IEEE Access
Multidisciplinary ⋮ Rapid Review ⋮ Open Access Journal

Received June 8, 2020, accepted June 12, 2020, date of publication June 16, 2020, date of current version June 25, 2020.

Digital Object Identifier 10.1109/ACCESS.2020.3002840

Saddle-Node Bifurcations of Power Systems in the Context of Variational Theory and Nonsmooth Optimization

PABLO DANIEL PAZ SALAZAR [1], (Graduate Student Member, IEEE), YAVDAT ILYASOV [2], LUÍS FERNANDO COSTA ALBERTO [3], (Senior Member, IEEE), EDUARDO COELHO MARQUES COSTA [1], AND MAURÍCIO B. C. SALLES [1], (Member, IEEE)

[1]Department of Electrical Automation and Energy Engineering, University of Sao Paulo (USP), São Paulo 05508010, Brazil
[2]Ufa Federal Research Centre, Institute of Mathematics, Russian Academy of Sciences, 450008 Ufa, Russia
[3]Department of Electrical and Computer Engineering, São Carlos School of Engineering, University of São Paulo, São Paulo 13566-590, Brazil

Corresponding author: Pablo Daniel Paz Salazar (pablopaz@usp.br)

This work was supported in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior–Brazil (CAPES) under Finance Code 001, in part by the National Council for Scientific and Technological Development (CNPq), Brazil, and in part by the Cooperativa Rural de Electrificación CRE Ltda., Santa Cruz, Bolivia.

ABSTRACT This paper presents a new concept for finding saddle-node bifurcation (SNB) points in voltage stability analysis of power systems by applying the extended functional method (EFM). This method enables the finding of the SNB point of power systems by directly calculating the extreme values of a nonsmooth variational functional, which is obtained in its turn by the so-called nonlinear generalized Collatz-Wilandt formula. The main theoretical result establishes the EFM applicability for finding the maximum loading capacity of power systems. The maximum loading capacity of the power system is shown to be located at the maximizing point of the nonsmooth function of bifurcations. The subgradient method for nonsmooth functions was applied. The EFM was performed on various IEEE test systems to find SNB points, and the results were compared with those obtained with the Continuation Power Flow (CPF) and the Point of Collapse (PoC) method. The simulation results show that the proposed method is robust and, unlike the PoC method, finds the SNB point even when good guessing of the turning point is not available. Tasks such as tracking an SNB point displaced by a contingency and infeasible power flow were performed successfully by using the EFM.

INDEX TERMS Variational analysis, voltage stability, saddle node bifurcation, nonsmooth optimization, maximum loadability, direct method.

I. INTRODUCTION

Bifurcations are nonlinear phenomena related to a qualitative and abrupt change in nonlinear systems' behavior as a consequence of the variation of one or more parameters. In the 1980s, static bifurcation was associated with loss of steady-state stability and voltage collapse in power systems [1], [2]. Later on, in the early 1990s, a more in-depth theory on static and dynamic bifurcations was developed for voltage stability analysis [3]–[6]. At that time, researchers realized that most of the events associated with voltage collapse were happening with heavily loaded systems as a long term phenomenon.

The associate editor coordinating the review of this manuscript and approving it for publication was Xiaorong Xie.

To date, there still is research interest in real-time applications to find a suitable tool to predict the collapse point with accuracy [7]–[15] and to improve short term voltage stability [16], [17].

In this way, a great variety of voltage stability problems were well modeled and analyzed from a static perspective taking into account a single slowly varying parameter: the load. In the static approach, the Power System is represented by a set of nonlinear algebraic equations (the *power flow* equations) and the maximum loading capacity is associated with a bifurcation of these equations as a consequence of load variation.

Several computational techniques were developed from a static perspective to predict the loadability limit in power systems. These techniques can be classified into two categories:

indirect or path-following methods and direct methods. For example, the continuation power flow methods (CPF) [18], [19] are indirect methods that are widely applied to determine the relationship between power demand and voltage profile (P-V curve). These methods compute the bifurcation point by increasing the load until the loadability limit or turning point is reached. On the other hand, direct methods attempt to directly compute the bifurcation point without computing the trajectory of the system from the initial load to the maximum one.

The *Point of Collapse* (PoC) is a direct method used to compute the SNB point solving a set of nonlinear algebraic equations, which are the bifurcation conditions, and usually require good initial guess of the SNB point to avoid non-convergence of the numerical methods [20]–[23].

In [24], [25], the calculation of SNBs was formulated in the context of Optimal Power Flow (OPF), while in [26], [27], a constrained OPF to calculate the maximum loadability of the system taking into account the limitation of reactive power generation was formulated. In [28], was demonstrated the equivalence between CPF and OPF to calculate SNBs and Limited Induced Bifurcations (LIB).

Recently, novel algorithms with improved computational times have been developed. In [29], for instance, the maximum loadability is computed via factored power flow, whereas [30], presents a fast method for finding LIBs.

The contribution of this paper is the proposal of a conceptually new direct method for computing SNB points (turning points) for voltage stability assessment. The method is based on the so-called *extended functional method* [31], which enables finding the SNB point of the systems of equations by a direct calculation of the extreme values of nonsmooth functions. Unlike the well-known direct methods (see, e.g., [32]), the proposed method is robust and does not require a good initial guess for ensuring convergence.

This paper is structured as follows: Section II presents a background on the EFM. Then, Section III presents the procedure to find SBNs. Finally, in section IV, various IEEE test systems are performed with the EFM and the results compared with CPF and PoC method.

II. BACKGROUND ON THE EXTENDED FUNCTIONAL METHOD

This section presents the Extended Functional Method, which consists of the theoretical background of the direct method proposed in this paper.

Consider the power flow equations given in the following abstract form:

$$f(x,\lambda) = g(x) - \lambda b = 0, \quad x \in Q \subseteq \mathbb{R}^n,\ \lambda \in \mathbb{R}. \tag{f}$$

Here $g : Q \to \mathbb{R}^n$ is a continuously differentiable function, Q is an open domain in $\mathbb{R}^n$, $b \in \mathbb{R}^n$, and λ represents the load parameter. We assume that $b_i > 0$ for all $i \in \{1, \dots n\}$. Obviously, if $b_i \neq 0$, for all $i \in \{1, \dots n\}$, this condition can be achieved by multiplying on (-1) equations in (1).

Hereafter, $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ stand for the Euclidean norm and the scalar product in $\mathbb{R}^n$, respectively; $\nabla_x := (\partial/\partial x_1 \dots \partial/\partial x_n)^T$.

