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Analyticity of the Density and Exponential Decay of Correlations in 2-d Bootstrap Percolation

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Abstract. We consider some deterministic cellular automata on the state space $\{0,1\}^{\mathbb{Z}^d}$, starting from the product of Bernoulli measures and evolving in discrete time according to the bootstrap percolation rules, in which a 0 changes to a 1 when it has at least ℓ neighbours which are 1. We prove that in case $\ell = 2d - 1$ the limiting measure has an exponential decay of correlations and the density function is analytic in $[0,1]$.

1. Introduction.

The Bootstrap Percolation model is a cellular automaton which evolves in the following way. A given configuration of 0's and 1's on the sites of the hypercubic lattice in d dimensions is updated by flipping to 1 each 0 with at least ℓ neighboring 1's and leaving the rest of that configuration unchanged. Here ℓ is a nonnegative integer no bigger than $2d$.

The initial configuration is chosen according to a product of Bernoullis with parameter p and updates are performed at discrete time units.

We are interested in the limiting behavior of the model as time goes to infinity. More precisely, we want to study the limiting measure obtained by evolving the initial product measure by the bootstrap percolation dynamics. (The limit exists since the dynamics can only increase the initial configuration.)

It is known from the works of van Enter [1] and Schonmann [5] that if $\ell \leq d$, then almost all initial configurations evolve towards the constant configuration with 1 at all sites. On the other hand, it is clear that when $\ell > d$, the limiting measure is different from this. For example, in the most simple of the latter cases, when $\ell = 2d$, then the initial configuration only changes once, the initial 0's completely surrounded by 1's flipping to 1, and thence nothing changes under the dynamics. Thus the limiting measure will be the initial product measure conditioned not to have 0's completely surrounded by 1's. So it is very close to being a product measure, the dependences are only local (of range 1) and quantities as the density function can be written down explicitly as polynomials in p.

The next simpler case is the object of this paper, namely, the case when $\ell = 2d - 1$. It is the only remaining case in $d = 2$ and without loss of generality we will restrict attention to this (for simplicity; all our arguments are immediately seen to hold qualitatively in general for $\ell = 2d - 1$).

In the situation in focus, a initial 0 is *protected* from ever flipping to 1 (or *stable*) if it initially belongs to either a *circuit*, which is a closed path of 0 sites, or to a *dumbbell*, which is two circuits joined by a path of 0 sites. The *only if* converse holds with probability one, as will be seen, so local quantities as the density function (the probability under the limiting measure that the origin is 1) and other correlations of the limiting measure can be written very simply and conveniently in terms of probabilities under the initial product measure that one or more sites belong to these quite simple structures.

A natural estimate on the decay of the tail of the distribution of the diameter of these structures will then yield smoothness properties of the density function and exponential decay of correlations of the limiting measure.

To get analyticity, the analysis of the structures described above by itself is not enough. We define thus a procedure to construct a subset of the closed percolation cluster of the origin (joined to part of its boundary and a few other sites, see Section 4) containing them, the analysis of which leads to the desired result.

2. The Model and Results.

The model considered in this paper is defined on the lattice $\mathbb{Z}^d$, where $\mathbb{Z}$ is the set of integers, and $d = 1, 2, \ldots$ is the dimension. The system evolves in discrete time $t = 0, 1, 2, \ldots$. To each element (site) of $\mathbb{Z}^d$, x, we associate at each instant of time t a random variable $\eta_t(x)$, which take values 0 or 1. We say that the site x is closed (resp. open) at time t if $\eta_t(x) = 0$ (resp. $\eta_t(x) = 1$). Elements $\eta_t \in \{0,1\}^{\mathbb{Z}^d} = \Omega$ will be called configurations. The system will be always started at $t = 0$, from a translation invariant product random field, i.e. the random variables $\eta_0(x)$, $x \in \mathbb{Z}^d$ are i.i.d. with $P(\eta_0(x) = 0) = p$, $P(\eta_0(x) = 1) = q = 1 - p$; $p \in [0,1]$ is called the initial density. The system evolves according to a deterministic rule:

$$\eta_{t+1}(x) = \begin{cases} 1 & \text{if} \quad \eta_t(x) = 1; \\ 1 & \text{if} \quad \eta_t(x) = 0 \quad \text{and} \quad \sum\limits_{y:\|x-y\|_1 = 1} \eta_t(y) \geq \ell; \\ 0 & \text{if} \quad \eta_t(x) = 0 \quad \text{and} \quad \sum\limits_{y:\|x-y\|_1 = 1} \eta_t(y) < \ell. \end{cases}$$

In this the article case $\ell = 2d - 1$ will be studied.

Let us define the following functions:

1) $\overline{\eta}(x) = (T\eta)(x) = \lim_{n\to\infty} \eta_n(x),\ x \in \mathbb{Z}^d$;

2) $\rho(p) = \mu_p(\overline{\eta}(0) = 1),\ p \in [0,1]$, where μ_p is the integration with respect to the initial product measure with the density p;

3) $\psi(p) = 1 - \rho(1-p) = \mu_{1-p}(\overline{\eta}(0) = 0)$.

(Remark. $\overline{\eta}$ is well defined by monotonicity)

Now we summarize our results and explain how the rest of the article is organized: In Section 3 we give a "geometric" description of lattice animals which leave the origin empty at all times. Then we prove that the diameter distribution of these structures has an exponentially decaying tail, and, as a corollary of this fact, we obtain that the limiting measure $\overline{\mu}_p$ has exponentially decaying correlations. In Section 4, analyticity of the density function $\psi(p)(\rho(p))$ on the interval $[0,1]$ is proved. Section 5 is devoted to a brief discussion of the shape of density function and the case of the higher dimensions.

As will be seen in Section 4, the estimates we use to get analitycity for ψ are of the same nature as the ones in the Section 3, but stronger, so we did not really need to introduce most of the objects and results in the latter section. Their appeal and simplicity on the other hand justify their inclusion, in our opinion.

3. Preliminary results and elementary properties.

Definition 3.1. Given an initial configuration $\eta_0 \in \{0,1\}^{\mathbb{Z}^2}$ a site $x \in \mathbb{Z}^2$ is called *stable* if and only if $\overline{\eta}(x) = 0$. □

Proposition 3.2. A site $x \in \mathbb{Z}^2$ is stable if and only if it is initially closed and has at least two stable neighbours.

The proof is obvious and we omit it.

Definition 3.3. We will say that collection of sites $\{x_i\}_{i=1}^k$, $x_i \in \mathbb{Z}^2$, $k \le \infty$, forms:

a) *a path* (an infinite path, if $k = \infty$) if sites x_i are distinct, and $||x_{i+1} - x_i||_1 = 1$ for $i = 1, 2, \dots, k$;

b) *a finite circuit*, if $k < \infty$, sites x_i $i = 1, \dots, k-1$ are distinct, $||x_{i+1} - x_i||_1 = 1$, $i = 1, \dots, k$, and $x_k = x_1$;

an infinite circuit, if $k = \infty$, and the collection $\{x_i\}_i$ can be represented as a double infinite path $\{x_i\}_{i \in \mathbb{Z}}$;

c) *a dumbbell*, if sites of $\{x_i\}_i$ form two non-intersecting finite circuits which are connected by a (finite) path or if they form a finite circuit attached to an infinite path (in the last case we say that the dumbbell is infinite). □

We say that a set of sites $\{x_i\}$ is closed (resp. open) if all its points are closed (resp. open).

Let $\mathcal{C}$ and $\mathcal{H}$ ($\overline{\mathcal{C}}$ and $\overline{\mathcal{H}}$ resp.) denote the sets of all finite closed circuits and all finite closed dumbbells of the initial configuration η_0 respectively (all closed circuits and all closed dumbbells resp.). Elements of the union $\mathcal{R} = \mathcal{C} \cup \mathcal{H}$ ($\overline{\mathcal{R}} = \overline{\mathcal{C}} \cup \overline{\mathcal{H}}$) will be called *finite rings* (rings resp.).

