Matemática Contemporânea, Vol. 9, 61–73
http://doi.org/10.21711/231766361995/rmc96

CONFORMAL SUB-RIEMANNIAN GEOMETRY IN DIMENSION 3

Elisha Falbel **Claudio Gorodski**
José Miguel Veloso

0. Introduction

Conformal geometry considers scale-invariant properties of Riemannian manifolds. See [14] for a modern introduction to conformal structures via second order frames. Complex structures are related to oriented conformal structures in such a way that, in real dimension two, both structures are equivalent.

Any 3-dimensional manifold has a contact structure ([13, 15]), that is, a non-integrable distribution characterized by $\theta \wedge d\theta \neq 0$ at every point, where θ is a 1-form annihilating the distribution. Contact structures first appeared in Mechanics, but they also appeared intermingled with complex structures in [16]. Poincaré examined the boundary of domains in $\mathbf{C}^2$ in an attempt to understand uniformization in the case of two complex variables. The structure on the boundary is nowadays abstracted in the concept of a CR-structure (Cauchy-Riemann structure).

It turns out that we may view the CR-structure as a conformal structure on the contact distribution. Webster defined the concept of pseudo-Hermitian structure in [19] and noted that it relates to CR-structures in the same way as Riemannian structures relate to conformal ones. In [12], the general case of a metric structure on a contact distribution is treated and in [10] the conformal geometry of this structure is analysed based on the treatment of the CR case ([3, 4, 6]).

A metric defined on a distribution is also called a sub-Riemannian structure.

In dimension 3 a contact sub-Riemannian structure is equivalent to a pseudo-Hermitian structure, and its conformal geometry is equivalent to a CR structure. In this paper we make explicit the relation between sub-Riemannian, pseudo-Hermitian and CR structures in the case of dimension 3. We analyse Cartan's CR invariant by expressing it in terms of sub-Riemannian data and compute it for all homogeneous sub-Riemannian manifolds classified in [9]. See Table 1.

1. Basic structures

Let $\mathcal{D}$ be a contact distribution defined on a 3-dimensional smooth manifold M, that is, there is a 1-form θ on M such that $\ker d\theta = \mathcal{D}$ and $\theta \wedge d\theta \neq 0$. We will consider the following structures:

Definition 1.1.

a. $(M, \mathcal{D}, J)$ is a *CR-structure* if J is a smoothly varying linear endomorphism on $\mathcal{D}$ which satisfies $J^2 = -1$.

b. $(M, \mathcal{D}, g)$ is a *sub-Riemannian structure* if g is a smoothly varying positive definite symmetric bilinear form on $\mathcal{D}$.

c. $(M, \mathcal{D}, [g])$ is a *conformal sub-Riemannian structure* if $[g]$ is a conformal class of sub-Riemannian metrics.

2. Sub-Riemannian and pseudo-Hermitian structures

Let $(M, \mathcal{D}, g)$ be a sub-Riemannian structure. The adapted coframe bundle is the bundle of positively oriented orthonormal adapted coframes θ, θ^1, θ^2 satisfying $d\theta = 2\theta^1 \wedge \theta^2$. If θ', $\theta^{1'}$, $\theta^{2'}$ is another adapted coframe, then

$$\begin{aligned} \theta' &= \theta \\ \theta^{i'} &= a^i_j \theta^j \quad \text{where } (a^i_j) \in SO(2) \end{aligned}$$

Theorem 2.1. ([11, 19]) *There exists a unique connection form ω and torsion* *forms* τ^1, τ^2 *such that*

$$\begin{aligned} d\theta^1 &= \theta^2 \wedge \omega + \theta \wedge \tau^1 \\ d\theta^2 &= -\theta^1 \wedge \omega + \theta \wedge \tau^2 \end{aligned}$$

with $\tau^1 \wedge \theta^1 + \tau^2 \wedge \theta^2 = 0$.

