DEFORMATION THEORY OF ONE-DIMENSIONAL SYSTEMS

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ABSTRACT. Something remarkable occurs with one-dimension dynamical systems, that is, maps acting on either an interval or circle. Maps are often not structurally stable, however their topological class is an infinite dimensional smooth manifold with finite codimension. This implies that the theory of deformation of those systems is quite rich. Recent developments suggest that the study of the existence, uniqueness and regularity of solutions for certain cohomological equation is crucial for a better understanding of these phenomena, and ergodic theory plays an important role in this study.

CONTENTS

1. TOPOLOGICAL CLASSES

Dynamical systems arise as models to things that evolve with time. However rarely your model is indeed exact, but you hope that at least some of the dynamical properties of your model resists to small perturbations, otherwise modeling itself would be a waste of time.

Perhaps the best known form of persistence of dynamical properties is *structural stability*. We say that a dynamical system $f\colon M \to M$ acting on a topological space M is structurally stable if a small perturbation $g\colon M \to M$ (on an appropriated topology on the set of all concerning maps) has the same topological dynamics of

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the original map, that is, there is a homeomorphism $h\colon M \to M$ that conjugates f and g,

$$g \circ h = h \circ f$$

and furthermore h is close to identity. That means that f and g are undistinguable from the topological point of view.

Structural stability is a strong property that is not satisfied by all dynamical systems. However, it is an important concept in many areas of mathematics and physics. Hyperbolic maps are an interesting class of dynamics that are structural stable. Those include Anosov maps and expanding maps acting on manifolds, as well as horseshoes (see Hasselblatt and Katok [27]) They have a quite complicated dynamics and yet they are structurally stable.

One may ask whether the conjugacy could be more regular than just a homeomorphism for hyperbolic systems. It turns out that is really *rare* that the conjugacy is smooth (let say, C^1). There are very simple smooth invariants, that is, dynamical properties that are preserved by smooth conjugacies, but that are not preserved by homeomorphisms.

Indeed, if we consider a fixed point $f(p) = p$ which is hyperbolic, that is $D_p f$ is a hyperbolic linear map, then for g close enough to f there is a unique fixed point q for g close to p, which is also hyperbolic. But a suitable small perturbation g of f can make the eigenvalues of $D_q g$ to be completely distinct from those of $D_p f$. However if a conjugacy h is smooth then those eigenvalues are identical. Hyperbolic maps may have loads of hyperbolic points, so this is a serious obstruction for additional regularity of the conjugacies. Indeed under certain circumstances the marked spectrum of periodic points is a *complete* smooth invariant (see Shub and Sullivan [54], Martens and de Melo [44] and Li and Shen [36]). See also de la Llave [17] and McMullen [46].

So, although small perturbations of a structurally stable map have exactly the same topological dynamics, the *geometric* features of the perturbation can be quite different. Not just the spectrum of a periodic point can change, but also the Haussdorf dimension of some invariant set, as well as its physical measures.

1.1. **Flexibility and "almost" structural stability in one-dimensional dynamics.**

Could we do a similar analysis of the deformation of the geometry for maps that are *not* structurally stable? The set of all maps that are conjugate to a map f is called the **topological class** of f, that we will denote by $\mathcal{T}_f$. It is noteworthy that finding "trivial" deformations of a dynamical system f_0 can be achieved with relative ease. We just pick a isotopy of diffeomorphisms h_t with $h_0 = Id$ and consider $f_t = h_t \circ f_0 \circ h_t^{-1}$. That is not an interesting kind of deformation since f_t is *a priori* smoothly conjugate to f_0.

However in one-dimensional setting, that is, maps acting on either an interval or circle, something remarkable happens. Maps are often *not* structurally stable, but

Often topological classes are a smooth Banach manifold with positive **finite** *codimension.*

In particular there is a rich theory of *deformations* of such dynamical systems, since we can find many smooth families

$$t \mapsto f_t \tag{1.1}$$

of maps inside a given topological class. So dynamicaly-defined objects as periodic points and their multipliers, invariant measures, and so on, also *deforms* along such family and we can ask how this occurs and sometimes even quantify it. One interesting application is *linear response theory* for physical measures. See Ruelle [52] and Baladi and S. [6][8].

The most basic question is perhaps how the conjugacies are **deformed**. If (1.1) is a deformation in the topological class of f then there is a family of conjugacies h_t satisfying

$$h_t \circ f_0 = f_t \circ h_t$$

and we can ask how smooth is the function

$$t \mapsto h_t. \tag{1.2}$$

If this function is indeed smooth it follows quite easily that many dynamically defined objects also move in a smooth way. For instance if p is a n-periodic point of f_0 then $h_t(p)$ is n-periodic point for f_t, so it also moves in a smooth way with respect to t. Moreover the Lyapunov exponent of this periodic point

$$t \mapsto \ln Df_t^n(h_t(p))$$

is also smooth.

Furthermore if μ_0 is a measure of maximal entropy of f_0 then $\mu_t = h_t^\star \mu_0$ is a measure of maximal entropy for f_t, and for a smooth observable ψ we have that

$$t \mapsto \mu_t(\psi) = \int \psi \circ h_t \ d\mu_t$$

is also smooth provided (1.2) is regular enough. Similar results for physical measures (linear response) are more involved (See Baladi and S. [6]).