Proposition 3.4

$$\{\overline{\eta}(x) = 0\} = \{x \in R \text{ for some } R \in \overline{\mathcal{R}}\}.$$

Proof. ($\supset$). Suppose $R \in \overline{\mathcal{R}}$. Every site in R is closed and (by Definition 3.3) has at least two closed nearest neighbours. So, none of them can flip to 1 (become open).

($\subset$). Let $x = 0$. By Proposition 3.2 one can find inductively two sequences of stable sites $x_1, x_2, \ldots, x_n, \ldots$ and $x_{-1}, x_{-2}, \ldots, x_{-n}, \ldots$ such that x_n and x_{-n} are nearest neighbours to x_{n-1} and x_{-n+1} respectively for all $n \geq 1$. Now we have four possibilities:

either

1) sequence $\{x_i\}_i$ and $\{x_{-i}\}_i$ intersect;

or

2) sequence $\{x_i\}_i$ intersects itself and sequence $\{x_{-i}\}_i$ intersects itself and 1) does not happen;

or

3) sequence $\{x_i\}_i$ intersects itself, $\{x_{-i}\}_i$ does not and 1) does not happen;

or

4) neither $\{x_i\}_i$ nor $\{x_{-i}\}_i$ intersects itself nor the other sequence.

In cases 1) and 4), the origin belongs to a circuit and cases 2) and 3) the origin belongs to a dumbbell. $\square$

Definition 3.5.

$\mathcal{R}_d = \{R \in \mathcal{R}$ such that $0 \in R$ and R has minimal radius$\}$. □

Let R_d be one of elements of $\mathcal{R}_d$ chosen in an arbitrary predetermined way. We define:

$$d = \begin{cases} \text{radius of } R_d, & \text{if } \mathcal{R}_d \neq \emptyset, \\ 0, & \text{if } \mathcal{R}_d = \emptyset, \end{cases}$$

whre the radius is the maximal L_1 distance from the origin of sites in R_d.

Proposition 3.6. The distribution of d has an exponentially decaying tail, i.e. there exists $\alpha > 0$, such that for all $n \geq 1$

$$\mu_p\{d \geq n\} \leq e^{-\alpha n}.$$

Proof. Let us define:

x_i, $i = 1,2,3,4$ – the i-th nearest neighbour site to the origin;

$e_i = (0, x_i)$, $i = 1,2,3,4$ – the i-th nearest neighbour bond touching the origin;

B_i, $i = 1, \ldots, 4$ – the connected graph obtained from the closed percolation cluster of the origin without using $\{x_j, j \neq i\}$, which we call ***i-th branch*** of R_d.

A circuit formed by four sites will be called a ***square*** and we observe that the closed square is the minimal stable ring; a path formed by three sites which are not on the one line we will call a ***corner*** and use notation Γ for it. A ***corner of a site*** x is any corner Γ such that $\Gamma \cup \{x\}$ is a square. An ***outward corner of a site*** x is any corner Γ of x such that $S_{||x||_1} \cap \Gamma = \emptyset$, where S_n is the sphere of radius n centered at the origin, that is $S_n = \{y \in \mathbb{Z}^2 : ||y||_1 \leq n\}$. We further say that a set of sites has a closed corner attached if one of its sites has a corner which is closed.

Let us define now the following events:

A_n^i = {there exists at least one closed path in B_i inside S_{n-2} connecting the originto ∂S_{n-2}, all such paths have no closed square attached},

$i = 1, \ldots, 4$, where $\partial S_{n-1} = \{x : ||x||_1 = n-2\}$.

Notice that $\{d \geq n\} \subset \bigcup_i A_n^i$. But

$$A_n^1 \subset A_{n-1}^1 \cap \{ X_n \text{ has no outward closed corner} \};$$

where X_n is the first site (in the predetermined order) of ∂S_{n-2} touched by any closed path of B_1 from the origin within S_{n-2}.

