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Non-injectivity of the lattice map for non-mixed Anderson t-motives, and a result towards its surjectivity

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Let M be an uniformizable Anderson t-motive and $L(M)$ its lattice. First, we prove by an explicit construction that for the non-mixed M, the lattice map $M \mapsto L(M)$ is not injective. Second, we show that $\exists$ lattices L_0 such that $L_0 \neq L(M)$ for pure M, but $\exists$ a non-pure M such that $L_0 = L(M)$. This is a result towards surjectivity of the lattice map. The t-motives used in the proofs are non-pure t-motives of dimension 2, rank 3. Finally, we start calculations in order to answer a question whether all non-pure t-motives of dimension 2, rank 3 are uniformizable, or not.

Keywords: Anderson t-motives; lattice map.

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0. Introduction

We shall consider in this paper only Anderson t-motives having the nilpotent operator N equal to 0. Let M be an uniformizable t-motive of rank r, dimension n. Let $L(M) \subset \mathbb{C}_\infty^n$ be its lattice, it is isomorphic to $\mathbb{F}_q[\theta]^r$. Is the lattice map $M \mapsto L(M)$ an injection; a surjection? Unlike the case of abelian varieties where there is a one-to-one correspondence between abelian varieties and lattices having a Riemann form, in the functional field case we know much less.

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This paper contains two results on the lattice map. First, we prove that for non-mixed t-motives, the lattice map is not injective. Second, we show that some lattices which are not images of the lattice map of duals of Drinfeld modules, are lattices of some non-pure t-motives.

The technique of both results is similar to the one of [6]. We consider the simplest nontrivial case of t-motives of rank 3, dimension 2, defined by explicit Eqs. (2.2) and (3.3). We solve equations defining the lattices of these t-motives.

Let us explain what was known on the lattice map earlier. First, for the case of Drinfeld modules, i.e. for $n = 1$, we have a one-to-one correspondence between Drinfeld modules (they are all uniformizable) and lattices in $\mathbb{C}_\infty$ [2].

Further, there is a result (see [6]) on a local surjectivity of the lattice map near $M = \mathfrak{C}_2^{\oplus n}$, where $\mathfrak{C}_2$ is the rank 2 Carlitz module (see 2.1 for its definition).

The most important result is the injectivity[a] of the lattice map on the set of mixed uniformizable Anderson t-motives ([9, Theorem 3.34b]). Recall that a t-motive is called mixed if it has a filtration such that its quotients are pure, of increasing weight. See [9, Definition 3.5b], for an exact definition of mixedness, and [4, Definition 5.5.2], for the definition of purity.

The first result this present paper: the condition of mixedness is necessary. We give two different examples of non-injectivity. First example: we give an explicit construction of a one-parametric family of non-isomorphic non-mixed $M(a)$, where $a \in \mathbb{C}_\infty$ is a parameter, such that all lattices $L(M(a))$ coincide. Second example: let L be a fixed lattice. We show (see (3.2)) that there exist a pure M_1 and a non-pure M_2 (hence M_1, M_2 are not isomorphic) such that $L = L(M_1) = L(M_2)$.

Let us describe the second result. By the theory of duality (see [5]), we have some information for the case $n = r - 1$. Namely, let L be a lattice of rank r in $\mathbb{C}_\infty^{r-1}$. If L^* — the dual of L — exists, then it is a lattice of rank r in $\mathbb{C}_\infty$. Hence, there exists a Drinfeld module M such that $L(M) = L^*$. The dual of M (in the meaning of [5, 10]), denoted by M^*, always exists, it is a pure t-motive of rank r, dimension $r - 1$. By the theory of duality [5], $L = L(M^*)$. Moreover, we have the following theorem:

Theorem 0.1 ([5, Corollary 8.4]). *All pure t-motives of rank r, dimension $r-1$ are uniformizable. The lattice map is a one-to-one correspondence between the set of pure t-motives of rank r, dimension $r - 1$, and the set of lattices of rank r in $\mathbb{C}_\infty^{r-1}$ having duals.*

There is a natural

Question 0.2. Let L be a lattice of rank r in $\mathbb{C}_\infty^{r-1}$ such that its dual L^* does not exist. Is L a lattice of a t-motive (necessarily non-pure)?