A point (x^*, λ^*) is called *saddle-node bifurcation* (SNB) of (1) in Q if there is a C^1-map $(-a, a) \ni s \longmapsto (x(s), \lambda(s)) \in Q \times \mathbb{R}$ [32]–[34] for some $a > 0$ such that

- $(x(s), \lambda(s))$ satisfies (1), $\forall s \in (-a, a)$, $(x(0), \lambda(0)) = (x^*, \lambda^*)$,
- $\frac{d}{ds}\lambda(0) = 0$,
- $\lambda(s) \in (-\infty, \lambda^*]$ *or* $\lambda(s) \in [\lambda^*, +\infty)$, $\forall s \in (-a, a)$.

A SNB point (x^*, λ^*) of (1) is said to be *maximal* in Q if $\bar{\lambda} \leq \lambda^*$ for any other SNB point $(\bar{x}, \bar{\lambda})$ of (1) in Q. The necessary condition

$$\begin{cases} f(x^*, \lambda^*) = 0, \\ J_x f(x^*, \lambda^*)\zeta^* = 0, \end{cases} \tag{1}$$

for (x^*, λ^*) to be a SNB point, is often called a *branching system* (see e.g., [32], [33]), being the core of the PoC method [20]. Hereafter, $J_x f(x, \lambda) = (\frac{\partial f}{\partial x_i})_{1 \leq i \leq n}$ is the Jacobian matrix of $f(x, \lambda)$. We call

$$\lambda(x) = \min_{i=1,\dots,n} \frac{g_i(x)}{b_i} = \min_{i=1,\dots,n} r_i(x), \quad x \in Q$$

functional of bifurcations of the EFM and denote $r_i(x) = g_i(x)/b_i$, $i = 1, \dots, n$. For $x \in Q$, we define

$$N(x) = \{i \in \{1, \dots, n\} : r_i(x) = \lambda(x)\} \tag{2}$$

and denote by $|N(x)|$ the number of elements in $N(x)$. Observe that for any $x \in Q$, $|N(x)| = n$ if and only if $(x, \lambda(x))$ satisfies to (1).

Following [35], [36], [36], [37], we introduce the convex hull

$$\begin{aligned} \partial\lambda(x) &:= co\{\nabla_x r_i(x) : i \in N(x)\} \\ &= \{z = \sum_{i \in N(x)} \zeta_i \nabla_x r_i(x) : \sum_{i \in N(x)} \zeta_i = 1,\ \zeta_i \geq 0,\ i \in N(x)\}. \end{aligned}$$

A point $\hat{x} \in Q$ is called stationary point of $\lambda(x)$ if $0 \in \partial\lambda(\hat{x})$. Observe,

$$\begin{aligned} \nabla_x r_i(x) &= \frac{1}{b_i}(\nabla_x g_i(x)) \\ &= \frac{1}{b_i}\nabla_x f_i(x, \lambda), \quad i = 1, \dots, n. \end{aligned}$$

Thus, if $\hat{x}$ is a stationary point of $\lambda(x)$, then there exist ζ_i for $i \in N(\hat{x})$ such that $\zeta_i \geq 0$, $i \in N(\hat{x})$, $\sum_i \zeta_i = 1$ and

$$\sum_{i=1}^{n} \zeta_i \frac{1}{b_i} \nabla_x f_i(\hat{x}, \lambda)|_{\lambda=\lambda(\hat{x})} = (J_x f(\hat{x}, \lambda(\hat{x})))^T \xi = 0,$$

where we set $\xi_i = \zeta_i/b_i$ for $i \in N(\hat{x})$ and $\xi_i = 0$ for $i \in \{1, \dots, n\} \setminus N(\hat{x})$. This implies in particular that $Ker J_x f(\hat{x}, \lambda(\hat{x})) \neq \emptyset$. Thus, we infer

- *If $\hat{x} \in Q$ is a stationary point of $\lambda(x)$ so that $|N(\hat{x})| = n$ and $\hat{\lambda} = \lambda(\hat{x})$, then $(\hat{x}, \hat{\lambda})$ satisfies branching system* (1).

Following [31], [36], [37], we shall seek a SNB point of (1) by means of the *nonlinear generalized Collatz-Wieland formula*:

$$\lambda^* = \max_{x\in Q} \lambda(x) = \max_{x\in Q} \min_{i=1,\ldots,n} \frac{g_i(x)}{b_i}. \tag{3}$$

A point (x^*) is called maximizer of (3) if $\lambda^* = \lambda(x^*)$. From [31], [36], [37] we have the following main lemma

Lemma 1: Suppose that $\lambda^* < +\infty$.

(1^o) Then (1) has no solutions in Q for any $\lambda > \lambda^*$.

(2^o) If there exists a maximizer $x^* \in Q$ of (3) such that $|N(x^*)| = n$ and $dimKer(J_x f(x^*, \lambda^*)) = 1$, then (x^*, λ^*) is a maximal SNB point of (1) in Q

Thus, the maximal SNB point of (1) can be found using (3) that is, by maximizing the functional of bifurcations $\lambda(x)$. Notice that $\lambda(x)$ *is a piecewise continuously differentiable function*, and (3) is a nonsmooth optimization problem. Thus, one can make the following conclusion:

- *The maximal SNB point of* (1) *and, therefore, the maximum loading capacity of the power system is located at the maximizing point of the nonsmooth function* $\lambda(x) = \min_i r_i(x)$.

Nonsmooth optimization deals with optimization problems where objective functions have discontinuous gradients. This theory has been intensively developing over the past few decades. There are various numerical methods for solving nonsmooth optimization problems: subgradient, cutting plane, bundle, gradient sampling methods, and others. (see e.g., [38]–[40]). These methods have their supporters and advantages, which may depend on the type of problems under consideration.

Since the power flow equations (1) has not yet been investigated under the framework of the theory of nonsmooth optimization, it makes sense to test the known methods of this theory to determine the optimal one. In the present paper, we test the subgradient method for (1). In this regard, it makes sense to emphasize that the continuation approach of finding bifurcations, as it has been shown in [37] (see also below Remark 1) is a particular case of the subgradient method.