We have:

$$\begin{aligned}\mu_p(A_n^1) &\leq \sum_{x \in \partial S_{n-2}} \mu_p(A_{n-1}^1, X_n = x,\ x \text{ has no outward closed corner}) \\ &= \sum_{x \in \partial S_{n-2}} \mu_p(A_{n-1}^1, X_n = x)\mu_p(x \text{ has no outward closed corner})\end{aligned} \tag{3.1}$$

since the events $E_1 = \{A_{n-1}^1, X_n = x\}$ and $E_2 = \{x \text{ has no outward closed corner}\}$ are independent: $E_1 \in \mathcal{F}_{S_{n-2}}$ and $E_2 \in \mathcal{F}_{S_{n-2}^c}$, where, given $\Lambda \subset \mathbb{Z}^2$, $\mathcal{F}_\Lambda$ is the σ-algebra generated by events dependent on sites in Λ.

On the other hand

$$\mu_p(x \text{ has no outward closed corners}) \leq c = 1 - (1-p)^3.$$

From (3.1) we have: $\mu_p(A_n^1) \leq c\,\mu_p(A_{n-1}^1)$.

This immediately implies that for $p > 0$ there exists $\alpha > 0$ such that $\mu_p(A_n^1) \leq e^{-\alpha n}$. In the neighborhood of $p = 0$, one can use the fact that the size of the closed percolation cluster of the origin has an exponential tail ([2]) to get $\alpha > 0$ independent of p. This finishes the proof. □

Let us define $\overline{\mu}_p$ as $\mu \circ T^{-1}$, where T was defined in Section 2.

Corollary 3.7. $\overline{\mu}_p$ has exponentially decaying correlations:

$$|\overline{\mu}_p(\eta(0)\eta(x)) - \overline{\mu}_p(\eta(0)) \cdot \overline{\mu}_p(\eta(x))| \leq C \cdot e^{-\alpha' \|x\|_1},$$

where $C, \alpha' > 0$.

Proof. First we notice that $\{\overline{\eta}(0) = 0\} = \{d_0 > 0\}$, where $\{0 < d_x \leq r\} = \{x$ belongs to a ring of diameter at most $r\}$.

Let $n = \lceil \|x\|_1/3 \rceil$ be fixed. We have:

$$|\overline{\mu}_p(\eta(0)\eta(x)) - \overline{\mu}_p(\eta(0)) \cdot \overline{\mu}_p(\eta(x))| =$$

$$= |[\mu_p\{0 < d_0 \leq n; 0 < d_x \leq n\} + \mu_p\{\{d_0 > n\} \text{ or } \{d_x > n\}\}+$$

$$-[\mu_p\{0 < d_0 \leq n\} \cdot \mu_p\{0 < d_x \leq n\}+$$

$$+2\mu_p\{d_0 > n\}\mu_p\{0 < d_x \leq n\} + \mu_p\{d_0 > n\}\mu_p\{d_x > n\}]| \leq$$

$$\leq 6\mu_p\{d_0 > n\} \leq 6e^{-\alpha n},$$

and we get the result by making $\alpha' = \alpha/3$ and $C = 6$. □

Corollary 3.8. ρ is smooth in $(0,1)$.

Proof. Here we follow the argument of Russo (see [4])

$$\psi(p) = 1 - \rho(1-p) = \mu_{1-p}\{d_0 > 0\} = \sum_{n>0} \mu_{1-p}\{d_0 = n\} =$$

$$= \sum_{n>0} \left(\sum_{\sigma \in \Lambda_n} \mu_{1-p}\{\sigma\} \right) = \sum_{n>0} \left(\sum_{\sigma \in \Lambda_n} p^{|\sigma|_c} q^{|S_n| - |\sigma|_c} \right) =$$

$$= \sum_{n>0} \sum_{\sigma \in \Lambda_n} p^m q^k$$

where $\Lambda_n = \{\sigma \in \{0,1\}^{S_n} \subset \Omega$ such that σ has a minimal ring of diameter n containing the origin $\}$; $|\sigma|_c$ = number of closed sites in σ; and $m = |\sigma|_c$, $b = |S_n| - |\sigma|_c$.