2. The structure of topological classes: a road map

How could we study the smooth structure of a topological class? First we need to model the space of all possible maps. To give a simple example, consider the set of all piecewise C^r expanding maps acting on an interval I. One can see this set as a convex set inside a space of piecewise C^r functions acting on$I$. That convex set itself has a smooth structure. See Grotta-Ragazzo and S. [26].

If the topological class $\mathcal{T}_f$ of a map f has a smooth structure, then for every vector v in the tangent space of f in $\mathcal{T}_f$ one can find a smooth family f_t such that $f_t \in \mathcal{T}_f$ for every t, $f_0 = f$, $v = \partial f_t|_{t=0}$ and

$$\partial_t f_t = v_t,$$

with $v_0 = v$. In particular since f_t is conjugated with f_0 there is a family of conjugacies h_t such that

$$h_t \circ f = f_t \circ h_t. \tag{2.3}$$

The next step is perhaps surprising. As we saw it the conjugacies are *not* smooth in general. However often

$$t \mapsto h_t(x)$$

is indeed *quite smooth* with $h_0(x) = x$. The most well-known situation are Beltrami paths in holomorphic dynamics, since families of conjugacies are often *holomorphic motions* (introduced in Mañé, Sad and Sullivan [42]. See de Faria and de Melo

[14] for holomorphic motions and other methods in holomorphic dynamics), but it seems to be a common phenomena in one-dimensional dynamics. So define

$$\alpha_t(x) = \partial_t h_t(h_t^{-1}(x)).$$

Of course

$$\partial_t h_t(x) = \alpha_t(h_t(x)). \tag{2.4}$$

It is remarkable that one can find α_t *a priori*, without knowing h_t itself. Indeed one can differentiate (2.3) with respect to t, and obtain

$$v_t = \alpha_t \circ f_t - Df_t \cdot \alpha_t. \tag{2.5}$$

The argument is simple. Let $y = h_t(x)$. We have

$$\begin{aligned}
\partial_t(h_t(f(x))) &= \partial_t(f_t(h_t(x))) \\
\partial_t h_t(f(x)) &= v_t(h_t(x)) + Df_t(h_t(x))\partial_t h_t(x) \\
v_t(h_t(x)) &= \partial_t h_t(f(x)) - Df_t(h_t(x))\partial_t h_t(x) \\
v_t(y) &= \partial_t h_t(f(h_t^{-1}(y))) - Df_t(y)\partial_t h_t(h_t^{-1}(y)) \\
v_t(y) &= \partial_t h_t(h_t^{-1}(f_t(y))) - Df_t(y)\partial_t h_t(h_t^{-1}(y)) \\
v_t &= \alpha_t \circ f_t - Df_t \cdot \alpha_t.
\end{aligned}$$

The reader can see that a *formal solution* α_t for this cohomological equation is

$$\alpha_t(x) = -\sum_{i=0}^{\infty} \frac{v_t(f_t^i(x))}{Df_t^{i+1}(x)}. \tag{2.6}$$

If f_t is invertible there is an analogous formal solution iterating backwards (see the next section, where we deal with Anosov maps). As a consequence we see that if v belongs to the tangent space at f of the topological class of f then the cohomological equation

$$v = \alpha \circ f - Df \cdot \alpha \tag{2.7}$$

does have a solution, at least formally, given by (2.6) for $t = 0$. Note that in simple settings, as for one-dimensional (piecewise) expanding maps, such formal solution *does* converge, and we hope that when some hyperbolic *inducing* is possible, one can find a *actual* solution of (2.7) using (2.6). We call α an *infinitesimal deformation* of f.

In good situations, if we can solve the cohomological equation along a family f_t we can *reverse* this argument. We can consider the ordinary differential equation (2.4) and, if it is *uniquely integrable* we can obtain a flow h_t that indeed conjugates f_t to f_0.

So we want to **characterize** *the vectors v tangent to the topological class $\mathcal{T}_f$ studying the* **existence, uniqueness and regularity** *of the solutions of the cohomological equation (2.7).*

There are many regularity conditions for α that imply unique integrability. The usual Lipchitz condition is one of them, but we are going to see that it is rare that α is Lipchitz in the hyperbolic setting. Weaker regularities such as Log-Lipchitz and Zygmund conditions are usually more useful. As a bonus, those regularities imply that the conjugacies h_t are Hölder and quasi-symmetric, respectively (See Reimann [51]).

3. A simple example: Anosov maps are structurally stable

Let M be a compact manifold with a Riemannian metric and $f\colon M \to M$ be a C^1 diffeomorphim such that M is a hyperbolic set. We say that f is an *Anosov* map. It is well-known that Anosov maps are structurally stable. We provide a fairly simple proof of this result using a new method that illustrates quite well some of the main steps in the previous section. We list facts one can prove using well-known methods in hyperbolic dynamics (see Hasselblatt and Katok [27]).

