 7.6. Let C be a multiplicative abelian category of K-modules satisfying Axiom A. Let R be an algebra object of C that is finitely generated as an F-algebra. Then R^K is finitely generated as an F-algebra.

Proof. Proceeding as in our proof of Theorem 7.4., we see that is enough to derive a contradiction from the assumption that R^K is not finitely generated, but that $(R/J)^K$ is finitely generated for every non-zero ideal J of R such that $J\varepsilon C$ and the map $J\varepsilon R\varepsilon C$. In doing this, we may also assume that R^K contains no zero divisor of R. In that case, the field of fractions of R^K, denoted by $[R^K]$, is a finitely generated field extension of F (see coments following this proof). We can represent R as a quotient S/Q where S is a graded algebra object of C that is finitely generated as an F-algebra, and Q is an ideal of S that is also a C-subobject of S. Then S^K is finitely generated as an F-algebra. We also know (from Thm.7.2., above) that $S^K/Q\cap S^K$ is finitely generated as an F-algebra and that R^K is integral over $S^K/Q\cap S^K$. Consider the following diagram



The extension at the bottom is integral, whence the one at the top is algebraic. It is also finitely generated (the extension $[R^K]/F$ is finitely generated). Thus, the extension $[R^K] / [S^K/Q\cap S^K]$ is finite algebraic. By general commutative algebra, the integral closure of $S^K/Q\cap S^K$ in $[R^K]$ is finitely generated as an $S^K/Q\cap S^K$-module. As $S^K/Q\cap S^K$ is noetherian we deduce that R^K(that is contained in the integral closure mentioned above) is finitely generated as an $S^K/Q\cap S^K$-module. Thus, in the extension $F\subset S^K/Q\cap S^K\subset R^K$, the left part is a finitely generated algebra extension and the right part is a finitely generated $S^K/Q\cap S^K$-module extension. Thus, R^K is a finitely generated F-algebra.

Q.E.D.

Let R be a K-module algebra such that R^K does not have any zero divisor. Let [R] denote the total ring of fractions of R, and let M be a maximal ideal of [R]. As R is finitely generated as an algebra [R]/M is a finitely generated field extension of F. As the map $R^K \to [R]/M$ is injective, we can identify $[R^K]$ with a subfield of [R]/M, thus we deduce that $[R^K]$ is a finitely generated field extension of F.

We will fix our attention on the following context. Let R_0 be a fixed rational K-module algebra and let C_0 be the abelian category of (R_0,K)-modules. Recall that an (R_0,K)-module is an F-vector space M, that is at the same time an R_0-module and a K-module and the actions are related by the formula $x.(fm)=x.f\ x.m$ for $x\varepsilon K$, $f\varepsilon R_0$, $m\varepsilon M$. We define morphisms of (R_0,K)-modules in the evident way. The category C_0 satisfies the conditions on C that we assumed before, if R is an algebra object in C_0, then $R\otimes_F R$ is again an algebra object in C_0 and the multiplication is a morphism of the category.

<u>Defn.7.7</u>. We say that K acts on R_0 in a linearly reductive or exact way if the category C_0 defined above satisfies Axiom B.

<u>Defn.7.8</u>. We say that K acts on R_0 in a geometrically reductive way if the category C_0 defined above satisfies Axiom A.

<u>Theorem 7.9</u>. In the context above, the following conditions a) b) and c) are equivalent, and condition c') implies all of them.

a) K acts on R_0 in a linearly reductive way.

b) For every surjective map $\lambda: M \to N$ of (R_0,K) - modules the map $\lambda/M^K: M^K \to N^K$ is surjective.

c) For every K-stable ideal J of R_0 and every surjective map $\lambda: M \to R_0/J$ of (R_0,K)-modules there is an $m \in M^K$ such that $\lambda(m)=1+J$.

c') For every K-stable ideal J of R_0 , every surjective map $\lambda: M \to R_0/J$ splits as a K-module map.