Now, let $p \in (0,1)$. Then

$$\frac{d^k}{dp^k}\mu_{1-p}(d_0 = n) = \sum_{\sigma\in\Lambda_n} \frac{d^k}{dp^k} p^m q^b \leq \sum_{\sigma\in\Lambda_n} p^m q^b \left(\frac{m}{p} + \frac{b}{q}\right)^k \leq$$

$$\leq C_k n^{2k} \sum_{\sigma\in\Lambda_n} p^m q^b = C_k n^{2k} \mu_{1-p}(d_0 = n).$$

Therefore the sum $\sum_{n>0} \frac{d^k}{dp^k}\mu_{1-p}(d_0 = n)$ is uniformly convergent in $(0,1)$, giving that $\psi(p)$ is k times differentiable. □

The argument for smoothness in the extremes of $(0,1)$ is different. Instead of pursuing this further here, we leave the matter for the next section where we get the stronger result of analyticity of ψ in $[0,1]$.

4. Analyticity of ρ.

Let A be the following event: $A = \{0$ is stable $\}$. Before we introduce some additional construction (V-lattice animals) we remind that on $\mathbb{Z}^2$ some ordering of the sites is fixed. Let us denote by $\bar\partial\Lambda$ the outer boundary of a set $\Lambda \subset \mathbb{Z}^2$, that is, $\bar\partial\Lambda = \{x \notin \Lambda : \|x - y\|_1 = 1.$ for some $y \in \Lambda\}$.

Also let $\bar S_n = \{(x_1,x_2) \in \mathbb{Z}^2 : |x_1| \vee |x_2| \leq n\}$ and define $Q(x) = x + \bar S_1$.

Definition 4.1. (A constructive description of V-lattice animal).

I) $V_0 = Q(0)$;

II) for $n \geq 0$, given V_n:

a) If the occurence of the event A (or $\overline{A}$) is determined in V_n we stop the procedure and set $V = V_n$,

b) If occurence of the event A (or $\overline{A}$) is not determined in V_n, take the first (in the given order) site, denoted x_n and called a *candidate*, from the following set

$$\partial\{c(V_n)\backslash \text{ (any branch}^* \text{ with a square)}\}$$

and, if there is a corner Γ_n of x_n such that $\{\Gamma_n \cup \{x_n\}\} \cap V_n = \emptyset$, we say that x_n is *fresh* and put

$$V_{n+1} = V_n \cup Q(x_n);$$

otherwise, put

$$V_{n+1} = V_n \cup \{x_n\}.$$

Here $c(\Lambda)$ is the subset of connected closed sites of Λ containing the origin; by branch* we mean the branch formed of closed sites in the sense of Section 3. Notice that in order to be fresh, a site must be a candidate.

III) If procedure never stops, then $V = \bigcup_n V_n$. □

We will say that a lattice animal is *admissible* if it is obtainable trough the procedure described in the Definition 4.1. We have:

$$\psi(p) = \mu_{1-p}(A) = \sum_{n \geq 1} \mu_{1-p}(|V| = n, A) =$$
$$= \sum_{n\geq 1} \sum_{|V|=n}^{(1)} p^{|V|_c}(1-p)^{|V|-|V|_c} = \sum_{n\geq 1} \sum_{m,b} a_{n,m,b} p^m (1-p)^b, \tag{4.1}$$

where the sum $\Sigma^{(1)}$ is taken over all V such that $|V| = n$, the event A occurs in V, and V is admissible, and $a_{n,m,b}$ is the number of admissible lattice animals of size n with m closed and b open sites in which the event A occurs.