The curvature form is

$$\Omega = d\omega$$

and we write then

$$\Omega = K\theta^1 \wedge \theta^2 + W_1\theta^1 \wedge \theta + W_2\theta^2 \wedge \theta \tag{1}$$

It will be important to collect the Bianchi identities in the following. First observe that we may choose θ^1 and θ^2 such that $\tau^1 = \tau_0\theta^1$ and $\tau^2 = -\tau_0\theta^2$. This defines a parallelism on the manifold in the case $\tau_0 \neq 0$. We have

$$\begin{aligned} -W_1 - \tau_{02} &= 2\tau_0\omega_1 \\ W_2 - \tau_{01} &= 2\tau_0\omega_2 \\ K_0 - W_{12} - W_1\omega_1 + W_{21} - W_2\omega_2 &= 0 \\ -\omega_{12} + \omega_{21} - (\omega_1)^2 - (\omega_2)^2 + 2\omega_0 &= K \\ -\omega_{10} + \omega_{01} - \omega_1\tau_0 - \omega_0\omega_2 &= W_1 \\ -\omega_{20} + \omega_{02} - \omega_0\omega_1 - \omega_2\tau_0 &= W_2 \end{aligned}$$

Here we used the convention $\alpha = \alpha_1\theta^1 + \alpha_2\theta^2 + \alpha_0\theta$ for a 1-form α on M and, in particular, $df = f_1\theta^1 + f_2\theta^2 + f_0\theta$ for a function f defined on M.

We draw some consequences from the Bianchi identities. If $\tau_0 = 0$ then $W_1 = W_2 = 0$. On the other hand, in the case that $\tau_0 \neq 0$ and K, τ_0, W_1, W_2 are constant, we have the relations

$$\omega_1 = -W_1/2\tau_0, \quad \omega_2 = W_2/2\tau_0 \quad \text{and} \quad \omega_0 = \frac{1}{2}(K + W_1^2 + W_2^2). \tag{2}$$

There are two possible cases: $W_1 = -W_2$ and $W_1 = W_2$. If $W_1 \neq 0$, then $\omega_0 = \tau_0$ and $\omega_0 = -\tau_0$ in each case, respectively.

We will show how the sub-Riemannian structure is related to the pseudo-Hermitian structure defined by Webster in [19]. In fact, in dimension 3, the structures are the same. A pseudo-Hermitian structure is defined fixing a distribution $\mathcal{D}$, a complex operator J in $\mathcal{D}$ and fixing a form $\theta = \boldsymbol{\theta}$ with kernel $\mathcal{D}$. The equations of Webster's connection are obtained by defining

$$\begin{aligned} \theta^1 &= \boldsymbol{\theta}^1 + i\boldsymbol{\theta}^2 \\ \tau^1 &= \boldsymbol{\tau}^1 + i\boldsymbol{\tau}^2 \\ \theta_1^1 &= -i\omega \end{aligned}$$

With this notation, we easily see that we have

$$\begin{aligned} d\theta &= i\theta^1 \wedge \theta^{\bar{1}} \\ d\theta^1 &= \theta^1 \wedge \theta_1^1 + \theta \wedge \tau^1 \end{aligned}$$

with conditions

$$\begin{aligned} \theta_1^1 + \theta_{\bar{1}}^{\bar{1}} &= 0 \\ \tau^1 \wedge \theta^{\bar{1}} &= 0 \end{aligned}$$

Here $\theta_{\bar{1}}^{\bar{1}} = \bar{\theta}_1^1$.

The curvature of this connection is

$$\Omega_1^1 = d\theta_1^1 = R\theta \wedge \theta^{\bar{1}} + W\theta^1 \wedge \theta - \bar{W}\theta^{\bar{1}} \wedge \theta. \tag{3}$$

Comparing the pseudo-Hermitian curvature (3) with the sub-Riemannian one (1), we obtain

$$\begin{aligned} R &= \frac{1}{2}K \qquad (4) \\ W &= -\frac{1}{2}W_2 - \frac{i}{2}W_1 \qquad (5) \end{aligned}$$

3. Sub-conformal and CR-structures

In this section we describe the bundles which are associated to the CR-structure and to the sub-conformal structure. Note that in dimension 3 both structures coincide. See [6] for details on the CR-case and [10] for the sub-conformal case.