Our second result (Theorem 3.2) is that for $r = 3$, for many such L the answer is "yes". This gives us evidence that the lattice map is surjective.

[a]This is only a rough statement.

Finally, we state the following question: what is the simplest example of a non-uniformizable t-motive M (i.e. what are the minimal values of r, n of M)? The simplest example known today is given in [1, Sec. 2.2]; see also [4, Example 5.9.9]. It has the minimal known values of $r = 4$ and $n = 2$.

From the first sight, this is really the simplest example, because if M has $r = 3$, $n = 2$ then M is dual to a Drinfeld module of rank 3 and hence is uniformizable.

But this is not completely true, because M having $r = 3$, $n = 2$ is dual to a Drinfeld module of rank 3 only if M is pure. Non-pure M having $r = 3$, $n = 2$ exist, and we do not know are they uniformizable, or not (and if they are non-uniformizable, we do not know their h^1, h_1).

So, we have the following

Problem 0.3. *Calculate* h^1, h_1 *of* $M_{np}(A)$ *defined by the formula* (1.6), *and to check, are they all uniformizable, or not.*

This can be made by the methods developed in [3, 8]. Conjecturally, all non-pure M having $r = 3$, $n = 2$ can be defined by this formula. Recall that if M is not uniformizable then it can happen that $h^1 \neq h_1$.

Remark. $M_{np}(A)$ defined by (1.6) depends on 3 parameters while t-motives of rank 4, dimension 2 considered in [3, 8] depend on 4 parameters. Hence, calculations for them will be simpler than the original calculations of [3, 8].

This paper is organized as follows. In Sec. 1, we give necessary definitions and notations. Section 2 contains the proof of non-injectivity of the lattice map. Section 3 contains the proof of the second result. Section 4 contains some related possibilities of further research. In Sec. 5, we start calculations which will lead us to solve Problem 0.3.

1. Definitions and Notations

We use standard definitions for Anderson t-motives. Let q be a power of a prime, $\mathbb{F}_q$ the finite field of order q, θ a transcendental element, $\mathbb{R}_\infty := \mathbb{F}_q((1/\theta))$ the finite characteristic analog of $\mathbb{R}$. It has a valuation v_∞ defined by $v_\infty(\theta) = -1$. Let $\overline{\mathbb{R}_\infty}$ be an algebraic closure of $\mathbb{R}_\infty$. Let $\mathbb{C}_\infty := \widehat{\overline{\mathbb{R}_\infty}}$ be the completion of $\overline{\mathbb{R}_\infty}$ with respect to (the only continuation of) v_∞. It is the finite characteristic analog of $\mathbb{C}$.

The Anderson ring $\mathbb{C}_\infty[T, \tau]$ is the ring of non-commutative polynomials over $\mathbb{C}_\infty$ in two variables T, τ with the following relations (here $a \in \mathbb{C}_\infty$):

$$aT = Ta; \quad \tau T = T\tau; \quad \tau a = a^q \tau.$$

It has subrings $\mathbb{C}_\infty[T]$, $\mathbb{C}_\infty\{\tau\}$.

Let A be a matrix with entries in $\mathbb{C}_\infty[T, \tau]$. We denote by $A^{(k)}$, where $k \in \mathbb{Z}$, the matrix obtained by elevation of all coefficients of all entries of A to the q^k-th power (T and τ are not elevated to a power).

Definition 1.1. An Anderson t-motive M is a left $\mathbb{C}_\infty[T,\tau]$-module satisfying conditions:

(1) M as a $\mathbb{C}_\infty[T]$-module is free of finite dimension (denoted by r);
(2) M as a $\mathbb{C}_\infty\{\tau\}$-module is free of finite dimension (denoted by n);
(3) The action of $T-\theta$ on $M/\tau M$ is nilpotent.

Let $e_* := \begin{pmatrix} e_1 \\ \cdots \\ e_n \end{pmatrix}$ be a basis of M over $\mathbb{C}_\infty\{\tau\}$. In order to define M, we need to define the product Te_*:

$$Te_* = A_0 e_* + A_1 \tau e_* + \cdots + A_k \tau^k e_*, \tag{1.1}$$

where $A_i \in M_{n\times n}(\mathbb{C}_\infty)$. Condition (3) is equivalent to $A_0 = \theta I_n + N$ where $N \in M_{n\times n}(\mathbb{C}_\infty)$ is nilpotent. We shall consider only M having $N = 0$.