III. ALGORITHM

To find the maximizing point of $\lambda(x)$, we applied the subgradient method using an approach introduced in [37], [41]. According to [39], [42], $\lambda(x)$ is a directionally differentiable function in Q with respect to any vector $d \in \mathbb{R}^n$, and the directional derivative is defined by

$$\lambda'(x; d) = \min_i \{\langle \nabla r_i(x), d\rangle : i \in N(x)\}. \tag{4}$$

Following [39], [42], we call a maximizer $\hat{d}(x) \in \mathbb{R}^n$ of

$$\hat{\sigma}(x) = \lambda'(x; \hat{d}(x)) = \max\{\lambda'(x; d) : ||d|| = 1\} \tag{5}$$

(if $\hat{\sigma}(x) > 0$) a *direction of steepest ascent* of $\lambda(x)$ at $x \in Q$. In the case $\hat{\sigma}(x) > 0$, we define a *gradient* of $\lambda(x)$ as follows

$$\nabla\lambda(x) := \lambda'(x; \hat{d}(x)) \cdot \hat{d}(x) \equiv \hat{\sigma}(x) \cdot \hat{d}(x).$$

Observe that by Demyanov-Malozemov's Theorem [39], $\nabla\lambda(x)$ is the nearest point from the origin 0_n to the convex set $\partial\lambda(x)$ (see 1). Introduce matrices

$$A_{N(x)} = \left(\nabla r_{i_k}(x)\right)^T_{1\le k\le |N(x)|}, \quad \Gamma_{N(x)} = A^T_{N(x)} A_{N(x)},$$

where $i_1, \ldots, i_N \in N(x)$ is an arrangement of the set $N(x)$ such that $i_1 < i_2 < \ldots < i_N$. In the case $|N(x)| = n$, we denote $A(x) := A_{N(x)}$. From [37], [41], it follows that the maximization problem (5) is equivalent to the following quadratic programming problem

$$\hat{\sigma}^2(x) = \min_\alpha \{\alpha^T \Gamma_{N(x)} \alpha : \alpha \in \mathbb{R}^{N(x)}, \quad \sum_{i=1}^{N(x)} \alpha_i = 1, \quad \alpha_i \ge 0, \ i = 1, \ldots, N(x)\}, \tag{6}$$

so that if $\hat{\alpha}(x)$ is a minimizer of (6), then

$$\nabla\lambda(x) = \sum_{k=1}^{N} \hat{\alpha}_k \nabla r_{i_k}(x) \equiv A_{N(x)} \hat{\alpha}(x), \tag{7}$$

and

$$\hat{d}(x) = \nabla\lambda(x)/\hat{\sigma}(x) \equiv \frac{A_{N(x)}\hat{\alpha}(x)}{||A_{N(x)}\hat{\alpha}(x)||} \tag{8}$$

is a maximizer of (5) [41].

An important property of $\hat{\sigma}(x)$ (see [35], [39]) is that if $\hat{\sigma}(x) > 0$, then there exist $\tau_0 > 0$ such that

$$\lambda(x + \tau\hat{d}(x)) > \lambda(x)$$

for any $\tau \in (0, \tau_0)$. Furthermore (see [37]) if $x^* \in Q$ is a maximizer of $\lambda(x)$, then the necessary conditions of optimality $\hat{\sigma}(x^*) = 0$ is satisfied and there exists $\xi^* \in (\overline{\mathbb{R}^+})^n \setminus 0$ such that $\xi^* \in Ker(J_x f^T(x^*, \lambda^*)^T)$.

Accordingly, one can introduce the iteration formula for the steepest ascent direction method (see e.g., [41])

$$x_{k+1} = x_k + t_k \frac{\hat{d}(x_k)}{\|\hat{d}(x_k)\|}.$$

Here $t_k > 0$ is a predetermined step size. However, finding the steepest ascent direction by (6) is time-consuming. To avoid this, the so-called method of quasi-direction of steepest ascent was proposed in [37]. The quasi-direction of steepest ascent $Y(x)$ is determined by solving a linear equations system, which is less time-consuming than solving (6).

Let us shortly describe the ideas of this method. For $\delta \in \mathbb{R}$, consider the following system of equations:

$$\begin{cases} \Gamma_{N(x)}\alpha = \delta \cdot 1_{|N(x)|}, \\ \sum_{i=1}^{|N(x)|} \alpha_i = 1, \end{cases} \tag{9}$$

where $\alpha = (\alpha_1, \ldots, \alpha_{|N(x)|})^T$.

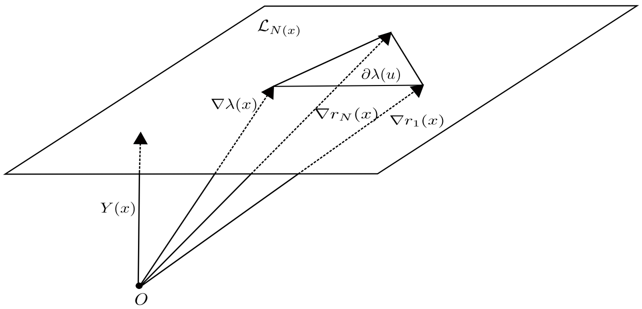

FIGURE 1. Quasi-direction of steepest ascent $Y(x)$ and direction of steepest ascent $\nabla\lambda(x)$.

Assume that $|N(x)| > 1$. Consider the affine space

$$\mathcal{L}_{N(x)} = \{v = \sum_{i \in N(x)} \beta_i \nabla r_i(x) : \sum_{i \in N(x)} \beta_i = 1\}.$$

Introduce the so-called quasi-direction of steepest ascent $Y(x) = A^T_{N(x)}\alpha(x)$, where $\alpha(x)$ satisfies (9). Observe, if $\delta > 0$, then (9) implies

$$\langle Y(x), \nabla r_i(x)\rangle = \langle A_{N(x)}\alpha, \quad \nabla r_i(x)\rangle = \langle \Gamma_{N(x)}\alpha, e_i\rangle \quad (10)$$

$\forall i \in N(x)$. From this, it follows that $Y(x)$ is an orthogonal vector to $\mathcal{L}_{N(x)}$ (see [37]), and thus $Y(x)$ is the nearest point from the origin $0_n \in \mathbb{R}^n$ to the affine space $\mathcal{L}_{N(x)}$. Recall that $\nabla\lambda(x)$ is the nearest point from the origin 0_n to the convex set $\partial\lambda(x)$. Since $\partial\lambda(x)$ lies on $\mathcal{L}_{N(x)}$, it follows that the necessary conditions of optimality $\hat{\sigma}(x^*) = 0$, i.e., $0 \in \partial\lambda(x)$, entails $Y(x^*) = 0$. See Fig. 1.

Furthermore, the following main lemma holds (see [37])

Lemma 2: Let $x \in Q$ and assume that (α, δ) is a solution of (9). Then

a) If $\delta = 0$, then $|N(x)| = n$ and $\alpha_k \neq 0$, $\forall k = 1, \ldots, n$.
b) If $\delta = 0$ and $\alpha_k > 0$, $\forall k = 1, \ldots, n$, then $\hat{\sigma}(x) = 0$, and there exists $\xi \in (\mathbb{R}^+)^n \setminus 0$ such that $\xi \in Ker(J_x f(x, \lambda))^T$ with $\lambda = \lambda(x, \xi)$.
c) If $\delta = 0$ and there exist subsets $N_1(x), N_2(x)$ such that $N_1(x) \cup N_2(x) = N(x)$ and $\alpha_k > 0, \forall k \in N_1(x)$, whereas $\alpha_k \leq 0, \forall k \in N_2(x)$, then $\nabla\lambda(x)$ lies on the boundary $\partial_{N_1(x)}\lambda(x)$ of $\partial\lambda(x)$.