Theorem 3.1. *There is a neighborhood V of f in $C^1(M)$ such that every $g \in V$ is an Anosov map. Indeed, there are linear projections*

$$\pi_s^g : T_xM \to T_xM,\ \pi_u^g : T_xM \to T_xM$$

on each tangent space T_xM such that

A. *We have $\pi_s^g \circ \pi_u^g = 0 = \pi_u^g \circ \pi_s^g$.*
B. $v = \pi_s^g(v) + \pi_u^g(v)$
C. *We have that $\pi_s^g : TM \to TM$ and $\pi_u^g : TM \to TM$ are continuous.*
D. *We have that*
$$g \in V \mapsto \pi_s^g \text{ and } g \in V \mapsto \pi_u^g$$
are continuous.
E. *$E_g^s(x) = \pi_s^g(T_xM)$ and $E_g^u(x) = \pi_u^g(T_xM)$ are the stable and unstable directions of g.*
F. *There are $C > 0$ and $\lambda \in (0,1)$ such that*
$$|D_xg^n(\pi_s^g(v))| \leq C\lambda^n|\pi_s^g(v)| \text{ and } |D_xg^{-n}(\pi_u^g(v))| \leq C\lambda^n|\pi_u^g(v)|$$
for every $n \geq 0$, $v \in T_xM$ and $g \in V$.
G. *There is $\epsilon_0 > 0$ such that for every $g \in V$ and every $x, y \in M$, with $x \neq y$, there is $k \in \mathbb{Z}$ such that $d(g^k(x), g^k(y)) > \epsilon_0$.*

Theorem 3.2. *Anosov maps are structurally stable.*

Proof. Let V be as in Theorem 3.1. Without loss of generality we can assume that for every $g \in V$ there is a smooth path f_t, with $t \in (-\delta, 1+\delta)$, such that $f_0 = f$ and $f_1 = g$ and $g_t \in V$ for every t.
For every (x, t_0) define

$$v_{t_0}(x) = \partial_t f_t(x)|_{t=t_0}.$$

Note that $v_{t_0}(x) \in T_{f_{t_0}(x)}M$ and

$$x \mapsto v_{t_0}(x)$$

is continuous for every t_0.

Define the vector field

$$\alpha_t(x) = \sum_{k=0}^{\infty} Df_t^k(f_t^{-k}(x))\cdot\pi_s^{f_t}(v_t(f_t^{-(k+1)}(x))) - \sum_{k=0}^{\infty} Df_t^{-(k+1)}(f_t^{k+1}(x))\cdot\pi_u^{f_t}(v_t(f_t^k(x)))$$

Note that

$$(x,t) \mapsto \alpha_t(x)$$

is continuous and it satisfies

$$v_t(x) = \alpha_t(f_t(x)) - Df_t(x) \cdot \alpha_t(x). \tag{3.8}$$

Given $x_0 \in M$, we claim that the initial value problem

$$\begin{cases} \dot{y} = \alpha_t(y), \\ y(0) = x_0. \end{cases} \tag{3.9}$$

has a unique solution $y\colon [0,1] \to M$. The existence follows from Peano existence theorem. To show the uniqueness, note that (3.8) implies

$$y^n(t) = f_t^n(y(t))$$

is a solution of initial value problem

$$\begin{cases} \dot{y}^n = \alpha_t(y^n), \\ y^n(0) = f^n(x_0). \end{cases} \tag{3.10}$$

for every $n \in \mathbb{Z}$. Since there is C such that

$$|\alpha_t(x)| \leq C$$

for every $(x,t) \in M \times [0,1]$, it follows that

$$d(f_t^n(y(t)), f^n(x_0)) = d(y^n(t), y^n(0)) \leq C|t|,$$

so if $y_\star$ and y are two solutions of (3.9) we have that

$$d(f_t^n(y(t)), f_t^n(y_\star(t))) \leq 2C|t|$$

for every $n \in \mathbb{Z}$. But Theorem 3.1.G implies that $y(t) = y_\star(t)$ for t small. That proves the claim. Denote by $h_t(x_0)$ the unique solution of (3.9). Then $f_t^n(h_t(x_0))$ and $h_t(f^n(x_0))$ are solutions of (3.10), so

$$h_t \circ f = f_t \circ h_t.$$

The continuity of

$$(x,t) \mapsto h_t(x)$$

and that h_t are homeomorphisms on M can be also obtained using similar arguments. □

Question 3.3. *How regular are the infinitesimal deformations α_t in this setting? We believe they are Zygmund, but we are able to prove that only for linear Anosov maps (See Grotta-Ragazzo and S. [26]).*

4. Historical remarks

There are many distinct Riemann surfaces homeomorphic to a given compact surface M with genus larger than 1. The set of all possible conformal structures (up to isotopy) on M is called the *Teichmüller space* of M. It turns out one can give a *differentiable structure* (indeed complex analytic) to the Teichmüller space and we can ask how its geometric properties (for instance the length of a closed geodesic) changes along a deformation. One can easily argue that the theory of Teichmüller spaces is one of the most influential developments in XXth century mathematics. J. Hubbard's book [29] is a quite good modern book on the subject. One of the main tools in this theory is quasi-conformal methods.

Teichmüller theory has a wide range of applications, including in the study of low-dimensional topology, hyperbolic geometry, as well as generalizations as the study of moduli spaces in many areas of mathematics, including complex analysis, algebraic geometry, and geometric group theory.

The *representation theory of groups* is a branch of mathematics that studies how groups can be represented as linear transformations on vector spaces. More formally, a representation of a group G in $GL(V)$ (or the projective linear group $PSL(V)$), where V is vector space is a homomorphism from G to the group of invertible linear transformations on V. This homomorphism is called a representation of the group, and it associates each element of G with a linear transformation on V. Of particular interest are *discrete* representations. For instance, those appear naturally when we represent a hyperbolic Riemann surface as a quotient space of the hyperbolic half-plane by the action of discrete subgroups of isometries. Indeed, the Teichmüller space can be seen exactly as the set of all possible suitable discrete representations of the fundamental group of the given Riemann surface (see Saito [53]).