Theorem 7.10. In the above context, the following conditions a) and b) are equivalent, and condition b') implies a) and b).

a) K acts on R_0 in a geometrically reductive way.

b) For every (R_0,K)-module M, every K-stable ideal J of R_0 and every surjective (R_0,K)-module map $\lambda: M \to R_0/J$ there is a $q>0$ and an m in $S^q(M)^K$ such that $S^q(\lambda): S^q(M) \to R_0/J$ sends m to 1+J.

b') For every (R_0,K)-module M, every K-stable ideal J of R_0 and every surjective (R_0,K)-module map $\lambda: M \to R_0/J$, there is a $q>0$ such that the map $S^q(\lambda)$ splits as a K-map.

The proofs of Theorem 2.3 and 2.4 yield proofs of Theorem 7.9 and 7.10. As we pointed out before, if the group K is linearly reductive then K acts in a linearly reductive way on every rational K-module algebra. Similarly for geometrically reductive groups.

In particular, we deduce.

Corollary 7.11. If K acts on R_0 in a geometrically reductive way and R_0 is finitely generated K-module algebra, then R_0^K is a finitely generated algebra.

The concept of linearly reductive action is stronger than the concept of geometrically reductive action, but there is a particular case in which both concepts coincide. Let G be an affine algebraic group and K a closed subgroup of G, then there is a natural action of K on P(G). We will prove in Section 8 that K acts in a linearly reductive way on P(G) if and only if it acts on P(G) in a geometrically reductive way. Using this equivalence we can interpret Theorem 6.1.as a transitivity theorem; a particular case of the following result.

Theorem 7.12. Let G be an affine algebraic group, K a closed subgroup of G, and R_0 an arbitrary G-module algebra. If K acts on P(G) in a geometrically reductive way and G acts on R_0 in a geometrically reductive way then K acts on R_0 in a geometrically reductive way.

Proof. Consider an arbitrary (R_0,K)-module M, let I be a K-stable ideal of R_0 and λ a surjective (R_0,K) - map from M to R_0/I. Using the result mentioned before about the equivalence of the concepts of linearly reductive actions and geometrically reductive actions of K on P(G), from the hypotheses and from Theorem 7.9. we deduce that the map $(id \otimes \lambda):(P(G) \otimes M)^K \to (P(G) \otimes R_0/I)^K$ is surjective.

Consider the map $\psi : R_0 \otimes (P(G) \otimes M)^K \to R_0 \otimes (P(G) \otimes R_0/I)^K$ given by $\psi = id \otimes id \otimes \lambda$. We endow $R_0 \otimes (P(G) \otimes M)^K$ as well as $R_0 \otimes (P(G) \otimes R_0/I)^K$ with the G-structure given by the diagonal action and the R_0 structure given by multiplication on the first tensor factor. Then, ψ is a (R_0,G)-module map. Define $E_M : (P(G) \otimes M)^K \to M$ as $E_M(\Sigma f_i \otimes m_i) = \Sigma f_i(1)m_i$, then E_M is a K-map, and the following diagram is commutative.

$$\begin{array}{ccc}
R_0 \otimes (P(G) \otimes M)^K & \xrightarrow{\psi} & R_0 \otimes (P(G) \otimes R_0/I)^K \\
\downarrow{\scriptstyle id \otimes E_M} & & \downarrow{\scriptstyle id \otimes E_{R_0/I}} \\
R_0 \otimes M & \xrightarrow{id \otimes \lambda} & R_0 \otimes R_0/I \\
\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu} \\
M & \xrightarrow{\lambda} & R_0/I
\end{array}$$