Let $(M, \mathcal{D}, [g])$ be a sub-conformal structure. We let E' be the line-bundle of all sub-Riemannian metrics in the conformal class $[g]$. Given a sub-Riemannian metric, that is, a section of this bundle, there exists a canonical contact form θ such that

$$d\theta = 2\theta^1 \wedge \theta^2 + h_i \theta^i \wedge \theta$$

where θ^i is an orthonormal coframe on $\mathcal{D}$. We could also consider the line bundle E of all contact forms associated to the sub-Riemannian metrics in this sense. It is clear then that there exists a fiber bundle isomorphism between those two bundles. To explicit this isomorphism, consider a trivialization of E', that is, a choice of a sub-Riemannian metric g and the corresponding trivialization of E, that is, the contact form θ. Then the bundle map is defined as $\lambda g \to \lambda\theta$. We will identify E with E' in the following considerations.

We will construct a bundle Y of 1-forms over the bundle E. We begin by defining the tautological form ω. Given a point e in E, consider a coframe θ^i as above. At e we consider the pull-back of θ^i and all forms defined by

$$\omega^i = \sqrt{\lambda} a^i_j \theta^j + v^i \omega \quad \text{where } (a^i_j) \in SO(2);$$

finally we define the form ϕ by imposing the equation

$$d\omega = 2\omega^1 \wedge \omega^2 + \omega \wedge \phi.$$

Observe that for each choice of ω^i, ϕ is then any form in the family

$$\phi = -\frac{d\lambda}{\lambda} + 2(v^1\omega^2 - v^2\omega^1) + s\omega$$

The bundle of all forms ω, ω^i, ϕ is denoted by Y. This is a G-structure with G the group of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ v^i & u^i_j & 0 \\ s & -2v^k u^k_j & 1 \end{pmatrix}$$

where $(u^i_j) \in SO(2)$.

In the case of CR-structures we also form the line bundle E of contact forms and denote by $\omega = \boldsymbol{\omega}$ the tautological form. Y will be the G-structure of all coframes satisfying the equation

$$d\omega = i\omega^1 \wedge \omega^{\bar{1}} + \omega \wedge \phi.$$

where $\omega^1 = \boldsymbol{\omega}^1 + i\boldsymbol{\omega}^2$, $\phi = \boldsymbol{\phi}$.

In dimension 3, a sub-conformal structure is equivalent to a CR-structure. In complex notation write the group G as the group of matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ v^1 & u & 0 & 0 \\ v^{\bar{1}} & 0 & \bar{u} & 0 \\ s & iuv^{\bar{1}} & -i\bar{u}v^1 & 1 \end{pmatrix}$$

where $u \in U(1)$.

We use now the convention $\alpha^{\bar{1}} = \bar{\alpha}^1$ and $\alpha_1 = \alpha^{\bar{1}}$ for α^1 a complex valued 1-form.

Theorem 3.1. ([3, 4, 6]) *On Y there exists a unique parallelism given by the forms ω, ω^i, ϕ ,ϕ^i_j, ϕ^i, ψ such that the following equations are satisfied*

$$\begin{aligned} d\omega &= i\omega^1 \wedge \omega^{\bar{1}} + \omega \wedge \phi \\ d\omega^1 &= \omega^1 \wedge \phi^1_1 + \omega \wedge \phi^1 \\ d\phi &= i\omega_{\bar{1}} \wedge \phi^{\bar{1}} + i\phi_{\bar{1}} \wedge \omega^{\bar{1}} + \omega \wedge \psi \\ d\phi^1_1 &= i\omega_1 \wedge \phi^1 - 2i\phi_1 \wedge \omega^1 - \frac{1}{2}\psi \wedge \omega \\ d\phi^1 &= \phi \wedge \phi^1 + \phi^1 \wedge \phi^1_1 - \frac{1}{2}\psi \wedge \omega^1 + Q\omega^{\bar{1}} \wedge \omega \end{aligned}$$

with the condition $\phi - \phi^1_{\bar{1}} - \phi^{\bar{1}}_{\bar{1}} = 0$.