On this and other circumstances, the set of all possible discrete representations has a smooth structure and we also can talk about infinitesimal deformations. As for infinitesimal conjugacies, the earliest reference we know of the concept (using the term infinitesimal deformations) is in the work of A. Weil on cohomology of groups [58]. This was very influential, in particular in studies of "infinitesimal rigidity". For instance the work of McMullen on rigidity [47] for certain 3-manifolds and quadratic-like maps. McMullen first proves rigidity for the infinitesimal conjugacy and then integrate this to obtain rigidity of the conjugacy. Indeed one can also see similarities with earlier work on deformations of complex manifolds by Kodaira and Spencer (see Kodaira [35])

Since its introduction by Mañé, Sad and Sullivan [42] *holomorphic motions* are often used to study the dynamics of rational maps, polynomial-like maps, transcendental functions and Fuschian groups. Overall, the deformation theory of rational maps and holomorphic motions is a rich and important area of complex dynamics.

Cohomological equations appear quite often in dynamics (see for instance Livšic [37], Avila and Kocsard [3] and de Faria, Guarino and Nussenzveig [16]), and in particular when dealing with deformations problems. Perhaps some of the earliest examples are the rigidity of real analytic perturbations of circle diffeomorphisms by Kolmogorov and Arnold [2] and the already cited work by A. Weil [58], both with a lasting impact.

In hyperbolic dynamics there is a long line of results on differentiability of topological entropy and Hausdorff dimension of hyperbolic invariant sets, where infinitesimal conjugacies appears (not explicitly with this name). See Katok, Knieper, Pollicott and Weiss [31]. Indeed, earlier work on structural stability of Anosov diffeomorphisms by Moser and Mather uses infinitesimal conjugacies, as in Moser [49]. Infinitesimal deformations and thermodynamical formalism appears in the study of deformations of compact hyperbolic manifolds in Flaminio [22]. The relation between infinitesimal deformations and the Weil-Petersson metric is the main theme in Fathi and Flaminio [20]. Indeed, one can see Teichmüller theory of a compact Riemann surface as a study of deformations of certain piecewise-moebius maps associated with a fuschian group, the so called Bowen-Series map [10] (see also Adler and Flatto[1]). This point of view is essential to McMullen's [48] interpretation of the Weil-Petersson metric as a dynamically defined object. Infinitesimal conjugacies were also used to study the action of pseudo-Anosov maps on representation groups (see Kapovich [30]).

In Teichmüller theory and complex dynamics (using quasiconformal maps approach) the differentiation of the conjugacy with respect to the parameter is often a consequence of powerful results on the analytic dependence (and formulas to its derivative) with respect to the parameter for solutions of Beltrami equation. That includes the measurable Riemann mapping theorem and holomorphic motions, so infinitesimal deformations often do not appear explicitly. In R. Mañé [41] work on the instability of Herman rings infinitesimal conjugacies play an essential role. M. Lyubich [40] gave an fairly complete picture for deformations of quadratic-like maps that were important to understand renormalization of those maps, and there infinitesimal deformations take the front row. This was the main influence for our contributions.

More recently Baladi and S. [6][7] studied deformations of piecewise expanding unimodal maps, mainly to apply it in the theory of linear response of those systems, and later on in the study of linear response for certain Collet-Eckmann maps.

5. Topological classes and infinitesimal deformations

From now on we will keep ourselves to the one-dimensional setting, where we are able to give far more complete answers. Suppose for instance that we have a smooth family f_t of piecewise expanding maps acting on an interval I. We can ask when there are no bifurcations on f_t, that is, when f_t is always belong to the topological class of f_0. It turns out that

Theorem 5.1 (Characterization of smooth deformations. Grotta-Ragazzo and S. [26])**.** *Under fairly general conditions the following statements are equivalent*

A. *f_t belongs to the topological class of f_0 for every t.*
B. *For every t the cohomological equation (2.5) has a continuous solution α_t.*

For k large enough consider the set $PE^k([a,b],\mathcal{D})$ of all C^k piecewise expanding maps on an interval $[a,b]$ whose discontinuities lie exactly on n points in

$$\mathcal{D} = \{x_1, \ldots, x_n\} \subset (a,b)$$

One can easily model $PE^k([a,b],\mathcal{D})$ as an infinite-dimensional Banach manifold.

Theorem 5.2 (Dimension of Topological classes. Grotta-Ragazzo and S. [26])**.** *For k large enough we have that the topological class of $f \in PE^k([a,b],\mathcal{D})$ is an infinite-dimensional smooth submanifold of codimension $2n+2$ in $PE^k([a,b],\mathcal{D})$. The tangent space of the topological class at f is exactly the subspace of piecewise smooth functions v having a continuous solution α for the cohomological equation*

$$v = \alpha \circ f - Df \cdot \alpha.$$

Question 5.3. *How far could we generalize this result for other kinds of one-dimensional maps, such as unimodal maps with non flat critical points, circle diffeomorphisms with break points, generalized interval exchange transformations with finite smoothness?*

Question 5.4. *Are the topological classes connected? Are they contractible?*

Question 5.5. *Can we endow the topological class with an interesting complex analytic structure?*

There are previous results on the smooth structure of topological classes, as the work by Lyubich [40] on quadratic-like maps. The connecteness of the topological classes of (real) quadratic-like maps follows from the Riemann Measurable Mapping Theorem and the quasisymmetric rigidity of those maps, a deep result by Lyubich [39] and Graczyk and Światek [24]. The proof that the hybrid class of quadratic (a sort of complexfication of the topologival class) is contractible is a more recent result by Avila and Lyubich [4]. We also cite the work of Goncharuk and Yampolsky [23] for analytic circle diffeomorphisms.