Let z be a complex number in a neighborhood N of an arbitrary closed interval I of $(0,1)$ and write

$$\psi(z) = \sum_{n\geq 1}\sum_{m,b} a_{n,m,b} z^m (1-z)^b =$$

$$= \sum_{n\geq 1}\sum_{m,b} a_{n,m,b} \left(\frac{z}{p}\right)^m \left(\frac{1-z}{1-p}\right)^b p^m (1-p)^b \leq \tag{4.2}$$

$$\leq \sum_{n\geq 1} c^n \sum_{m,b} a_{n,m,b} p^m (1-p)^b = \sum_{n\geq 1} c^n \mu_{1-p}(|V| = n, A),$$

where $c > 1$. By taking N close enough to I, we can bring c arbitrarily close to 1.

Lemma 4.2. There exists $\beta > 0$ such that

$$\mu_{1-p}(|V| = n, A) \leq e^{-\beta n},$$

and β doesn't depend on p.

Lemma 4.3. If $V = n$ then V at least $n/25$ fresh sites.

Proof. Let $\bar{Q}(x)$ denote $x + \bar{S}_2$. We will pick a site x in V and show that $\bar{Q}(x)$ must contain a fresh site. This is true for any site in $\bar{Q}(0)$ because the origin is fresh. So we pick a site x in V outside $\bar{Q}(0)$. Let us consider the successive sites of V in $\bar{Q}(x)$ added in the induction steps order. Sometimes more than one site is added in a single step; in this case, a set of sites of the form $Q(z)$ is added for some z, which is necessarily fresh and we set the convention that z is added before the remaining sites of $Q(z)$, which are added in the given order of $\mathbb{Z}^2$.

It is clear that the first one, say z and added at the end of step k, belongs to the boundary of $\bar{Q}(x)$. If z is not fresh, then clearly $Q(x) \cap V_{k+1} = \emptyset$. So $Q(x)$ is unchecked at the beginning of the next step and so it remains as long as successive sites of V added to the boundary of $\bar{Q}(x)$ are not fresh or a candidate site of $Q(x)$ is not added. Eventually a site of $Q(x)$ must be added (for x has to be added some time), say it does for the first time at the end of step k'. We conclude that if no site in $V_{k'}$ at the boundary of $\bar{Q}(x)$ is fresh, then $Q(x) \cap V_{k'} = \emptyset$ and the added site (at the boundary of $Q(x)$) must be a candidate, for otherwise it belongs to the boundary of $Q(z)$ for a (fresh) site z in the boundary of $\bar{Q}(x)$, and it must be fresh, for it has a corner in (besides belonging to) $Q(x)$. □

Proof of Lemma 4.2.

Let us consider the times in the induction steps for the costruction of V when fresh sites are added, defined as follows.

$$\tau_0 = 0,$$

$$\tau_n = \inf\{k > \tau_{n-1} : \text{ the procedure reached k steps and } x_k \text{ is fresh}\} \quad \text{for } n \geq 1,$$

where we set $\inf \emptyset = \infty$.

In $\{\tau_k < \infty\}$, we denote by A_k the event that one of the unchecked corners Γ_k, chosen in an arbitrary order, together with x_k forms a square which is *not* closed.

Now, by the preceding lemma, $\{|V| = n, A\} \subset \{\tau_k < \infty\}$, with $k = O(n)$, and the last event is contained in

$$\bigcup_{j=1}^{k-1} \bigcap_{i \neq j, i=1}^{k-1} \{\tau_i < \infty, A_i\},$$

and, for $j = k-1$,

$$\mu(\bigcap_{i \neq j, i=1}^{k-1} \{\tau_i < \infty, A_i\}) = \prod_{i=1}^{k-2} \mu(A_i | \{\tau_i < \infty\}, A_{i-1}, \ldots, A_0)$$

and similarly for the other $j < k-1$.

But, clearly, the conditioning events above depend only on the configuration in V_i, while A_i does not, so, upon noticing that V_i is a *stopping set* (that is, the event $\{V_i = v\}$ depends only on the occupation configuration in v for all possible configurations v) for all i, each of the probabilities in the above product equals the (unconditional) probability that a given square is not closed, namely, $c := 1 - p^4$.