The vertical maps μ are the maps given by the R_0-module action. When we endow M and R_0/I with the given (R_0,K)-module structures and $R_0 \otimes M$ as well as $R_0 \otimes R_0/I$ with the diagonal K-module structure and the R_0-module structure given by multiplication on the first tensor factor, all the vertical maps are (R_0,K)-module maps. Now, by looking at the surjective morphism $S(\psi)$ of (R_0,K)-algebras from the symmetric algebra built on $R_0 \otimes (P(G) \otimes M)^K$ to the symmetric algebra built on $R_0 \otimes (P(G) \otimes R_0/I)^K$ that is induced by ψ, we deduce that there is an x in $S^q(R_0 \otimes (P(G) \otimes M)^K)^G$ such that $S^q(\psi)(x) = 1 \otimes 1 \otimes (1+I)$, with $1 \otimes 1 \otimes (1+I)$ in $(R_0 \otimes (P(G) \otimes R_0/I)^K)^G$. Then, if we call $y = S^q(\mu(id \otimes E_M))(x)$, we have that y is in $S^q(M)^K$ and $S^q(\lambda)(y) = \mu(id \otimes E_{R_0/I})(1 \otimes 1 \otimes (1+I)) = 1+I$.

Q.E.D.

The converse of Theorem 7.12. is not true. A counter-example will be given after Corollary 8.8. Thus the only case in which we can expect to deduce reductivity assumptions for the action of K on R from reductivity assumptions for the action of G on R, is when the quotient G/K is affine. If we look not at reductivity assumptions but only at the finiteness of rings of invariants, there is more that can be said.

The following example shows that if K is a subgroup of G, $P(G)^K$ can be finitely generated even if K is not exact in G. Take $G=SL_2(F)$ and $K=\left\{\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix} / b\varepsilon F\right\}$. Then the quotient G/K is isomorphic to $F^2-(0,0)$ that is not affine, but has finitely generated ring of polynomial functions. This is a particular case of a situation that has been extensively studied (see for example G. Hochschild and G. Mostow "Unipotent Groups in Invariant Theory" Proc. Nat. Acad. Scie. USA, Vol 70, No. 3, pp 646-648, March 1973) and we will limit ourselves to the following comments (that were pointed out to the author by G.Hochschild). Let G be an affine algebraic group and K a closed subgroup. Let R_0 be a G-module algebra and assume that G acts on R_0 in a geometrically reductive way. The G-module structure on R_0 induces a P(G)-comodule structure $\rho:R_0\to R_0\otimes P(G)$ related to the G-action by $\rho(r) = \Sigma r_i \otimes f_i$ and $x.r=\Sigma f_i(x)r_i$ for $x\varepsilon G$. If we endow $R_0 \otimes P(G)$ with the G-module structure $x.(r \otimes f) = x.r \otimes f.x^{-1}$, it is easy to show that $\rho(R_0) = (R_0 \otimes P(G))^G$ and that ρ is injective. Now, consider the action of G on $R_0 \otimes P(G)$ given by $x*(r \otimes f) = r \otimes x.f$. The following computation shows that, with respect to these structures ρ is a G-module map. If $\rho(r) = \Sigma r_i \otimes f_i$ and $x \epsilon G$, we have $\rho(x.r) = \rho(\Sigma f_i(x) r_i) = \Sigma f_i(x)\rho(r_i)$. If $\rho(r_i) = \sum_j r_{ij} \otimes f_{ij}$ and $\Delta(f_i) = \sum_k f'_{ik} \otimes f''_{ik}$, we have $\sum_{ij} r_{ij} \otimes f_{ij} \otimes f_i$ =

$= \sum_{ik} r_i \otimes f'_{ik} \otimes f''_{ik}$. Now, $\rho(x.r) = \sum_{ij} f_i(x)(r_{ij} \otimes f_{ij})$ =

$= \sum_{ik} f''_{ik}(x)(r_i \otimes f'_{ik}) = \sum_i r_i \otimes \sum_k f'_{ik} f''_{ik}(x) = \Sigma r_i \otimes x.f_i$ =

$= x*\Sigma r_i \otimes f_i = x*\rho(r)$.