There are also recent results on the structure of topological classes of real analytic one-dimensional maps by Clark and van Strien [12]. All these works crucially employs complex dynamics methods. In contrast, we use real dynamics and specially ergodic theory. In Baladi and S. [6] there are some related results for piecewise expanding unimodal maps.

6. The regularity of infinitesimal deformations

Turns out that α is continuous, however one can asks if it is more regular than that. Since the conjugacies are often *not* Lipchitz, α can not be Lipchitz itself.

Let f be a $C^{2+\beta}$ expanding map on the circle and $v\colon \mathbb{S}^1 \to \mathbb{R}$ be a $C^{1+\alpha}$ function. So there is an unique bounded function $\alpha\colon \mathbb{S}^1 \to \mathbb{R}$ satisfying

$$v = \alpha \circ f - Df \cdot \alpha.$$

The solution α is given by the formula

$$\alpha(x) = -\sum_{i=0}^{\infty} \frac{v(f^i(x))}{Df^{i+1}(x)}.$$

We have

Theorem 6.1 (de Lima and S. [18])**.** *For $\delta \in \mathbb{R}$ close to zero we have*

$$\alpha(x+\delta) - \alpha(x) = \delta\Big(\sum_{i=0}^{N(x,\delta)} \phi(f^i(x))\Big) + O(\delta),$$

where

$$\phi = \frac{Dv + D^2 f \cdot \alpha}{Df}$$

and $N(x,\delta)$ is the unique positive integer that satisfies

$$\frac{1}{|Df^{N(x,\delta)+1}(x)|} \leq |\delta| < \frac{1}{|Df^{N(x,\delta)}(x)|}.$$

This looks quite technical, however the main consequence is that there is a deep connection between the regularity of α with the dynamical behavior of the Birkhoff sum of an observable ϕ. It immediately follows that

Corollary 6.2 (de Lima and S. [18])**.** *The infinitesimal conjugacy α is a Zygmund function, that is, there is C such that*

$$|\alpha(x+\delta) + \alpha(x-\delta) - 2\alpha(x)| \leq C|\delta|.$$

Remark 6.3. In particular α is Log-Lipchitz, that is, there is $C > 0$ such that

$$|\alpha(x+\delta) - \alpha(x)| \leq -C|\delta| \log |\delta|$$

for small δ (see Zygmund [60]). The Log-Lipchitz continuity implies that (2.4) is uniquely integrable, since α satisfies the Osgood condition for unique integrability. Indeed the Zygmund condition in dimension one implies that the flow is quasisymmetric (See Reimann [51]), so the conjugacies are quasisymmetric.

Remark 6.4. The observable ϕ has a very odd-looking formula. However it has a simple but deep meaning. Indeed, suppose that f_t is a smooth family of deformations of $f_0 = f$, that is, we have conjugacies $h_t \circ f = f_t \circ h_t$ with $h_t(x) = x$. So $h_t(x)$ is the smooth continuation of x, in the sense that the (topological) dynamics of $h_t(x)$ with respect to f_t is exactly the same than the dynamics of x with respect to f. Let $\partial_t f_t|_{t=0} = v$. If we want to see how the Lyapunov exponent changes along this deformation, one can easily see that

$$\partial_t \ln Df_t^k(h_t(x))|_{t=0} = \sum_{j=0}^{k-1} \phi(f^k(x)).$$

For piecewise expanding map there are known obstructions for α to be Zygmund, but we have the following

Theorem 6.5. *Every infinitesimal conjugacy α is Log-Lipchitz continuous, that is, there is $C > 0$ such that*

$$|\alpha(x+\delta) - \alpha(x)| \leq -C|\delta| \log |\delta|$$

for small δ.

Remark 6.6. It is known [26] that for piecewise expanding maps for which some discontinuities have simple dynamics there is an obstruction for α to be Zygmund. In particular the topological class and the quasi-symmetric class can be distinct. The distinction between those classes was already observed for for certain circle diffeomorphisms with Liouville rotation number [2], dissipative Lorenz maps by Martens, Palmisano and Winckler [45] and critical circle maps by de Faria and Guarino [15]. This is in contrast with a similar situation for real-analytic quadratic-like maps when α is always Zygmund (Lyubich [40]).

Question 6.7. *Characterize the obstructions for α to be Zygmund for various classes of maps such as piecewise expanding maps, circle diffeomorphisms and Lorenz maps.*

7. Statistical properties of infinitesimal deformations

In the setting of the previous section (expanding maps acting on the circle) one may ask if either Zygmund or Log-Lipchitz regularities are sharp. Could α be Lipchitz? it turns out it is not. The key is Theorem 6.1. It says that the regularity of α is deeply connected with the statistical properties of the observable ϕ. Indeed

$$\int \phi \; d\mu = 0,$$

where μ is the absolutely continuous f-invariant probability. Denote

$$\sigma^2(\phi) = \lim_n \frac{1}{n} \int \Big(\sum_{k<n} \phi \circ f^k \Big)^2 \; dm.$$

Theorem 7.1 (de Lima and S. [18])**.** *The following statements are equivalent*

A. $\sigma^2(\phi) > 0$.

B. *There is a periodic point p such that*

$$\sum_{k<n} \phi(f^k(p)) \neq 0,$$

where n is the period of p.