We also have one more equation:

$$d\psi = -\psi \wedge \phi + 2i\phi^1 \wedge \phi_1 + \rho \wedge \omega \tag{6}$$

It is not difficult to see that $\rho = 2Q_{\bar{1}}\theta^1 + 2Q_1\theta^{\bar{1}}$. See Cheng ([5]).

The Cartan connection on Y is the $su(2,1)$ valued form

$$\pi = \begin{pmatrix} -\frac{1}{3}(\phi_1^1 + \phi) & \omega^1 & 2\omega \\ -i\phi_1 & \frac{1}{3}(2\phi_1^1 - \phi) & 2i\omega_1 \\ -\frac{1}{4}\psi & \frac{1}{2}\phi^1 & \frac{1}{3}(\phi + \phi_{\bar{1}}^{\bar{1}}) \end{pmatrix}$$

and the curvature form for this connection is

$$\Pi = \begin{pmatrix} 0 & 0 & 0 \\ -i\bar{Q}\omega^1 \wedge \omega & 0 & 0 \\ -\frac{1}{4}\rho \wedge \omega & \frac{1}{2}Q\omega^{\bar{1}} \wedge \omega & 0 \end{pmatrix}$$

(see ([6])).

In the case $\dim M = 2n+1 \geq 5$, let $(M, \mathcal{D}, [g])$ be a nondegenerate conformal sub-Riemannian structure. We let E, as before, be the line-bundle of all sub-Riemannian metrics in the conformal class $[g]$. Given a sub-Riemannian metric, there exists a canonical contact form θ such that

$$d\theta = h_{ij}\theta^i \wedge \theta^j + h_i\theta^i \wedge \theta$$

where θ^i is a dual basis of an orthonormal basis of $\mathcal{D}$, and $\det(h_{ij}) = 1$.

Over the bundle E we construct a bundle Y of forms analogously to the 3-dimensional case. The construction is very similar to the construction of the corresponding bundle in the case of CR-structures ([6]). The main difference from that case is that the bundle Y is not a G-structure.

Consider the tautological form ω. Given a point $e = \lambda g$ in E, where g is a section of E, consider a coframe θ^i as above. On e we consider the pull-back θ^i and all forms defined by

$$\omega^i = \sqrt{\lambda}a_j^i\theta^j + v^i\omega \quad \text{where} \quad (a_j^i) \in O(2n);$$

finally we define the form ϕ by imposing the equation

$$d\omega = \omega \wedge \phi + h_{ij}\omega^i \wedge \omega^j$$

Although the bundle Y is not a G-structure, we were able to find a parallelism for Y introducing as a main tool (ad_H) acting on $gl(2n)$, where $H = (h_{ij})$ is the varying antisymetric form. The parallelism is determined through smoothly varying algebraic conditions. Whoever is familiar with Chern's treatment of the CR case should imagine the formidable details necessary to implement these ideas. The following theorem is proved in [10].

Theorem 3.2. ([10]) *There exists a unique parallelism of Y given by the forms ω, ω^i, ϕ, ω^i_j, ϕ^i, ψ satisfying the equations*

$$
\begin{aligned}
&d\omega = h_{ij}\omega^i \wedge \omega^j + \omega \wedge \phi \\
&d\omega^i = -\frac{1}{2}\phi \wedge \omega^i - \omega^i_j \wedge \omega^j - \phi^i \wedge \omega \\
&d\phi = 2h_{ij}\phi^i \wedge \omega^j + b_{ij}\omega^i \wedge \omega^j + \omega \wedge \psi \\
&d\omega^i_j + \omega^i_k \wedge \omega^k_j + h_{kj}\omega^i \wedge \phi^k - h_{ki}\omega^j \wedge \phi^k - \frac{1}{2}b_{kj}\omega^k \wedge \omega^i + \frac{1}{2}b_{ki}\omega^k \wedge \omega^j \\
&\quad -h_{kj}\phi^i \wedge \omega^k + h_{ki}\phi^j \wedge \omega^k + h_{ij}\omega^k \wedge \phi^k = S^i_{jkm}\omega^k \wedge \omega^m + (V^i_{jk}\omega^k + W^i_{jk}\phi^k) \wedge \omega \\
&d\phi^i - \frac{1}{2}\phi \wedge \phi^i - \phi^j \wedge \omega^i_j + \frac{1}{2}\psi \wedge \omega^i = \\
&\quad \frac{1}{2}(V^i_{jk} - V^i_{kj})\omega^k \wedge \omega^j + W^i_{jk}\phi^k \wedge \omega^j + P^i_j\omega^j \wedge \omega + R^i_j\phi^j \wedge \omega + U^i\psi \wedge \omega \\
&d\psi + \psi \wedge \phi - 2h_{ij}\phi^i \wedge \phi^j + (2h_{ij}R^i_k + 4b_{jk})\phi^k \wedge \omega^j + 2h_{ij}U^i\psi \wedge \omega^j = \rho \wedge \omega + Q_{ij}\omega^i \wedge \omega^j
\end{aligned}
$$