As $x*(y.(r \otimes f)) = y.r \otimes x.f.y^{-1} = y.(x*(r \otimes f))$ the map ρ induces an isomorphism of F-algebras from R_0^K to $(R_0 \otimes P(G)^K)^G$. Now assume that R_0 and $P(G)^K$ are finitely generated as F-algebras. Then $R_0 \otimes P(G)^K$ is a finitely generated (R_0,G)-module algebra. By our assumption on G we deduce that $(R_0 \otimes P(G)^K)^G$, and thus R_0^K is finitely generated as an F-algebra. Thus , in this situation, $P(G)^K$ is a universal object with respect to the finite generation of rings of K-invariants.

8. Orbit Varieties

Defn.8.1. Let K be an affine algebraic group acting on a variety X. A categorical quotient of X by K is a pair (Y,ϕ), where Y is a variety and $\phi: X \to Y$ is a morphism such that:

i) If x and x' are in the same orbit of X , then $\phi(x) = \phi(x')$.

ii) Given any variety Z and a morphism ψ: $X\to Z$ which is constant on orbits, there is a unique morphism $\chi:Y\to Z$ such that $\chi\phi=\psi$.

If moreover $\phi^{-1}(y)$ consists exactly of one orbit for every y in Y, then (Y,ϕ) is called an orbit space.

The definition above and the one given in Section 5 (Defn.5.1.) coincide when X is an affine variety. The following theorem is well known in the case where the group K is geometrically reductive, see [11], and our proof sketched here is very similar to the one presented there.

Theorem 8.2. Let K be an affine algebraic group acting on the affine variety X in a geometrically reductive way (meaning that K acts on P(X) in a geometrically reductive way). Then, there is an affine variety Y and a morphism $\phi:X\to Y$ such that

i) ϕ is K-invariant

ii) ϕ is surjective

iii) If U is open in Y the map ϕ^*: $O_Y(U)\to O_X(\phi^{-1}(U))$ is an isomorphism of $\underline{O}_Y(U)$ onto $\underline{O}_X(\phi^{-1}(U))^K$

iv) If W is a closed K-invariant subset of X then $\phi(W)$ is closed in Y.

v) If W_1 and W_2 are disjoint closed invariant subsets of X, then $\overline{\phi(W_1)}\cap\overline{\phi(W_2)}=\phi$. (Here $\overline{}$ denotes Zariski closure).

Proof. We known that $P(X)^K$ is a finitely generated F-algebra (Theorem 7.6.). Define Y to be the affine variety associated with $P(X)^K$ and ϕ the map induced by the inclusion $P(X)^K \hookrightarrow P(X)$. The proof of i) is straightforward. (If $x_1K = x_2K$ for $x_1, x_2 \varepsilon X$ for every $f \varepsilon P(X)^K$ $f(\phi(x_1)) = f(\phi(x_2)$, as the elements of $P(X)^K$ separate the points of Y we deduce that $\phi(x_1) = \phi(x_2)$.).To prove ii) consider an element y of Y, and let $f_1,\dots,f_r$ generate the maximal ideal corresponding to y in $P(Y) = P(X)^K$. Using Lemma 7.1., we deduce that $\Sigma f_i P(X) \neq P(X)$, so that there is a maximal ideal of $P(X)$ containing $\Sigma f_i P(X)$. If x is the point of X corresponding to that ideal, then $\phi(x) = y$. Assertion iii) can be proved as in [11]. Let f be an element of $P(X)^K$, then it is enough to check iii) for $U=Y_f$, and in this case $(P(X)^K)_f = (P(X)_f)^K$. It is clear that iv) is a consequence of v). v) Consider the ideals I_1 and I_2 of W_1 and W_2 in $P(X)$. We have $I_1+I_2 = P(X)$. Consider the map $I_2 \rightarrow P(X)/I_1$ given by the projection. This is a surjective $(P(X),K)$-module algebra map. There is a function g in I_2^K such that $g+I_1 = 1+I_1$,i.e. $g-1 \varepsilon I_1$. Then g is K-invariant and $g(W_1)=1$, $g(W_2)=0$. Regarding g as an element of $P(Y)$ we have that $g(\phi(W_1))=1$ $g(\phi(W_2))=0$, hence $\overline{\phi(W_1)} \cap \overline{\phi(W_2)} = \phi$.