Using this, the following convergence iteration by quasi-direction of steepest ascent method can be introduced:

$$x_{k+1} = x_k + \tau_k \frac{Y(x_k)}{\|Y(x_k)\|}$$

where $\tau_k > 0$ is a predetermined step size [37].

Below, the finding of the maximal SNB point of (1) will be carried out with a given accuracy. Let us give the corresponding definitions. Following [37], we say that x is a solution of (1) with accuracy $\varepsilon > 0$, if

$$|r_i(x) - \lambda(x)| < \varepsilon \quad \text{for all } i = 1, 2, \ldots, n. \quad (11)$$

Denote $N_\varepsilon(x) = \{i \in [1 : n] : |r_i(x) - \lambda(x)| < \varepsilon\}$. Then x is a solution of (1) with accuracy $\varepsilon > 0$ if and only if $|N_\varepsilon(u)| = n$.

Let $\varepsilon_0 > 0$, $\delta_0 > 0$. We call $x^*_{(\varepsilon_0,\delta_0)}$ the δ_0- SNB point of (1) with accuracy ε_0 if

(i) $|N_{\varepsilon_0}(x^*_{(\varepsilon_0,\delta_0)})| = n$,
(ii) $\Gamma_{N_{\varepsilon_0}(x^*_{(\varepsilon_0,\delta_0)})}\alpha = \delta \cdot 1_{|N_{\varepsilon_0}(x^*_{(\varepsilon_0,\delta_0)})|}$, for $\delta \in (0, \delta_0)$ and $\alpha \in \mathbb{R}_+^{|N_{\varepsilon_0}(x^*_{(\varepsilon_0,\delta_0)})|}$ such that $\sum_{1 \leq i \leq |N_{\varepsilon_0}(x^*_{(\varepsilon_0,\delta_0)})|} \alpha_i = 1$.

The corresponding pseudo-code of the quasi-direction of the steepest ascent (QDSA) algorithm for finding the δ_0- SNB point with a given accuracy ϵ_0 is presented in the next section. The algorithm enables the finite cyclic reduction of ϵ and δ. The convergence of this algorithm is discussed in [37].

Remark 1: Assume $N(x) = n$, then

$$\lambda(x) = \frac{g_i(x)}{b_i}, \quad \forall i = 1, \ldots, n,$$

which means that x satisfies (1) with $\lambda = \lambda(x)$ and thus, the point x lies on the branch of the solutions of (1). On the other hand, if $N(x) = n$ and (α, δ) solves (9), then:

$$A(x)A^T(x)\alpha = \delta 1_n$$

Now denoting $\dot{x} = \frac{A^T(x)\alpha}{\delta}$, we obtain (see [41]) the Davidenko-Abbott system

$$J_x f(x, \lambda))\dot{x} = -f_\lambda(x, \lambda), \quad (12)$$

which lies at the core of the continuation methods (see e.g. [32], [33]). Notice that equality $Y(x) = \dot{x}\delta$ implies that the quasi-direction of steepest ascent $Y(x)$ is collinear with the tangent vector $\dot{x}$ to the curve of the branch of the solutions of (1). Thus, the predictor step by the tangent vector in the continuation approach is, in fact, a particular case of the quasi-direction of the steepest ascent method when $|N(x)| = n$.

A. PSEUDO-CODE FOR THE QUASI-DIRECTION OF STEEPEST ASCENT ALGORITHM (QDSA)

IV. STUDY CASES

In this section, we present the performance of the EFM applied on various tasks and the comparison with CPF and PoC method. The IEEE 14, 30, 57 and 118 test systems are used for simulations ignoring reactive power limits of generators. The study cases are divided into five sections:

- Testing various initial points x^0.
- Testing large λ cases.
- Tracking performance of the proposed method.
- Infeasible Power Flow ($0 < \lambda < 1$).
- The EFM providing x^0 for PoC.

A. TESTING VARIOUS INITIAL POINTS x^0

The state vector x is composed of n unknown variables ($n_{bus} + n_{pq} - 1 = n$), where n_{bus} is the number of buses and n_{pq} is the number of PQ type buses. The first n_{pq} elements of the vector x represent voltage modules $\bar{V} := [v_1, \ldots, v_{n_{pq}}]$, followed by voltage angles $\bar{\theta} := [\theta_1, \ldots, \theta_{n_{bus}-1}]$, so that $x = [\bar{V}, \bar{\theta}]^T$.

Algorithm 1 The QDSA Pseudo-Code

1: Set an initial point x^0 and accuracies $\epsilon_0 > 0$ and $\delta_0 > 0$ For $k = 0, 1, 2 \ldots.$ until the saddle-node bifurcation point (x^*, λ^*) is found.

2: Compute

$$\lambda(x^k) = \min_i r_i(x^k)$$
$$\mu(x^k) = \max_i r_i(x^k)$$

Compute

$$\epsilon = (\mu(x^k) - \lambda(x^k))/2$$

3: Input the set of indexes

$$N_\epsilon(x^k) = \{i \in \{1 : n\} : |r_i(x^k) - \lambda(x^k)| < \epsilon\},$$
$$N = |N_\epsilon(x^k)|$$

4: Find δ^k and α^k by solving

$$\mathcal{M}_{N_\epsilon(x^k)} t^k = q_N,$$

where

$$\mathcal{M}_{N_\epsilon(x^k)} = \begin{pmatrix} \Gamma_{N_\epsilon(x^k)} & -1_{N_\epsilon(x^k)} \\ 1^T_{N_\epsilon(x^k)} & 0 \end{pmatrix}.$$
$$\Gamma_{N_\epsilon(x^k)} = A^T_{N_\epsilon(x^k)} A_{N_\epsilon(x^k)}, \quad t^k = \begin{pmatrix} \alpha^k \\ \delta^k \end{pmatrix}, \; q_N = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

5: If $\delta^k < \delta_0$, then go to Step 6, otherwise

a) Compute the quasi-direction of steepest ascent

$$y^k = \frac{Y^k}{\|Y^k\|}, \quad Y^k = A^T_{N_\epsilon(x^k)} \alpha^k$$

b) Find the step length τ^k by the *golden search rule* applied to

$$\tau^k = arg \max\{\min_i r_i(x^k + \tau y^k)\}, \quad \tau > 0$$

c) Compute $x^{k+1} = x^k + \tau^k y^k$ and return to Step 2

6: If $\epsilon < \epsilon_0$, then go to step 7, otherwise $\epsilon = \epsilon/2$ and return to step 3

7: If $N = n$, Output the bifurcation point $x^*_{(\epsilon,\delta)}$ and $\lambda^*_{(\epsilon,\delta)}$, otherwise $\delta_0 = \delta_0/2$ and return to step 6

Four initial guess points x^0 are used to start the algorithm:

- $x^0 = [\bar{1}, \bar{0}]^T$ (Flat start)
- $\lambda(x^0) = 1$ (Equilibrium base case point)
- $x^0 = [\bar{0.5}, \frac{-\pi}{10}]^T$
- $x^0 = [\bar{0.5}, \frac{-\pi}{6}]^T$

In all scenarios, the performance is compared with the Continuation Power Flow and PoC Methods, using the open-access Matlab software PSAT [43].