with conditions

$$
\begin{aligned}
\omega^i_j &= -\omega^j_i \\
(b_{ij}) &\in g \\
(S^i_{jim}) &\in (g')^{\perp} \\
V^i_{ji} &= 0 \\
P^i_i &= 0
\end{aligned}
$$

where $g = \ker(ad_H) \cap so\,(2n)$, $(g')^{\perp} = \operatorname{im}(ad_H) \cap sim(2n)$, for $H = (h_{ij})$, ρ is a 1-form, and the conditions

$$
\begin{aligned}
h_{ij} &= -h_{ji} \\
b_{ij} &= -b_{ji} \\
S^i_{jkm} = -S^j_{ikm} &= -S^i_{jmk} \\
S^i_{jkm} + S^i_{kmj} + S^i_{mjk} &= 0
\end{aligned}
$$

$$V^i_{jk} = -V^j_{ik}$$
$$W^i_{jk} = -W^j_{ik}$$

(sim$(2n)$ denotes the vector space of $2n \times 2n$ real symmetric matrices).

The parallelism defined by Chern for the CR-structure in [6] corresponds to the case in the above Theorem where $H = J$ and $(\omega^i_j) \in g$. In that case the bundle Y is a G-structure and in fact Chern finds a Cartan connection.

4. Cartan's invariant

Cartan's invariant is Q and it is defined on Y. A change in coframes

$$\theta' = \lambda\theta$$
$$\theta^{1'} = \sqrt{\lambda}u\theta^1 + v^1\theta$$

transforms Cartan's invariant

$$Q = Q'(\lambda)^2u^2 \tag{7}$$

Definition 4.1. We say a CR-structure is *umbilic* at $p \in M$ if $Q(p) = 0$. Otherwise, we say the structure is *non-umbilic* at p.

To obtain an invariant defined on the base manifold M, we first observe that, if the point is umbilical, then the invariant $Q = 0$ is well defined on the base manifold at p. If the point is not umbilical, we shall find a parallelism on M at p by imposing $Q'(p) = i$. This condition fixes a unique coframe θ , θ^1 at p. Invariants A, B and C are thus obtained on M from the equation

$$d\theta^1 = A\theta^1 \wedge \theta^{\bar{1}} + B\theta \wedge \theta^1 + C\theta \wedge \theta^{\bar{1}} \tag{8}$$

Remark 4.1.

a. If a CR-structure is umbilical on a neighbourhood of a point, then the CR-structure is locally CR-equivalent to S^3 near that point.

b. If two nowhere umbilical CR-structures have the same constant invariants A, B and C, then they are locally CR-equivalent.

5. Cartan's invariant via sub-Riemannian invariants

It is now our goal to write Cartan's invariant in terms of the sub-Riemannian data. Following Webster [19] and a correction in [1], we embbed the sub-Riemannian structure into the sub-conformal structure by fixing a section

$$\begin{aligned} \phi_1^1 &= \theta_1^1 + \frac{1}{4} R\theta \\ \phi^1 &= \tau^1 + \frac{i}{4} R\theta^1 + E\theta \\ \psi &= G\theta + i(\bar{E}\theta^1 - E\theta^{\bar{1}}) \end{aligned} \tag{9}$$

where $E = \frac{R_1}{6} + \frac{2i\bar{W}}{3}$

Recall the last equation from Theorem 3.1

$$d\phi^1 - \phi \wedge \phi^1 - \phi^1 \wedge \phi_1^1 + \frac{1}{2}\psi \wedge \omega^1 = Q\omega^{\bar{1}} \wedge \omega \tag{10}$$

and substitute relations (9) into (10) to get

$$Q = \tau_0 \theta^{\bar{1}}_{10} - \tau_0 \theta^1_{10} - \frac{i}{2} R\tau_0 + E\theta^1_{1\bar{1}} \tag{11}$$