Let $V = \mathbb{C}_\infty^n$ and $L \subset V$ is isomorphic to $\mathbb{F}_q[\theta]^r$.

Definition 1.2. L is called a lattice of rank r if:

(1) The $\mathbb{C}_\infty$-span of L is V;
(2) The $\mathbb{R}_\infty$-span of L has dimension r over $\mathbb{R}_\infty$ (i.e. elements of a basis of L over $\mathbb{F}_q[\theta]$ are linearly independent over $\mathbb{R}_\infty$).

Two lattices $L_1 \subset V_1$, $L_2 \subset V_2$ are isomorphic if there exists a $\mathbb{C}_\infty$-linear isomorphism $\varphi : V_1 \to V_2$ such that $\varphi(L_1) = L_2$.

Let us consider the properties of the lattices of t-motives. Let us fix a basis e_* of M over $\mathbb{C}_\infty\{\tau\}$, and let A_i be from (1.1). The basis e_* defines the following action of T on $\mathbb{C}_\infty^n$:

$$T(Z) = \theta Z + A_1 Z^{(1)} + \cdots + A_k Z^{(k)},$$

where $Z \in \mathbb{C}_\infty^n$ is a matrix column (recall that $N = 0$, hence in our case $A_0 = \theta I_n$).

The following theorems are proved in [2] for the case $n = 1$, and in [1] for the case of any n.

Theorem 1.3. *For a fixed e_* there exists the only map* $\exp = \exp_M : \mathbb{C}_\infty^n \to \mathbb{C}_\infty^n$ *defined by the formula*

$$\exp(Z) = Z + C_1 Z^{(1)} + C_2 Z^{(2)} + \cdots, \tag{1.2}$$

where $Z \in \mathbb{C}_\infty^n$ is a matrix column and $C_i \in M_{n\times n}(\mathbb{C}_\infty)$, making the following diagram commutative:

$$\begin{array}{ccc} \mathbb{C}_\infty^n & \overset{exp}{\to} & \mathbb{C}_\infty^n \\ \theta\downarrow & & \downarrow \quad Z \mapsto T(Z) \\ \mathbb{C}_\infty^n & \overset{exp}{\to} & \mathbb{C}_\infty^n \end{array} \tag{1.3}$$

Theorem 1.4. $L(M) := Ker \exp$ *is a $\mathbb{F}_q[\theta]$-submodule of $\mathbb{C}_\infty^n$ of dimension $\le r$.*

Definition 1.5. M is called uniformizable if the dimension of $L(M)$ as a $\mathbb{F}_q[\theta]$-module is r.

Theorem 1.6. *If M is uniformizable then $L(M)$ is a lattice in $\mathbb{C}_\infty^n$. It is well-defined, i.e. if we change the basis e_* by another basis e'_* then we get an isomorphic lattice.*

We shall consider M of rank 3, dimension 2. Let us give explicit forms of (1.1) for them. First, let M be pure. Its Eq. (1.1) is

$$T\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} 0 & -a_1 \\ 1 & -a_2 \end{pmatrix}\tau\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\tau^2\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}. \tag{1.4}$$

We denote it by $M_p(a_1, a_2)$ (subscript "p" means pure).

Theorem 1.7. *$M_p(a_1, a_2)$ is pure, it is the dual of a Drinfeld module of rank 3 defined by the equation*

$$Te_1 = \theta\ e_1 + a_1\tau e_1 + a_2\tau^2 e_1 + \tau^3 e_1 \tag{1.5}$$

[10, *Sec.* 5; 5; 7, (12.2.2)]. *If a_1, a_2 are fixed then there are only finitely many a'_1, a'_2 such that $M_p(a_1, a_2) = M_p(a'_1, a'_2)$* [2]. *All pure M of rank 3, dimension 2 have this equation* [5].

Let now M be a t-motive such that its Eq. (1.1) is

$$T\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + A\tau\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\tau^2\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \tag{1.6}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 1 \end{pmatrix}. \tag{1.7}$$

M is not pure. We denote it by $M_{np}(A)$.