TABLE 1. Comparison between methods; IEEE 14-bus system.

	EFM		CPF-PSAT			PoC-PSAT	
Case	λ	Time [s]	λ	Time [s]	Step Size	λ	Time [s]
$x^0 = [\bar{1}, \bar{0}]^T$	3.87	0.61	3.87	0.41	0.5	3.87	0.14
$\lambda(x^0) = 1$	3.87	0.84	3.87	0.31	0.5	3.87	0.14
$x^0 = [\bar{0.5}, \frac{-\pi}{10}]^T$	3.87	0.73	NC[a]		0.5	NC[a]	
$x^0 = [\bar{0.5}, \frac{-\pi}{6}]^T$	3.87	0.73	NC[a]		0.5	NC[a]	

[a] Not Converged;

TABLE 2. Comparison between methods; IEEE 30-bus system.

	EFM		CPF-PSAT			PoC-PSAT	
Case	λ	Time [s]	λ	Time [s]	Step Size	λ	Time [s]
$x^0 = [\bar{1}, \bar{0}]^T$	2.52	1.7	2.52	1.52	0.5	2.52	0.29
$\lambda(x^0) = 1$	2.52	4.9	2.52	1.42	0.5	2.52	0.29
$x^0 = [\bar{0.5}, \frac{-\pi}{10}]^T$	2.52	2.7	NC[a]		0.5	NC[a]	
$x^0 = [\bar{0.5}, \frac{-\pi}{6}]^T$	2.52	4.5	NC[a]		0.5	NC[a]	

[a] Not Converged;

TABLE 3. Comparison between methods; IEEE 57-bus system.

	EFM		CPF-PSAT			PoC-PSAT	
Case	λ	Time [s]	λ	Time [s]	Step Size	λ	Time [s]
$x^0 = [\bar{1}, \bar{0}]^T$	1.55	9.1	1.55	0.937	0.1	1.55	0.15
$\lambda(x^0) = 1$	1.56	11.1	1.55	0.837	0.1	1.55	0.15
$x^0 = [\bar{0.5}, \frac{-\pi}{10}]^T$	1.55	112.8	NC[a]		0.5	NC[a]	
$x^0 = [\bar{0.5}, \frac{-\pi}{6}]^T$	1.55	115.6	NC[a]		0.5	NC[a]	

[a] Not Converged;

TABLE 4. Comparison between methods; IEEE 118-bus system.

	EFM		CPF-PSAT			PoC-PSAT	
Case	λ	Time [s]	λ	Time [s]	Step Size	λ	Time [s]
$x^0 = [\bar{1}, \bar{0}]^T$	2.05	74.5	2.05	1.12	0.5	2.05	0.2
$\lambda(x^0) = 1$	2.05	76.8	2.05	1.02	0.5	2.05	0.2
$x^0 = [\bar{0.5}, \frac{-\pi}{10}]^T$	2.05	235.7	NC[a]		0.5	NC[a]	
$x^0 = [\bar{0.5}, \frac{-\pi}{6}]^T$	2.05	256.8	NC[a]		0.5	NC[a]	

[a] Not Converged;

The EFM was implemented in MatLab R2015. For the Continuation Power Flow simulations on PSAT, the following settings were applied:

- Stop criterion: At Bifurcation point.
- Correction method: Perpendicular intersection.
- Adaptive Step Size with different initial values in order to not exceed 50 iterations (default).

For the PoC on PSAT, all simulations begin running CPF to get a close initial guess since otherwise, none of the cases converge. Simulations were run on an Intel i7 of 3.2 GHz CPU and 16 GB of RAM.

As shown in Tables 1,2,3, and 4, the EFM was the only one that converged successfully and found the solutions using initial guess points $x^0 = [\bar{0.5}, \frac{-\pi}{10}]^T$ and $x^0 = [\bar{0.5}, \frac{-\pi}{6}]^T$. No case converged with the PoC method using a flat start, and the converged cases used a suitable initial point provided by the CPF method.

Fig. 2 shows the schematic representation of the iterative procedure of the EFM to find the SNB using a flat start as the initial guess point.

B. TESTING LARGE λ CASES

Six random load level scenarios of the IEEE 30-bus system were simulated, decreasing the system's total load in the subsequent scenario to increase the voltage Stability

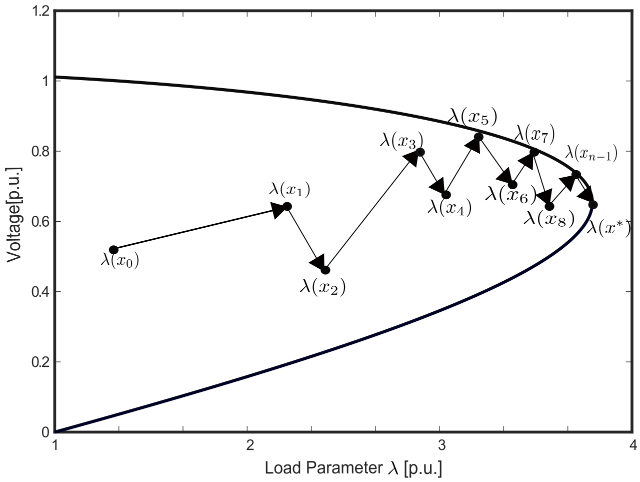

FIGURE 2. Representation of the iterative procedure to find the SNB by the EFM.

TABLE 5. Testing large λ cases; $x^0 = [\bar{1}, \bar{0}]^T$ (IEEE 30-bus system).