C. *There is $\ell > 0$ such that*

$$\lim_{\delta \to 0} \mu\Big(x \in \mathbb{S}^1 \colon \frac{\alpha(x+\delta)-\alpha(x)}{\sigma(\phi)\ell\delta\sqrt{-\log|\delta|}} \leq y\Big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{t^2}{2}}\ dt$$

D. *α is not a Lipchitz function on any subset with positive Lebesgue measure.*

E. *α is a continuous and nowhere differentiable function.*

There is more general but similar result for piecewise expanding maps of the interval in Grotta-Ragazzo an S. [26][25].

Remark 7.2. The condition B is generic on v. Indeed one can easily see that

$$\sum_{k<n} \phi(f^k(p))$$

depends only on the values of v along the the orbit of the periodic point p. From this one can conclude after a short argument that the linear functionals

$$v \mapsto \psi_p(v) = \sum_{k<n} \phi(f^k(p))$$

are linear independent when p runs over all periodic orbits. So the subspace of all v such that

$$\sum_{k<n} \phi(f^k(p)) = 0$$

for all periodic points p has *infinite codimension.* See Grotta-Ragazzo an S. [26] for details.

Remark 7.3. One can obtain a Law of Iterated Logarithm if A-E holds.

8. Derivative of α in the distributional sense

The results in the previous section on the modulus of continuity of an infinitesimal deformation are indeed a consequence of a deep connection with the ergodic behaviour of the piecewise expanding map f. In fact we have

Theorem 8.1 (Grotta-Ragazzo and S. [26])**.** *There is $g \in L^\infty(m)$ such that*

$$D\alpha = g + \sum_{k=0}^{\infty} \phi \circ f^k$$

in the sense of distributions.

That is, for every C^∞ function ψ with support in the interior of I we have

$$\int \alpha D\psi\ dm = \int g\psi\ dm - \sum_k \int \phi \circ f^k \psi\ dm.$$

To be fair, one needs to be more careful with the convergence of the r.h.s. if f does not have an unique absolutely continuous mixing invariant measure, but we address these details in [26].

Question 8.2. *Could we have similar results in higher dimension for structurally stable maps and flows?*

Question 8.3. *Could we have similar results for not necessarily expanding maps, as non-uniformly expanding maps as unimodal Collet-Eckermann maps?*

9. Pseudo-Riemannian metric in the topological class

C. McMullen [48] reinterpreted the Weil-Petersson metric in Teichmuller space using thermodynamical formalism on Bowen-Series maps and indeed defined a similar metric in the space of expanding Blaschke product. We can do something analogous (see Grotta-Ragazzo and S. [26]).

Indeed, let v_i, with $i = 1, 2$, be vectors in the tangent space of the topological class at a piecewise expanding map f. Then there are log-Lipchitz vectors α_i satisfying the cohomological equation. Let

$$\phi_i = \frac{Dv_i + D^2 f \cdot \alpha_i}{Df}.$$

Then

$$\int \phi_i \ d\mu = 0$$

for every absolutely continuous f-invariant probability μ, and we can define the pseudo-metric on the tangent space of f as

$$\sigma_f^2(v_1, v_2) = \lim_{n\to+\infty} \frac{1}{n} \int \big(\sum_{k=0}^{n-1} \phi_1 \circ f^k\big)\big(\sum_{k=0}^{n-1} \phi_2 \circ f^k\big) \ dm.$$

One can check that is is well defined. We call it *pressure* pseudo-metric following Bridgeman, Canary, Labourie, Sambarino [11]. It is a quite weird definition, but this pseudo-riemannian metric on the topological class has deep connections with the dynamics. For instance

Theorem 9.1. *Let f_t, $t \in [0, 1]$, be a smooth curve of orientation preserving expanding maps on the circle. Suppose that its length in the pressure metric is zero, that is,*

$$\int \sigma_{f_t}(\partial_t f_t, \partial_t f_t) \ dt = 0.$$

Then f_0 is smootly conjugate with f_1.

Proof. In this case $\sigma_{f_t}(\partial_t f_t, \partial_t f_t) = 0$ for every t. So the Hölder function ϕ_t is f_t-cohomologous to zero, that is, there is a Hölder function ψ_t satisfying

$$\phi_t = \psi_t \circ f_t - \psi_t.$$

Let p be a n-periodic point of f_0. Let h_t be the continuous family of homeomorphisms such that

$$h_t \circ f_0 = f_t \circ h_t,$$

with $h_t(x) = x$. Then $h_t(p)$ is a n-periodic point of f_t and

$$\begin{aligned}\partial_t \ln(Df_t^n(h_t(p))) &= \sum_{k=0}^{n-1} \partial_t \ln(Df_t(h_t(f_0^k(p)))) \\ &= \sum_{k=0}^{n-1} \phi_t(h_t(f_0^k(p))) \\ &= \psi_t(h_t(f_0^n(p)) - \psi_t(h_t(p)) = 0.\end{aligned}$$

for every t, so the multiplier of the analytic continuation of the period point p along this family f_t is constant. From this one can prove that the conjugacy h_t is absolutely continuous. It follows Shub and Sullivan [54] that h_t is smooth (see also Martens and de Melo [44] and Li and Shen [36]) □

By a similar argument one can show that if f_0 and f_1 are smoothly conjugated then there is a smooth path with zero length between them.