Conjecture 1.8. *All non-pure M of rank 3, dimension 2 have an equation of the form (1.6). If A of type (1.7) is fixed then there are only finitely many A' of type (1.7) such that $M_{np}(A) = M_{np}(A')$.*

Corollary 1.9. *The set of non-pure M of rank 3, dimension 2 is three-dimensional.*

We shall work with reducible M defined by (1.6), see Eqs. (2.2) and (3.3).

We see that L is a functor (a map on the level of sets) from the set of uniformizable t-motives to the set of lattices, both up to isomorphisms. Is it injective or surjective?

Let us prove the result of non-injectivity: we construct explicitly a set of non-isomorphic uniformizable t-motives such that their lattices are isomorphic.

2. Construction

We need more notations and definitions. First, the Carlitz module $\mathfrak{C}$ is a t-motive having $n = r = 1$, it is defined by the formula

$$Te = \theta e + \tau e,$$

where e is the only element of a basis of $\mathfrak{C}$ over $\mathbb{C}_\infty\{\tau\}$. Let us denote $\theta_{ij} := \theta^{q^i} - \theta^{q^j}$ and for $j \geq 1$

$$c_j := \frac{1}{\theta_{j,j-1}\theta_{j,j-2} \cdot \dots \cdot \theta_{j1}\theta_{j0}}.$$

We have (Carlitz):

$$\exp_{\mathfrak{C}}(z) = z + c_1 z^q + c_2 z^{q^2} + \cdots$$

We denote by π_1 a generator of the lattice $L(\mathfrak{C})$, it is unique up to multiplication by $\mathbb{F}_q^*$. It is the finite characteristic analog of $2\pi i \in \mathbb{C}$ which is defined up to ± 1.

Further, the Carlitz module of rank 2 (denoted by $\mathfrak{C}_2$) is a t-motive having $n = 1$, $r = 2$ (i.e. it is a Drinfeld module of rank 2) defined by the formula

$$Te = \theta e + \tau^2 e.$$

For even $j \geq 2$, we denote

$$c_{2,j} := \frac{1}{\theta_{j,j-2}\theta_{j,j-4} \cdot \dots \cdot \theta_{j2}\theta_{j0}}$$

We have

$$\exp_{\mathfrak{C}_2}(z) = z + c_{2,2} z^{q^2} + c_{2,4} z^{q^4} + \cdots.$$

We denote by π_2 a generator of the lattice $L(\mathfrak{C}_2)$, it is unique up to multiplication by $\mathbb{F}_{q^2}^*$.

We choose and fix $\omega \in \mathbb{F}_{q^2} - \mathbb{F}_q$, i.e. $(1, \omega)$ is a basis of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$. We choose and fix π_1, π_2, hence $(\pi_2, \omega\pi_2)$ is a basis of $L(\mathfrak{C}_2)$ over $\mathbb{F}_q[\theta]$.

Let $M_0 := \mathfrak{C}_2 \oplus \mathfrak{C}$, and let $L_0 := L(M_0) \subset \mathbb{C}_\infty^2$ be its lattice. We have: elements

$$l_1 := \begin{pmatrix} 0 \\ \pi_1 \end{pmatrix}, \quad l_2 := \begin{pmatrix} \pi_2 \\ 0 \end{pmatrix}, \quad l_3 := \begin{pmatrix} \omega\pi_2 \\ 0 \end{pmatrix}, \tag{2.1}$$

are a basis of L_0 over $\mathbb{F}_q[\theta]$.

Let $a \in \mathbb{C}_\infty$, and let A from (1.7) be $\left(\begin{smallmatrix} 0 & a \\ 0 & 1 \end{smallmatrix}\right)$. We denote the corresponding $M_{np}(A)$ by $M(a)$. Then Eq. (1.6) becomes

$$T \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix} \tau \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tau^2 \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}. \tag{2.2}$$

$M(a)$ is reducible, it enters in an exact sequence

$$0 \to \mathfrak{C} \to M(a) \to \mathfrak{C}_2 \to 0. \tag{2.3}$$

Arguments similar to [9], Example 3.9 show that $M(a)$ is not mixed (if $a \neq 0$), because wt $(\mathfrak{C}) = 1 >$ wt $(\mathfrak{C}_2) = \frac{1}{2}$.