	EFM		CPF-PSAT			PoC-PSAT	
Case	λ	Time [s]	λ	Time [s]	Step Size	λ	Time [s]
1	2.52	1.7	2.52	1.52	0.5	2.52	0.29
2	4.85	1.9	4.85	1.08	0.5	4.85	0.2386
3	6.00	2.1	6.00	0.68	0.5	6.00	0.2758
4	7.14	1.4	7.14	2.73	0.5	7.15	0.2385
5	9.67	1.3	9.68	1.73	2	NC[a]	NC[a]
6	13.87	1.7	13.87	2.6	2	WA[b]	WA[b]

[a] Not Converged;
[b] Wrong Answer

Margin (VSM) and to stress the methods. Table 5 shows that the PoC is the fastest method among all (with initial guess provided by the CPF method). On the other hand, EFM has shown better performance than CPF for large λ.

The CPF is time-consuming for cases with large λ (Base Case far from SNB), while the EFM seems to have a constant-time performance independent of λ size.

Even with the initial guess provided by CPF, two cases did not converge using the PoC method, and one converged case yielded a wrong answer.

C. TRACKING PERFORMANCE OF THE EXTENDED FUNCTIONAL METHOD

We compared the EFM and the PoC method tracking an SNB point shifted by a network contingency using the SNB point before perturbation as the initial guess for finding the new SNB. The CPF method does not work in this case because it is dependent on the base case convergence. For testing tracking performance, two perturbations were introduced in Case 1 of the IEEE 30-bus system. The first one causes the SNB point displacement due to the outage of two capacitors (buses 10 and 14). This contingency is considered a small perturbation since λ decreases less than 2% (from 2.52 to 2.48). The outage of line 2-5 causes the second displacement of the SNB. This contingency is considered a significant perturbation since λ decreases considerably (from 2.48 to 2.02). Table 6 shows that both the EFM and PoC method tracks the first SNB displacement successfully, being PoC faster than EFM. The second displacement is tracked successfully only by the EFM. Fig. 3 shows the schematic representation of the tracking procedure.

TABLE 6. Tracking performance of the EF method - IEEE 30-bus system.

		EFM		PoC-PSAT	
Case	x^0	λ	Time [s]	λ	Time [s]
Cont. 1	Base Case SNB	2.48	1.1	2.48	0.268
Cont. 2	Cont.1 SNB	2.02	1.4	NC[a]	NC[a]

[a] Not Converged;

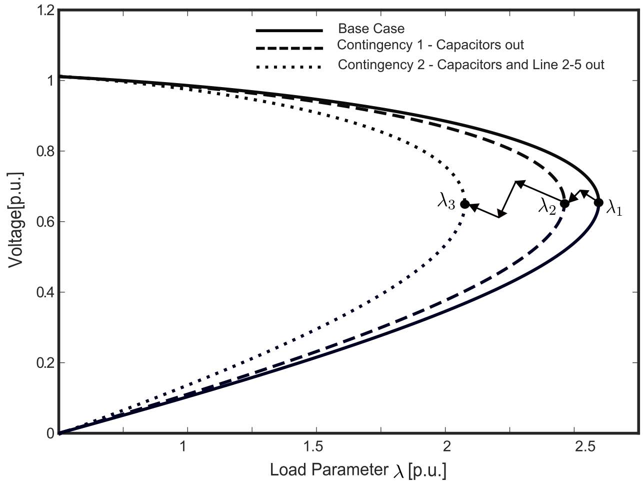

FIGURE 3. Tracking SNB point scheme.

D. INFEASIBLE POWER FLOW ($0 < \lambda < 1$)

We tested an infeasible power flow in which the given load level is beyond the SNB point, and a physical solution does not exist. In this case, SNB point exists with a load level smaller than the given load level, so that the SNB point exists in the range $0 < \lambda < 1$. For testing an infeasible power flow, a severe contingency to case 1 of the IEEE 30-bus-system was applied. Generators 2, 3, 5, and 13 were set as PQ buses, generators 8 and 11 were set with 1 [p.u.] of voltage output, capacitors 10 and 24, lines 2-5, 6-8, and 14-15 were set out of service. This simulation converged successfully with the EFM yielding $\lambda = 0.84$. PoC method did not converge. Fig. 4 shows how the PV curve decreases to the infeasible region ($0 < \lambda < 1$) due to the severe contingency.

E. EFM PROVIDING x^0 *FOR PoC METHOD*

Finally, we tested the EFM, providing suitable x^0 for the PoC method. We tested Cases 5 and 6 of the IEEE 30-bus system, in which the PoC method did not converge with the initial guess point provided by the CPF method. The stop criterion for the EFM to provide suitable initial guess point are $|N(x^k)| = n$ and $\mu(x^k) - \lambda(x^k) < 5$. It is worth noting that the initial right eigenvector and λ for the PoC method can be provided by the EFM as well. Using the mentioned stop criterion, the EFM provided a suitable initial guess point for PoC convergence. Table 7 shows the total time of the two cases using both the EFM and PoC method.

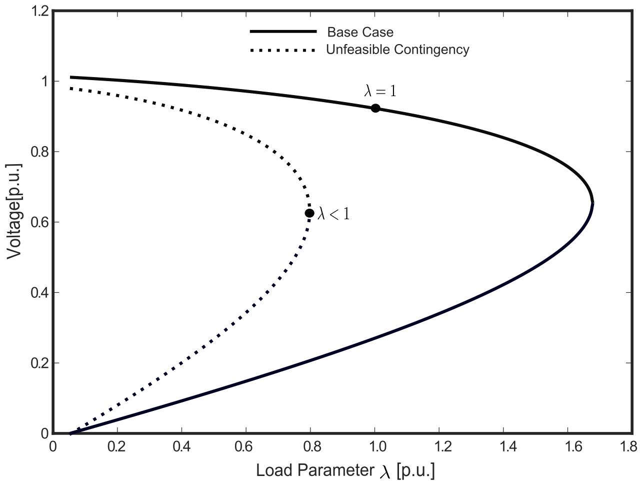

FIGURE 4. Infeasible contingency case.

TABLE 7. EFM providing x^0 for PoC method - IEEE 30-bus system.

Case	λ	EFM Time [s]	PoC-PSAT Time [s]	Total Time [s]
5	9.68	0.5	0.3	0.8
6	13.87	0.9	0.25	1.15

V. CONCLUSION

We presented a new concept for finding saddle-node bifurcation points based on the Extended Functional Method. It was demonstrated that it applies to find the maximum loadability of power systems. Moreover, we introduced the nonsmooth functional $\lambda(x)$, which stationary points correspond to the saddle-node bifurcation point. This means that a full range of nonsmooth optimization methods can be applied for finding bifurcation points of power systems. Among these methods, the subgradient method for nonsmooth functions (QDSA algorithm) has been applied for finding the maximal bifurcation point of power systems. As a result, the developed method shows flexibility for different tasks, such as tracking SNB's, calculating infeasible power flow cases, and providing initial guess points for the PoC method. The developed method shows robust convergence starting with non-conventional initial guess points wherewith classical methods do not converge. Since the technique is not path-following, it fits into the direct method class but without the need for the left/right eigenvector guess point.