Question 9.2. *What are the geometric properties of this pseudo-metric, as its sectional curvature and its (likely) lack of completeness?*

Regarding this question, there is some results in Lopes and Ruggiero [38] and in Pollicott and Sharp [50] in similar settings.

10. Deformations and Renormalization

There are interesting cases, mainly in one-dimensional dynamics, when a conjugacy is surprisingly much smoother than expected. In those cases the conjugacy is indeed C^1 either in the whole or at least on part of the phase space, as on a dynamically meaningful Cantor set. It is the phenomena of *rigidity and universality*, that occurs for dynamical systems that are far from the hyperbolic setting, as circle diffeomorphisms (Herman [28], Yoccoz [59], Khanin and Sinai [33], Goncharuk and Yampolsky [23]), circle homeomorphims with break points (Khanin, Kocić and Mazzeo [32], Cunha and S. [13], and more recently Berk and Trujillo [9]), generalized interval exchange transformations (Marmi, and Moussa and Yoccoz [43]), and infinitely renormalizable unimodal (Sullivan [56], McMullen [47] and Lyubich [40]), critical circle homeomorphisms (see for instance Khmelev and Yampolsky[34] and Estevez, de Faria and Guarino[19]), and multimodal maps (S. [55]).

Renormalization is the key to those results. For all these classes of dynamical systems one can define a *renormalization operator* acting on the corresponding maps. The renormalization operator is a dynamical system *acting on dynamical systems.*

To keep the exposition as simple as possible, consider the most classical of all renormalizations, the *period-doubling* renormalization, introduced by Feigenbaum [21] and Coullet and Tresser [57]. Consider an even unimodal map $f\colon I \to I$, with $I = [-1, 1]$ and quadratic critical point. We say that f is period-doubling *renormalizable* if there exists an interval $J = [-\beta, \beta] \subset I$ so that

- The interiors of J and $f(J)$ are disjoint.
- $f^2(J) \subset J$.
- $f^2(\partial J) \subset \partial J$.

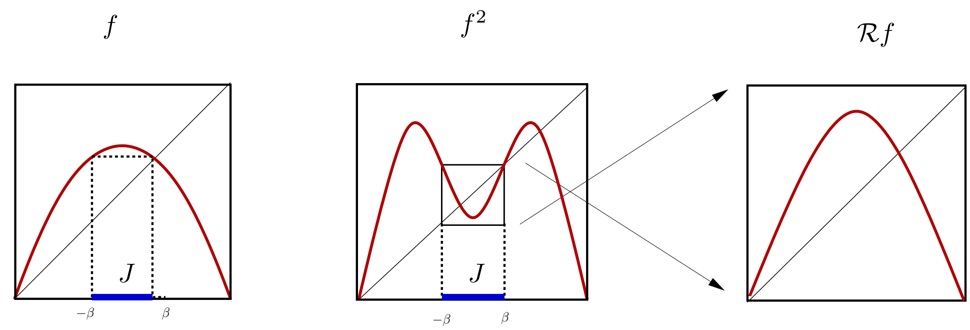

FIGURE 1. Construction of the renormalization operator.

If we look to the map $f^2\colon J \to J$, we see that it is also a unimodal map defined in this small interval (see Fig. 1). We can obtain a unimodal map in the original interval $[-1,1]$ defining the map

$$\mathcal{R}f\colon [-1,1] \to [-1,1]$$

as

$$\mathcal{R}f(x) = -\frac{1}{\beta} f^2(\beta x).$$

The map $\mathcal{R}f$ is called the *period-doubling renormalization* of f. The operator $\mathcal{R}$ acts on the set of renormalizable unimodal maps. We would like to understand its dynamics.

The renormalization $\mathcal{R}f$ of a unimodal map f could be renormalizable again, so we could define $\mathcal{R}^2 f$. If $\mathcal{R}^2 f$ is renormalizable again, we could define $\mathcal{R}^3 f$ and so on. If $\mathcal{R}^n f$ is defined for every n we say that f is *infinitely period-doubling renormalizable*. What happens with $\mathcal{R}^n f$ when n goes to infinity?

The following result, which puts together contributions by Sullivan [56], McMullen [47] and Lyubich [40], describes the dynamics of the period-doubling renormalization operator:

Theorem 10.1. *The renormalization operator has the following properties.*

A. *There exists a unique unimodal map $f^\star$ satisfying*

$$\mathcal{R}f^\star = f^\star.$$

B. *In an appropriate space of functions the renormalization operator is a smooth operator and $f^\star$ is a hyperbolic fixed point with codimension one stable manifold.*

C. *The connected component of the stable manifold of $f^\star$ is exactly the topological class of $f^\star$.*

10.1. **Infinitesimal deformations and Renormalization.** One of the many challenges in this result, solved by Lyubich [40], is to find a suitable space of maps (indeed germs) where $\mathcal{R}$ acts as an complex analytic operator, and show that the topological class of $f^\star$ is a codimension one manifold there. Indeed, this result explains the universality of the infinite sequence of bifurcations that appears in families of unimodal maps. To be more precise, a *quadratic-like map*

$$f\colon U \to V$$

is a complex analytic map with just one (quadratic) critical point, where U and V are topological open disks on $\mathbb{C}$ satisfying $\overline{U} \subset V$. Due to a result by Douady and Hubbard we know their dynamical behaviour is quite similar to a quadratic polynomial. The *filled-in Julia set* $K(f)$ of f is the non empty compact set

$$K(f) = \cap_k f^{-k}V.$$

We have

Theorem 10.2 (Lyubich [40]). *The topological class of a quadratic-like map f with connected filled-in Julia set is a codimension one manifold in a suitable space of quadratic-like maps. Futhermore the tangent space of the topological class at f is the set of all complex analytic functions v defined in a neighborhood of $K(f)$ such that there is a quasiconformal vector filed α satisfying*

$$v = \alpha \circ f - Df \cdot \alpha \qquad (10.11)$$

in a neighboorhood of $K(f)$ and $\overline{\partial}\alpha = 0$ on $K(f)$.