We shall prove:

Proposition 2.1. *There exists a neighborhood U of $0 \in \mathbb{C}_\infty$ such that $\forall\, a \in U$ we have*: *$M(a)$ is uniformizable and $L(M(a)) = L_0$.*

Proposition 2.2. *Conjecture* 1.8 *holds for these $M(a)$, i.e. for a fixed a there are only finitely many a' such that $M(a) = M(a')$.*

This implies that the lattice map is not injective.

Proof of Proposition 2.1. We need the notion of a Siegel matrix of a lattice. It is defined exactly like in the case of abelian varieties. Let $l_1, \ldots, l_r$ be a basis of L over $\mathbb{F}_q[\theta]$ such that $l_1, \ldots, l_n$ is a basis of V over $\mathbb{C}_\infty$. The Siegel matrix $S \in M_{(r-n)\times n}(\mathbb{C}_\infty)$ of L with respect to a basis $l_1, \ldots, l_r$ is defined by the formula

$$\begin{pmatrix} l_{n+1} \\ \cdots \\ l_r \end{pmatrix} = S \begin{pmatrix} l_1 \\ \cdots \\ l_n \end{pmatrix}.$$

Two lattices L_1, L_2 are isomorphic iff there exist bases l_{1*}, l_{2*} of L_1, L_2 such that their Siegel matrices S_1, S_2 are equal.

Equation (2.1) implies that S of L_0 is $(0, \omega)$. Let us show that $\forall\, a \in \mathbb{C}_\infty$ a Siegel matrix of $L(M(a))$ is the same. C_i from (1.2) for $M(A)$ satisfy the recurrence relation

$$C_m = \frac{A_1 C_{m-1}^{(1)} + A_2 C_{m-2}^{(2)}}{\theta_{m0}},$$

where $C_m = 0$ for $m < 0$, $C_0 = I_2$, A_1, A_2 are from (1.6), i.e. for our case $A_1 = \begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. We get by induction that

$$\begin{aligned} C_m = C_m(a) &= \begin{pmatrix} c_{2,m} & * \\ 0 & c_m \end{pmatrix} \quad \text{for } m \text{ even}, \\ C_m = C_m(a) &= \begin{pmatrix} 0 & * \\ 0 & c_m \end{pmatrix} \quad \text{for } m \text{ odd}. \end{aligned} \tag{2.4}$$

(see (3.6) for the formulas for (*), here we do not need them).

For sufficiently small a we have: $M(a)$ is uniformizable.[b] We see that there is a basis $l_1(a), l_2(a), l_3(a)$ of $L(M(a))$ over $\mathbb{F}_q[\theta]$ such that

$$l_2(a) = l_2 = \begin{pmatrix} \pi_2 \\ 0 \end{pmatrix}, \quad l_3(a) = l_3 = \begin{pmatrix} \omega\pi_2 \\ 0 \end{pmatrix}, \text{ while } l_1(a) \neq l_1 = \begin{pmatrix} 0 \\ \pi_1 \end{pmatrix}.$$

We get that the Siegel matrix of $L(M(a))$ with respect to $l_1(a), l_2(a), l_3(a)$ is the same $(0, \omega)$. □

Proof of Proposition 2.2. We need a few facts on $GL_2(\mathbb{C}_\infty\{\tau\})$. Let

$$X := X_0 + X_1\tau + \cdots + X_k\tau^k \in M_{2\times 2}(\mathbb{C}_\infty\{\tau\}), \tag{2.5}$$

where $X_i = \begin{pmatrix} x_{i11} & x_{i12} \\ x_{i21} & x_{i22} \end{pmatrix} \in M_{2\times 2}(\mathbb{C}_\infty)$. We have:

$\exists Y \in M_{2\times 2}(\mathbb{C}_\infty\{\tau\})$ such that $XY = 1 \implies |X_0| \neq 0 \implies X$ is not a zero divisor from both right and left $\implies YX = 1$, i.e. $X \in GL_2(\mathbb{C}_\infty\{\tau\})$.

