Sheaves on Quantales and their Truth Values

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Sheaf Theory is a well-established area of research with applications that goes from Algebraic Geometry [9] and Logic [10] to recent developments in Machine Learning [1]. A presheaf on a locale (or, equivalently, a complete Heyting algebra) L is a functor $F : L^{op} \to Set$. Given $u, v \in L$ such that $v \leq u$, we consider restriction maps $\rho^u_v : F(u) \to F(v)$ and denote $\rho^u_v(t) = t_{\restriction_v}$, for any $t \in F(u)$. Quantales are a non-idempotent and non-commutative generalization of locales, introduced by C.J. Mulvey [2]. Explicitly, a quantale Q is a complete lattice endowed with a binary associate operation $\odot$ that distributes over arbitrary suprema. A quantale Q is *semicartesian*, whenever $u \odot v \leq u, v$. Examples of semicartesian quantales include locales, the poset of ideals of a commutative ring, and the interval $[0, \infty]^{op}$ (where $\odot = +$). There are distinct definitions of sheaves on quantales [3–7], generalizing different ways that one can approach sheaf theory.

In [8,11], we proposed the following novel definition of a sheaf on a semicartesian (commutative) quantale Q:

A presheaf $F : Q^{op} \to Set$ is a **sheaf** if for any cover $u = \bigvee_{i\in I} u_i$ of any element $u \in Q$, the following diagram is an equalizer

$$F(u) \xrightarrow{e} \prod_{i\in I} F(u_i) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{(i,j)\in I\times I} F(u_i \odot u_j)$$

where

$$e(t) = \{t_{\restriction_{u_i}} \ : \ i \in I\}, \quad p((t_k)_{k\in I}) = (t_{i_{\restriction_{u_i \odot u_j}}})_{(i,j)\in I\times I}$$
$$q((t_k)_{k\in I}) = (t_{j_{\restriction_{u_i \odot u_j}}})_{(i,j)\in I\times I}$$

This definition recovers cohomological [13] and logical aspects [8,12] of sheaf theory. In this talk, we present the category $Sh(Q)$ whose objects are sheaves on Q and whose morphisms are natural transformations. The main results concerning the logical properties of this category are:

- $Sh(Q)$ is not a topos in general, although it behaves like one in a certain sense.
- The lattice of external truth values of $Sh(Q)$, that is, the lattice of subobjcts of the terminal sheaf, is canonically isomorphic to the quantale Q.
- Assuming certain extra conditions on Q, there is a sheaf that essentially classifies a class of subobjects in the category $Sh(Q)$. In other words, there is a candidate for an "internal truth value object" in $Sh(Q)$.

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The above places this work as part of a greater project towards a monoidal closed but non-cartesian closed version of elementary toposes. We hope to obtain a category more general and strongly related to a topos (whose internal logic is intuitionistc) but with an internal logic that has a linear flavor.

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