We note two main features on the EFM, which are worth highlighting:

- Unlike the CPF method, the computing time is not proportional to λ size.
- Unlike the PoC method, the EFM does not need an initial guess point close to the solution for ensuring convergence. Instead, the EFM converges using an initial guess point away from the solution.

Since we have demonstrated that the proposed technique applies to power systems and has exciting features, we believe that other methods for nonsmooth functions are worthy of researching in forthcoming works and could yield improved results.

REFERENCES

[1] A. Arapostathis, S. Sastry, and P. Varaiya, "Analysis of power-flow equation," *Int. J. Electr. Power Energy Syst.*, vol. 3, no. 3, pp. 115–126, Jul. 1981.

[2] H. Kwatny, A. Pasrija, and L. Bahar, "Static bifurcations in electric power networks: Loss of steady-state stability and voltage collapse," *IEEE Trans. Circuits Syst.*, vol. 33, no. 10, pp. 981–991, Oct. 1986.

[3] I. Dobson and H.-D. Chiang, "Towards a theory of voltage collapse in electric power systems," *Syst. Control Lett.*, vol. 13, no. 3, pp. 253–262, 1989.

[4] H.-D. Chiang, I. Dobson, R. J. Thomas, J. S. Thorp, and L. Fekih-Ahmed, "On voltage collapse in electric power systems," *IEEE Trans. Power Syst.*, vol. 5, no. 2, pp. 601–611, May 1990.

[5] V. Ajjarapu and B. Lee, "Bifurcation theory and its application to nonlinear dynamical phenomena in an electrical power system," *IEEE Trans. Power Syst.*, vol. 7, no. 1, pp. 424–431, Dec. 1992.

[6] C. A. Canizares, "On bifurcations, voltage collapse and load modeling," *IEEE Trans. Power Syst.*, vol. 10, no. 1, pp. 512–522, Apr. 1995.

[7] K. Vu, M. M. Begovic, D. Novosel, and M. M. Saha, "Use of local measurements to estimate voltage-stability margin," *IEEE Trans. Power Syst.*, vol. 14, no. 3, pp. 1029–1035, Aug. 1999.

[8] L. Bao, Z. Huang, and W. Xu, "Online voltage stability monitoring using var reserves," *IEEE Trans. Power Syst.*, vol. 18, no. 4, pp. 1461–1469, Nov. 2003.

[9] D. Q. Zhou, U. D. Annakkage, and A. D. Rajapakse, "Online monitoring of voltage stability margin using an artificial neural network," *IEEE Trans. Power Syst.*, vol. 25, no. 3, pp. 1566–1574, Aug. 2010.

[10] B. Leonardi and V. Ajjarapu, "Development of multilinear regression models for online voltage stability margin estimation," *IEEE Trans. Power Syst.*, vol. 26, no. 1, pp. 374–383, Feb. 2011.

[11] Y. Wang, I. R. Pordanjani, W. Li, W. Xu, T. Chen, E. Vaahedi, and J. Gurney, "Voltage stability monitoring based on the concept of coupled single-port circuit," *IEEE Trans. Power Syst.*, vol. 26, no. 4, pp. 2154–2163, Nov. 2011.

[12] B. Leonardi and V. Ajjarapu, "An approach for real time voltage stability margin control via reactive power reserve sensitivities," *IEEE Trans. Power Syst.*, vol. 28, no. 2, pp. 615–625, May 2013.

[13] F. Hu, K. Sun, A. Del Rosso, E. Farantatos, and N. Bhatt, "Measurement-based real-time voltage stability monitoring for load areas," *IEEE Trans. Power Syst.*, vol. 31, no. 4, pp. 2787–2798, Jul. 2016.

[14] R. E. C. Gutiérrez and J. M. Ramirez, "Voltage collapse detection based on local measurements," *Electr. Power Syst. Res.*, vol. 107, pp. 77–84, Feb. 2014.

[15] A. Chandra and A. K. Pradhan, "Online voltage stability and load margin assessment using wide area measurements," *Int. J. Electr. Power Energy Syst.*, vol. 108, pp. 392–401, Jun. 2019.

[16] T. Han, Y. Chen, J. Ma, Y. Zhao, and Y.-Y. Chi, "Surrogate modeling-based multi-objective dynamic VAR planning considering short-term voltage stability and transient stability," *IEEE Trans. Power Syst.*, vol. 33, no. 1, pp. 622–633, Jan. 2018.

[17] T. Han, Y. Chen, and J. Ma, "Multi-objective robust dynamic VAR planning in power transmission girds for improving short-term voltage stability under uncertainties," *IET Gener., Transmiss. Distrib.*, vol. 12, no. 8, pp. 1929–1940, Apr. 2018.

[18] V. Ajjarapu and C. Christy, "The continuation power flow: A tool for steady state voltage stability analysis," *IEEE Trans. Power Syst.*, vol. 7, no. 1, pp. 416–423, 1992.

[19] H.-D. Chiang, A. J. Flueck, K. S. Shah, and N. Balu, "CPFLOW: A practical tool for tracing power system steady-state stationary behavior due to load and generation variations," *IEEE Trans. Power Syst.*, vol. 10, no. 2, pp. 623–634, May 1995.

[20] F. L. Alvarado, "Direct detection of voltage collapse conditions," in *Proc. Bulk Power Syst. Voltage Phenomena-Voltage Stability Secur.*, Oct. 1989, pp. 523–538.

[21] V. Ajjarapu, "Identification of steady-state voltage stability in power systems," *Int. J. Energy Syst.*, vol. 11, no. 1, pp. 43–46, 1991.

[22] C. A. Canizares and F. L. Alvarado, "Point of collapse and continuation methods for large AC/DC systems," *IEEE Trans. Power Syst.*, vol. 8, no. 1, pp. 1–8, Feb. 1993.

[23] I. Dobson and L. Lu, "New methods for computing a closest saddle node bifurcation and worst case load power margin for voltage collapse," *IEEE Trans. Power Syst.*, vol. 8, no. 3, pp. 905–913, Aug. 1993.