Further, if for $X \in GL_2(\mathbb{C}_\infty\{\tau\})$ we have $\forall\, i\ x_{i21} = 0$ then

$$\sum_{i=0}^{k} x_{i11}\tau^i, \sum_{i=0}^{k} x_{i22}\tau^i \in \mathbb{C}_\infty\{\tau\}^* \implies \forall\, i > 0\ x_{i11} = x_{i22} = 0. \tag{2.6}$$

Let us assume that $M(a) = M(a')$. Multiplication by T in $M(a')$ is given by

$$T\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} + \begin{pmatrix} 0 & a' \\ 0 & 1 \end{pmatrix}\tau\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\tau^2\begin{pmatrix} e_1' \\ e_2' \end{pmatrix}. \tag{2.7}$$

An isomorphism between $M(a)$ and $M(a')$ is given by a matrix X from (2.5) of change of basis (where x_{***} are indeterminate coefficients), i.e.

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = X\begin{pmatrix} e_1' \\ e_2' \end{pmatrix}. \tag{2.8}$$

We substitute (2.8) to (2.2), and we multiply (2.7) by X from the left. We get matrix equalities:

$$TX\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}X\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} + AX^{(1)}\tau\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}X^{(2)}\tau^2\begin{pmatrix} e_1' \\ e_2' \end{pmatrix}, \tag{2.9}$$

$$XT\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} = X\begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} + XA'\tau\begin{pmatrix} e_1' \\ e_2' \end{pmatrix} + X\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\tau^2\begin{pmatrix} e_1' \\ e_2' \end{pmatrix}, \tag{2.10}$$

[b]Most likely $M(a)$ is uniformizable for all a. We do not need this fact. See Remark 3.3 on uniformizability of $M_t(a)$, and Problem 0.3 and the above considerations, for a general statement.

where $A = \begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix}$, $A' = \begin{pmatrix} 0 & a' \\ 0 & 1 \end{pmatrix}$. Equality of coefficients at τ^m of (2.9) and (2.10) is

$$\theta X_m + AX_{m-1}^{(1)} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X_{m-2}^{(2)} = \theta^{q^m} X_m + X_{m-1}A'^{(m-1)} + X_{m-2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.11}$$

where $m = 0, \ldots, k+2$ and $X_i = 0$ if $i \notin 0, \ldots, k$.

Equation (2.11) shows that if $x_{m21} = x_{m-1,21} = 0$ then $x_{m-2,21} = 0$. By induction from up to down we get that $\forall\, m\ x_{m21} = 0$. Hence, (2.6) implies that if $k \geq 1$ then $X_k = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$. Further, for $m = k+2$ (2.11) becomes

$$X_k \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} X_k^{(2)} \iff x_{k11} \in \mathbb{F}_{q^2}, \quad x_{k12} = x_{k21} = 0. \tag{2.12}$$

This means that $k = 0$, $X = X_0 = \begin{pmatrix} x_{011} & 0 \\ 0 & x_{022} \end{pmatrix}$. Finally, (2.11) for $m = 1$ is

$$\begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix} X_0^{(1)} = X_0 \begin{pmatrix} 0 & a' \\ 0 & 1 \end{pmatrix}, \text{ i.e. } \begin{pmatrix} 0 & x_{022}^q a \\ 0 & x_{022}^q \end{pmatrix} = \begin{pmatrix} 0 & x_{011}a' \\ 0 & x_{022} \end{pmatrix}.$$

This means that $x_{022} \in \mathbb{F}_q$ and $M(a)$ is isomorphic to $M(a')$ iff $a'/a \in \mathbb{F}_{q^2}$. □

3. Lattices Belonging to the Image of the Lattice Map

We need a definition of the dual lattice. An invariant (i.e. not depending on coordinates) definition is given for example in [5, Sec. 2; 7, (12.9.2); 9]. We shall need only the definition in terms of Siegel matrices. Namely, let $L \subset \mathbb{C}_\infty^n$ be a lattice, $l_* = (l_1, \ldots, l_r)$ a $\mathbb{F}_q[\theta]$-basis of L and S the Siegel matrix of L with respect to the basis l_*.

By definition, the dual lattice L^* is a lattice of rank r in $\mathbb{C}_\infty^{r-n}$ having a Siegel matrix S^t in some basis. This notion is well-defined, i.e. it does not depend on a choice of l_* and hence on a Siegel matrix.