Indeed, Avila, Lyubich and de Melo [5] were able to show that for quadratic-like maps that are real on the real line, if $K(f)$ has empty interior (the most interesting case) and v is real in the real line then v is tangent to the topological class if and only if one can find a Zygmund vector field α that satisfies (10.11) on the orbit of the critical point.

10.2. **Action of $D\mathcal{R}$ on infinitesimal deformations.** Renormalization theory for maps with critical points (as unimodal maps) had an extraordinary development in the last decades. However there is one area that resisted almost every attack. Most of the results are only proved for real analytic maps since one needs complex dynamics methods (quasiconformal methods) in an essential way. Similar results for unimodal maps of the form $|x|^\beta + c$, where $\beta \notin 2\mathbb{N}$, are yet out of reach, besides almost every expert agreeing that those results must also hold for these class of maps.

We must compare this with the study of piecewise expanding maps, circle diffeomorphisms and generalized interval exchange transformation, for which real dynamics alone is sufficient to attain significant advancements.

In S. [55] we prove the hyperbolicity of the renormalization operator acting on infinitely renormalizable multimodal maps using an infinitesimal approach. In particular the main intermediate result is

Theorem 10.3. *Let f be a multimodal map with quadratic critical points, infinitely renormalizable with bounded combinatorics. Then*

$$\sup_{k\geq 0} |D_f(\mathcal{R}^k f)\cdot v|_\infty < \infty$$

if and only if the cohomological equation (10.11) has a Zygmund solution α.

Here $D_f(\mathcal{R}^k f)$ denotes the derivative of the operator

$$f \mapsto \mathcal{R}^k f.$$

So the study of hyperbolicity of the operator was reduced to the study of the existence and regularity of the solutions of this cohomological equation. While the proof of the Theorem 10.3 also relies on complex dynamics methods, it places the study of solutions to cohomological equations in a central position, just as is the case for renormalization results for circle diffeomorphisms and generalized interval exchange transformations. We believe that this is the path toward a more unified approach to renormalization in one-dimensional dynamics.

We end this survey with a result that was used for unimodal, multimodal [55] and even Fibonacci renormalization but whose proof is so simple that it can be easily adapted for all cited classes where renormalization appears. It says that having a solution for the cohomological equation indeed makes the action of the derivative of the renormalization operator much easier to understand.

Theorem 10.4. *Let f be a unimodal map that is period-doubling renormalizable on the interval $[-\beta, \beta]$ and v such that there is a solution α for the cohomological equation (10.11) over $[-1, 1]$. Then*

$$D_f(\mathcal{R}f) \cdot v = \hat{\alpha} \circ \mathcal{R}f - D(\mathcal{R}f) \cdot \hat{\alpha}$$

on $[-1, 1]$, with

$$\hat{\alpha}(x) = \frac{1}{\beta}\alpha(\beta x) - \frac{1}{\beta}\alpha(\beta)x.$$

Note that if we know something on the regularity of α on $[-1, 1]$ one can easily deduce the (lack of) growth or contraction of $D_f(\mathcal{R}^k f) \cdot v$ along iterated renormalizations. For instance if f is infinitely renormalizable and α is Zygmund on $[-1, 1]$ then it easily follows that

$$\sup_{k \geq 0} |D_f(\mathcal{R}^k f) \cdot v|_{C^0[-1,1]} < \infty.$$

If α is β-Hölder this will give us a upper bound for the growth of the norm of $D\mathcal{R}^k_f \cdot v$.

One of the problems to adapt the methods we used for piecewise expanding maps in the renormalization setting is the *lack of hyperbolicity* of infinitely renormalizable maps. The physical measure of an infinitely renormalizable unimodal map has zero entropy. Baladi and S. [8] were able to use tower inducing techniques to study the existence and regularity of the solutions of (10.11) for certain Collet-Eckmann unimodal maps, but those maps are non-uniformly expanding, so they are far more hyperbolic than infinitely renormalizable maps.

Question 10.5. *Study the existence, regularity and uniqueness of solutions of the cohomogical equation (10.11) for increasily less hyperbolic one-dimensional maps such as Misiurewicz maps, Collet-Eckmann maps, Benedicks-Carlesson maps, multimodal maps satisfying some summability condition, maps with parabolic point and (let´s hope so) infinitely renormalizable maps (unimodal, multimodal, generalized interval exchange transformations).*

We pose this question for real maps with finite smothness and arbitrary non flat critical points. We believe that at least for maps with some hyperbolicity ergodic theory methods and in particular towers/inducing will be crucial. We also believe that Question 10.5 will be the central technical step to study the smooth structure of the topological class of such maps (Questions 5.3 and 5.4).

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