Q.E.D.

<u>Corollary 8.3</u>. Let K,X,Y and be as above.Then (U,ϕ) is a categorical quotient of $\phi^{-1}(U)$ by K for every open subset U of Y.

<u>Corollary 8.4</u>. In the above notation, $\phi(x_1) = \phi(x_2)$ if and only if $\overline{O(x_1)} \cap \overline{O(x_2)} \neq \phi$. ($O(x)$ denotes the orbit of x).

<u>Corollary 8.5</u>. In the situation above, if the orbits of K on X are closed, then Y is an orbit space.

Finally the following result can be proved in the same way as in [11].

<u>Theorem 8.6</u>. Let X, K, Y, and ϕ be as in Theorem 8.2., and let X' be the subset of X defined by $X' = \{ x \in X / 0(x)$ is closed and dim $0(x)$ has the maximum value$\}$. Then there is an open subset Y' of Y such that $\phi^{-1}(Y') = X'$ and (Y', ϕ) is an orbit space for the action of K on X'.

Now let us assume that X is an arbitrary affine variety and K is an affine algebraic group acting on X in a geometrically reductive way and such that the map $\alpha: X \times K \to X \times X$ given by $\alpha(x,k) = (x, xk)$ is an isomorphism onto its image. Assume moreover that there is an abstract group L acting transitively from the left on X as a group of variety automorphisms commuting with the K-action. Then, all the K-orbits on X are closed. This is because there is always one closed orbit and L permutes the orbits transitively. Using Corollary 8.5., we conclude that the orbit space of X with respect to the action of K exists and is affine. The following Theorem is a generalization of results in [2].

<u>Theorem 8.7</u>. Let X be an affine variety and K an affine algebraic group acting on X from the right in such a way that the map $\alpha: X \times K \to X \times X$, where $\alpha(x,k) = (x, xk)$, is an isomorphism onto its image. Suppose moreover that there is an abstract group L acting on X as a group of affine va-

riety automorphisms commuting with the action of K. Then the following conditions are equivalent:

a) The orbit space X/K exists and is affine.

b) P(X) is injective as a K-module.

c) K acts on X in a linearly reductive way.

d) K acts on X in a geometrically reductive way.

Proof. a) ⇒ b). This was proved in Section 5.

b) ⇒ c). This is a consequence of Lemma 3.2. and the fact that P(X) is injective as a K-module if the functor from K-modules to F-spaces given by $M \to (P(X) \otimes M)^K$ is exact.

c) ⇒ d). Obvious.

d) ⇒ a). See coments before the statement of the Theorem.

Q.E.D.

Corollary 8.8. Let G be a group and K a closed subgroup of G. Then the following conditions are equivalent:

a) G/K is affine.

b) P(G) is injective as a K-module.

c) K acts on G in a linearly reductive way.

d) K acts on G in a geometrically reductive way.

As to the counterexample to the converse of Theorem 7.12 any three algebraic groups $K \subset G \subset L$ such that L/K is affine and G/K or L/K not affine will do. Take for example

G a Borel subgroup of L and K trivial or take $K = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \;/\; a \neq 0 \quad a \in F \right\}$, $G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \;/\; ac \neq 0 \quad a,b,c \in F \right\}$, $L = GL(2,F)$.

The last Corollary also provides examples of groups K that act on a K-module algebra R_0 in a geometrically reductive way but are not geometrically reductive. It is enough to construct a pair $K \subset G$ such that G/K is affine but K is not reductive.

Finally, it is clear from our definitions that K is geometrically reductive if and only if K acts in a geometrically reductive way on every K-module algebra R_0 (take $R_0 = F$ with the trivial action). As a consequence of the comments that follow Theorem 6.1., to check the geometric reductivity of K, it is enough to prove that K acts on every K-module algebra of the form P(G) with $K \subset G$ in a geometrically reductive way.

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