Not all lattices have dual, because the [Definition 1.2, condition (2)] for S^t is not always satisfied. Really, let $n = 1$. A matrix $S = \begin{pmatrix} s_1 \\ \cdots \\ s_{r-1} \end{pmatrix} \in M_{(r-1)\times 1}(\mathbb{C}_\infty)$ is a Siegel matrix of a lattice of rank r in $\mathbb{C}_\infty$ iff

$$1, s_1, \ldots, s_{r-1} \text{ are linearly independent over } \mathbb{R}_\infty, \tag{3.1}$$

while its transposed $S^t = (s_1 \quad \ldots \quad s_{r-1}) \in M_{1\times(r-1)}(\mathbb{C}_\infty)$ is a Siegel matrix of a lattice of rank r in $\mathbb{C}_\infty^{r-1}$ iff

$$\exists\, i \text{ such that } s_i \notin \mathbb{R}_\infty. \tag{3.2}$$

For $r > 2$ (3.2) is weaker than (3.1), i.e. all lattices of rank r in $\mathbb{C}_\infty$ have duals, but not all lattices of rank r in $\mathbb{C}_\infty^{r-1}$ have duals.

Remark 3.1. The same phenomenon occurs for Siegel matrices of other sizes. For example, let $S = \{s_{ij}\} \in M_{2\times 2}(\mathbb{C}_\infty)$ be a matrix such that $v_\infty(a_{11}) \notin \mathbb{Z}$,

$v_\infty(a_{21}) \notin \mathbb{Z}$, $v_\infty(a_{11}) - v_\infty(a_{21}) \notin \mathbb{Z}$. Then obviously S is a Siegel matrix of a lattice $(\mathbb{F}_q[\theta])^4 \subset \mathbb{C}_\infty^2$. But if $1, a_{11}, a_{12}$ are linearly dependent over $\mathbb{R}_\infty$ then S^t is not a Siegel matrix of a lattice. We think that it is necessary to modify the Definition 1.2 of a lattice, in order to get the surjectivity of the lattice map.

According to Theorem 0.1, all lattices of rank r in $\mathbb{C}_\infty^{r-1}$ having dual are lattices of pure t-motives. Let us answer Question 0.2. We consider the case $r = 3$. We denote by $L(s_{11})$ the lattice having a Siegel matrix $S(s_{11}) := (s_{11}, \omega) \in M_{1\times 2}(\mathbb{C}_\infty)$ (recall that $\omega \in \mathbb{F}_{q^2} - \mathbb{F}_q$ is fixed). It is really a lattice, because $\omega \notin \mathbb{R}_\infty$, i.e. (3.2) holds.

We denote the field $\mathbb{F}_{q^2}((\theta^{-1})) = \{$the $\mathbb{R}_\infty$-linear envelope of 1 and $\omega\}$ by $\mathbb{R}_{\infty,2}$. If $s_{11} \in \mathbb{R}_{\infty,2}$ then $L(s_{11})$ has no dual.

Theorem 3.2. *For all sufficiently small $s_{11} \in \mathbb{C}_\infty$ there exists a t-motive M defined by* (1.6) *such that $L(M) = L(s_{11})$.*

Proof. Let $a \in \mathbb{C}_\infty$, and let A from (1.7) be $\begin{pmatrix} 0 & 0 \\ a & 1 \end{pmatrix}$. We denote the corresponding $M_{np}(A)$ by $M_t(a)$ (the subscript "t" means "transposed"). Then Eq. (1.6) becomes

$$T\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ a & 1 \end{pmatrix}\tau\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\tau^2\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}. \tag{3.3}$$

These $M_t(a)$ enter in an exact sequence

$$0 \to \mathfrak{C}_2 \to M_t(a) \to \mathfrak{C} \to 0, \tag{3.4}$$

hence they are mixed. Their C_m from (1.2) are denoted by $C_{t,m} = C_{t,m}(a)$, they are transposed of $C_m(a)$ of (2.4):

$$C_{t,m}(a) = \begin{pmatrix} c_{2,m} & 0 \\ d_m(a) & c_m \end{pmatrix} \quad \text{for } m \text{ even}, \quad C_{t,m}(a) = \begin{pmatrix} 0 & 0 \\ d_m(a) & c_m \end{pmatrix} \quad \text{for } m \text{ odd}. \tag{3.5}$$