Thus

$$\mu(|V| = n, A) \leq (k-1)c^{k-2}. \qquad \square$$

This is a desired bound for p away from 0. Since $c(V)$ is contained in the percolation cluster of closed sites of the origin, standard percolation arguments at low density give us the desired bound for p close to 0.

Now, going back to (4.2), Lemma 4.2 implies that ψ converges uniformly on a region of the complex plane containing $(0,1)$. For the neighborhoods of 0 and 1, we follow Kesten ([3], p 250) to find

$$\sum_{m,b} a_{n,m,b} \leq 4^n.$$

It follows that ψ converges uniformly on complex neighborhoods of 0 and 1. We summarize everything in the following

Theorem 4.4. The density function ψ is analytic function of p in the intereval $[0,1]$. $\square$

We close this section by remarking that the approach used in this section with V does not succeed if used with the minimal structure R_d since the event that such a structure has size n depends on order n^2 sites, which spoils the estimate in the lattice animal expansion (4.2).

5. Concluding remarks.

I. Shape of the density function.

We have

$$\text{a)} \quad \frac{\partial\psi(p)}{\partial p}\Big|_{p=0} = 0; \qquad \text{b)} \quad \frac{\partial\psi(p)}{\partial p}\Big|_{p=1} = 1,$$

Moreover $\psi(p)$ is convex in the neighbourhood of 0 and concave in the neighbourhood of 1.

Indeed, (4.1) implies that $a_1 = a_2 = a_3 = 0$ (since the minimal V which determines A is the closed square) and it implies a). On the other hand $\psi(p) > 0$ for $p > 0$, so if $\tilde{a}_{k_0,b'}$ is the first non-zero coefficient, we get

$$\psi(p) = \sum a_{n,m,b}\, p^m(1-p)^b = p^{k_0} \sum_{k_0 \geq 0} a_{n,m,b}\, p^{m-k_0}(1-p)^b$$

and in the neighbourhood of the origin $\psi(p)$ has the sign of $\tilde{a}_{k_0,b}$ so we get that $\tilde{a}_{k_0,b} > 0$ which implies convexity of ψ at $p = 0$.

Case b) follows from the Russo's formula (see [2]): Let A be an increasing event which depends on only finitely many sites of $\mathbb{Z}^2$. Then

$$\frac{d}{dp} P_p(A) = E_p(N(A)),$$

where $N(A)$ is the number of sites which are pivotal for A. So, if $p = 1$ we have $N(A) = 1$ and get b). Moreover $\psi(p)$ is monotonous, $\psi(b) \leq p$ and $\frac{d\psi}{dp}\big|_{p=1} = 1$ which implies concavity of ψ at $p = 1$.

It is natural to conjecture that ψ has a global S shape, that is, its second derivative changes sign once and only once in the interval $(0,1)$.

II. Case $d > 2$.

It is clear that if $\ell = 2d-1$ then the origin is stable if and only if it belongs or to a circuit or to a dumbbell. So, it remains just to repeat all proofs.

For the case $d < \ell < 2d-1$ (only possible in $d > 2$), the structure of the closed clusters of stable sites is more complicated, with branching. In this context we do not even know how to address the question of existence of an infinite minimal such structure with positive probability, except in two situations, namely when $p > 1-p_c$, that is, in the non-percolative regime of the 0's, where one can derive analyticity of the density and exponential decay of correlations from the exponential decay of the distribution of the size of the percolation cluster of 0's, to which a minimal structure or a construct like V will (basically in the case of V) belong to and in for p close enough to 0, where one gets the exponential estimates from standard cluster expansion arguments. A positive answer in the intermediate region would possibly be interesting, though our feelings go in the opposite direction. Nor can we give a full description of the minimal stable structures in this case as we did for the case treated in this paper in Proposition 3.4.

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