To simplify the exposition we will suppose from now on that K, τ_0, W_1, W_2 are constants. If $\tau_0 = 0$ then $Q = 0$. Otherwise, substituting the sub-Riemannian data (4), (5) and (2) into (11), we have

$$Q = i\{\frac{3}{4}\tau_0 K + \frac{1}{6\tau_0}(W_1^2 + W_2^2)\}$$

As we observed before, to obtain the parallelism in the non-umbilical case we set $Q' = i$ in (7) so that

$$\lambda^2 u^2 = \frac{3}{4}\tau_0 K + \frac{1}{6\tau_0}(W_1^2 + W_2^2)$$

Now there are two cases to consider, whether the right hand side of the equation above is positive or negative. We have respectively $u = 1$ and $u = i$ and find $C = \frac{u\tau_0}{\bar{u}|\lambda|}$ from (8). Similarly, $A = \frac{u}{4\tau_0\sqrt{|\lambda|}}(W_2 + iW_1)$ and $B = -\frac{i}{2|\lambda|}(K + 2W_1^2)$. It is better to work with the following expression as Cartan's invariant

$$\mathcal{C} = \frac{u}{\bar{u}}\frac{1}{C^2} = \frac{3}{4}\frac{K}{\tau_0} + \frac{1}{3}\frac{W_1^2}{\tau_0^3},$$

type	G	A	B	τ_0	K	W_1	W_2	$\mathcal{C}$
(1)	H^3	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	0	0	0	0	0
(2a)	$\widetilde{Euc_2^+}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{a}{4}$	$\frac{a}{2}$	0	0	$\frac{3}{2}$
(3a)	$Poinc_2^+$	$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{a}{4}$	$-\frac{a}{2}$	0	0	$-\frac{3}{2}$
(4ad)	$\widetilde{SU(1,1)}$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{pmatrix}$	$\frac{a-d}{4}$	$-\frac{a+d}{2}$	0	0	$-\frac{3}{2}\frac{1+s^2}{1-s^2}$
(5ad)	$\widetilde{SU(1,1)}'$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{pmatrix}$	$\frac{a+d}{4}$	$\frac{a-d}{2}$	0	0	$\frac{3}{2}\frac{1-s^2}{1+s^2}$
(6ad)	$SU(2)$	$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{pmatrix}$	$\frac{a-d}{4}$	$\frac{a+d}{2}$	0	0	$\frac{3}{2}\frac{1+s^2}{1-s^2}$
(7ab)	$\Sigma_+(b)$	$\begin{pmatrix} 0 & 1 & -b \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{a}{4}$	$a(\frac{1}{2} - b^2)$	$a^{3/2}b\frac{\sqrt{2}}{4}$	$-a^{3/2}b\frac{\sqrt{2}}{4}$	$\frac{3}{2} - \frac{b^2}{3}$
(8ab)	$\Sigma_-(b)$	$\begin{pmatrix} 0 & -1 & b \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{a}{4}$	$-a(\frac{1}{2} + b^2)$	$a^{3/2}b\frac{\sqrt{2}}{4}$	$a^{3/2}b\frac{\sqrt{2}}{4}$	$-\frac{3}{2} - \frac{b^2}{3}$