[24] C. A. Canizares, "Calculating optimal system parameters to maximize the distance to saddle-node bifurcations," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 45, no. 3, pp. 225–237, Mar. 1998.
[25] C. A. Cañizares, "Applications of optimization to voltage collapse analysis," in *Proc. IEEE-PES Summer Meeting*, San Diego, CA, USA, Oct. 1998, pp. 1–7.
[26] C. D. Vournas, M. Karystianos, and N. G. Maratos, "Bifurcation points and loadability limits as solutions of constrained optimization problems," in *Proc. Power Eng. Soc. Summer Meeting*, 2000, pp. 1883–1888.
[27] W. Rosehart, C. Roman, and A. Schellenberg, "Optimal power flow with complementarity constraints," *IEEE Trans. Power Syst.*, vol. 20, no. 2, pp. 813–822, May 2005.
[28] R. J. Avalos, C. A. Canizares, F. Milano, and A. J. Conejo, "Equivalency of continuation and optimization methods to determine saddle-node and limit-induced bifurcations in power systems," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 56, no. 1, pp. 210–223, Jan. 2009.
[29] C. Gomez-Quiles, A. Gomez-Exposito, and W. Vargas, "Computation of maximum loading points via the factored load flow," *IEEE Trans. Power Syst.*, vol. 31, no. 5, pp. 4128–4134, Sep. 2016.
[30] L. S. Neves, L. F. C. Alberto, and H.-D. Chiang, "A fast method for detecting limit-induced bifurcation in electric power systems," *Electr. Power Syst. Res.*, vol. 180, Mar. 2020, Art. no. 106101.
[31] Y. S. Il'yasov, "Bifurcation calculus by the extended functional method," *Funct. Anal. Appl.*, vol. 41, no. 1, pp. 18–30, Jan. 2007.
[32] R. Seydel, *Practical Bifurcation and Stability Analysis*, vol. 5. New York, NY, USA: Springer-Verlag, 2009.
[33] H. B. Keller, "Numerical solution of bifurcation and nonlinear eigenvalue problems," *Appl. Bifurcation Theory*, vol. 4, pp. 359–384, Feb. 1977.
[34] H. Kielhöfer, *Bifurcation Theory: An Introduction With Applications to PDEs*, vol. 156. New York, NY, USA: Springer-Verlag, 2006.
[35] V. F. Demyanov, G. E. Stavroulakis, L. N. Polyakova, and P. D. Panagiotopoulos, *Quasidifferentiability and Nonsmooth Modelling in Mechanics, Engineering and Economics*, vol. 10. New York, NY, USA: Springer-Verlag, 2013.
[36] Y. Il'yasov, "On a generalized collatz-wielandt formula and finding saddle-node bifurcations," 2020, *arXiv:2003.12556*. [Online]. Available: http://arxiv.org/abs/2003.12556
[37] Y. Il'yasov and A. Ivanov, "Computation of maximal turning points to nonlinear equations by nonsmooth optimization," *Optim. Methods Softw.*, vol. 31, no. 1, pp. 1–23, Jan. 2016.
[38] A. Bagirov, N. Karmitsa, and M. M. Mäkelä, *Introduction to Nonsmooth Optimization: Theory, Practice and Software*. Cham, Switzerland: Springer, 2014.
[39] V. F. Dem'yanov and V. N. Malozemov, *Introduction to Minimax*. Chelmsford, MA, USA: Courier Corporation, 1990.
[40] K. C. Kiwiel, *Methods Descent for Nondifferentiable optimization*, vol. 1133. Berlin, Germany: Springer-Verlag, 2006.
[41] A. Ivanov and Y. S. Il'yasov, "Finding bifurcations for solutions of nonlinear equations by quadratic programming methods," *Zhurnal Vychislitel'noi Matematiki Matematicheskoi Fiziki*, vol. 53, no. 3, pp. 350–364, 2013.
[42] R. Rockafellar, "Directional differentiability of the optimal value function in a nonlinear programming problem," in *Sensitivity, Stability Parametric Analysis*. Berlin, Germany: Springer-Verlag, 1984, pp. 213–226.
[43] F. Milano, "An open source power system analysis toolbox," *IEEE Trans. Power Syst.*, vol. 20, no. 3, pp. 1199–1206, Aug. 2005.

PABLO DANIEL PAZ SALAZAR (Graduate Student Member, IEEE) received the B.S. degree in electromechanical engineering from Universidad Autonoma Gabriel René Moreno, Santa Cruz, Bolivia, in 2008, and the M.S. degree in electrical engineering from the University of São Paulo, São Paulo, Brazil, in 2018, where he is currently pursuing the Ph.D. degree in electrical engineering. His research interests include smart grids, renewable energy, voltage stability, and energy storage.

YAVDAT ILYASOV received the Ph.D. degree from Moscow State University, in 1989, and the Doctor of Sciences degree from RAS, in 2000. He has been the Head of the Department, Institute of Mathematics, Russian Academy of Sciences, Ufa, Russia, since 2008. He is currently a Visiting Professor of IME, Universidade Federal de Goiás, Goiania, Brazil. His research interests include nonlinear analysis, solvability of nonlinear partial differential equations, variational methods, the theory of bifurcations, and numerical methods.

LUÍS FERNANDO COSTA ALBERTO (Senior Member, IEEE) received the M.S. and Ph.D. degrees in electrical engineering from the University of São Paulo, São Carlos, Brazil, in 1997 and 2000, respectively. He is currently a Full Professor with the Electrical Engineering Department, School of Engineering of São Carlos, University of São Paulo (USP), São Carlos. He held one-year appointments with Cornell University as a Visitor, in 2004 and 2011. He is a coauthor (with H.-D. Chiang) of the book *Stability Regions of Nonlinear Dynamical Systems: Theory, Estimation, and Applications* (Cambridge University Press, Cambridge, U.K., 2015). His research interests include analysis of nonlinear circuits and power systems. His contributions include the development of stability theory, stability region theory, invariance principles, and direct methods for stability analysis of power systems. He was a recipient of the Instituto de Engenharia Award as the Best Student of the Engineering School at USP, in 1995.

EDUARDO COELHO MARQUES COSTA received the Ph.D. degree in electrical engineering from the University of Campinas-Unicamp, Campinas, Brazil, in 2013. He is currently an Assistant Professor with the Polytechnic School, University of São Paulo. His research interests include power transmission systems, electromagnetic transients, power system modeling, and identification methods.

MAURÍCIO B. C. SALLES (Member, IEEE) received the M.Sc. degree from the State University of Campinas (UNICAMP), Sao Paulo, Brazil, and the Ph.D. degree from the University of Sao Paulo (USP), in 2009. He is currently an Assistant Professor with the Polytechnic School, University of Sao Paulo, since 2010. From 2006 to 2008, he joined the Research Team, Institute of Electrical Machines, RWTH Aachen University. From 2014 to 2015, he was a Visiting Scholar with the Harvard John A. Paulson School of Engineering and Applied Sciences. In 2018, he was an Invited Professor with the Ecole Centrale de Lille, France. His main research interests include distributed generation, power system dynamics, control and stability, renewable energy, energy storage and electricity markets. He is one of the founders of the Laboratory of Advanced Electric Grids–LGrid.