Table 1: 3-dimensional sub-homogeneous spaces: all of them are Lie groups G; $\{X_1, X_2, Y = [X_1, X_2]\}$ is a basis of the Lie algebra of G, $\{X_1, X_2\}$ is a basis of the distribution and A is the matrix of ad_Y restricted to the distribution; B is the matrix of the inner product on the distribution; τ_0, K, W_1 and W_2 are sub-Riemannian invariants; a, b and d are positive parameters; we may assume $a \geq d$ for types (4) and (6); each one of types (1), (2), (3), (7b) and (8b) gives rise to a unique homogeneous conformal sub-Riemannian manifold (or, what is the same, homogeneous CR-manifold), but each one of types (4), (5) and (6) gives rise to a one-parameter family of homogeneous conformal sub-Riemannian manifolds indexed by $s = (d/a)^{1/2}$, and $s \in (0, 1]$ for types (4) and (6) and $s > 0$ for type (5); $\mathcal{C}$ is Cartan's CR invariant.

since it allows to detect some umbilical cases with non-vanishing torsion. We next compute the value of this invariant for the examples of homogeneous sub-Riemannian manifolds classified in [9]. The final results are summarized in Table 1.

References

[1] Burns Jr., D.M. and Epstein, *A global invariant for CR-manifolds*, Invent. Math., 92 (1988), 333-348.

[2] Burns Jr., and Shnider, S., *Spherical hypersurfaces in complex manifolds*, Invent. Math., 33 (1976), 223-246.

[3] Cartan, É., *Sur la géometrie pseudo-conforme des hypersurfaces de l'espace de deux variables complexes*, I. Ann. Math. Pura Appl., 11 (1932), 11-17.

[4] Cartan, É., *Sur la géometrie pseudo-conforme des hypersurfaces de l'espace de deus variables complexes*, II. Ann. Sc. Ex. Norm. Super. Pisa., 1 (1932), 333-354.

[5] Cheng, J.H., *On the curvature of CR-structures and its covariant derivatives*, Math. Z., 196 (1987), 203-209.

[6] Chern, S.S. and Moser, J., *Real hypersurfaces in complex manifolds*, Acta Math., 133 (1974), 219-271.

[7] Ehlers, F., Neumann, W. D. and Scherk, J., *Links of surface singularities and CR space forms*, Comm. Math. Helv., 62 (1987), 240-264.

[8] Falbel, E. and Gorodski, C., *On contact sub-Riemannian symmetric spaces*, To appear in Ann. Scient. Éc. Norm. Sup.

[9] Falbel, E. and Gorodski, C., *Sub-Riemannian homogeneous spaces in dimensions 3 and 4*, To appear in Geom. Dedicata.

[10] Falbel, E. and Veloso, J. M., *A parallelism for conformal sub-Riemannian geometry*, Preprint, IMEUSP, (1993).

[11] Falbel, E., Verderesi, J.A. and Veloso, J. M., *Constant curvature models in sub-Riemannian geometry*, Matemática Contemporânea, Soc. Bras. Mat., 4 (1993), 119-125.

[12] Falbel, E., Verderesi, J. A. and Veloso, J. M., *The equivalence problem in sub-Riemannian geometry*, Preprint, IMEUSP, (1993).

[13] Gromov, M., *Stable maps of foliations in manifolds*, Izv. Akad. Nauk. C.C.C.P., 33 (1969), 707-734.

[14] Kobayashi, S., *Transformation Groups in Differential Geometry*, Springer-Verlag, (1972).

[15] Martinet, J., *Formes de contact sur les variétés de dimension 3*, Volume 209 of Lecture Notes in Mathematics, Springer-Verlag, (1971), 142-163.

[16] Poincaré, H., *Les fonctions analytiques de deux variables et la représentation conforme*, Rend. Circ. Mat. Palermo, 23 (1907), 185-220.

[17] Strichartz, R.S., *Sub-Riemannian geometry*, J. Diff. Geometry, 24 (1986), 221-263.

[18] Strichartz, R. S., *Corrections to 'sub-Riemannian geometry'*, J. Diff. Geometry, 30 (1989), 595-596.

[19] Webster, S.M., *Pseudo-Hermitian structures on a real hypersurface*, J. Diff. Geometry, 13 (1978), 25-41.

Elisha Falbel
Cláudio Gorodski
Instituto de Matemática e Estatística
Universidade de São Paulo
Caixa Posta 66281
São Paulo, SP 05389-970, Brasil

José Miguel Veloso
Departamento de Matemática – CCEN
Universidade Federal do Pará

Belém, PA 66059, Brazil