The expression for $d_m(a)$ is the following, it can be easily found by induction:

$$d_m(a) = \frac{[d_{m-1}(a)]^q}{\theta_{m0}} \quad \text{for } m \text{ even}, \quad d_m(a) = \frac{a \cdot c_{2,m-1}^q}{\theta_{m0}} + \frac{[d_{m-1}(a)]^q}{\theta_{m0}} \quad \text{for } m \text{ odd:}$$

$$d_0(a) = 0; \quad d_1(a) = \frac{a}{\theta_{10}}; \quad d_2(a) = \frac{a^q}{\theta_{21}\theta_{20}}; \quad d_3(a) = \frac{a}{\theta_{31}\theta_{30}} + \frac{a^{q^2}}{\theta_{32}\theta_{31}\theta_{30}};$$

$$d_4(a) = \frac{a^q}{\theta_{42}\theta_{41}\theta_{40}} + \frac{a^{q^3}}{\theta_{43}\theta_{42}\theta_{41}\theta_{40}};$$

$$d_5(a) = \frac{a}{\theta_{53}\theta_{51}\theta_{50}} + \frac{a^{q^2}}{\theta_{53}\theta_{52}\theta_{51}\theta_{50}} + \frac{a^{q^4}}{\theta_{54}\theta_{53}\theta_{52}\theta_{51}\theta_{50}}; \tag{3.6}$$

etc., we do not need its exact form. It is easy to find elements of a basis of $L(M_t(a))$, i.e. $Z = \binom{z_1}{z_2} \in \mathbb{C}_\infty^2$ which are solutions of

$$\sum_{m=0}^{\infty} C_{t,m}(a) Z^{(m)} = 0. \tag{3.7}$$

The solutions are

$$l_1 := \begin{pmatrix} 0 \\ \pi_1 \end{pmatrix}, \quad l_2 := \begin{pmatrix} \pi_2 \\ z_{22} \end{pmatrix}, \quad l_3 := \begin{pmatrix} \omega\pi_2 \\ z_{32} \end{pmatrix}. \tag{3.8}$$

To find z_{22} and z_{32} we denote

$$D(a) := \sum_{m=1}^{\infty} d_m(a) \pi_2^{q^m}; \quad D_\omega(a) := \sum_{m=1}^{\infty} d_m(a)(\omega\pi_2)^{q^m}.$$

Both $D(a)$, $D_\omega(a)$ are power series in a:

$$D(a) = \mathfrak{d}_0 a + \mathfrak{d}_1 a^q + \mathfrak{d}_2 a^{q^2} + \cdots$$

$$D_\omega(a) = \bar{\omega}\mathfrak{d}_0 a + \omega\mathfrak{d}_1 a^q + \bar{\omega}\mathfrak{d}_2 a^{q^2} + \cdots$$

where $\bar{\omega} = \omega^q$ is the conjugate of ω and

$$\mathfrak{d}_0 = \frac{\pi_2^q}{\theta_{10}} + \frac{\pi_2^{q^3}}{\theta_{31}\theta_{30}} + \frac{\pi_2^{q^5}}{\theta_{53}\theta_{51}\theta_{50}} + \cdots,$$

$$\mathfrak{d}_1 = \frac{\pi_2^{q^2}}{\theta_{21}\theta_{20}} + \frac{\pi_2^{q^4}}{\theta_{42}\theta_{41}\theta_{40}} + \cdots,$$

$$\mathfrak{d}_2 = \frac{\pi_2^{q^3}}{\theta_{32}\theta_{31}\theta_{30}} + \cdots$$

Calculating $v_\infty(\mathfrak{d}_i)$, we see

First, that $\mathfrak{d}_0 \neq 0$, because $v_\infty(\pi_2) = -q^2/(q^2-1)$ and $v_\infty(\mathfrak{d}_0) = -q/(q^2-1)$.

Second, that $D(a)$ and $D_\omega(a)$ converge for all $a \in \mathbb{C}_\infty$.

Equation (3.7) becomes

$$D(a) + \exp_{\mathfrak{C}}(z_{22}) = 0; \quad D_\omega(a) + \exp_{\mathfrak{C}}(z_{32}) = 0. \tag{3.9}$$

Remark 3.3. Since $\exp_{\mathfrak{C}}$ is surjective, we get, as a by-product, that $M_t(a)$ is uniformizable $\forall\, a \in \mathbb{C}_\infty$.

Now and below we shall consider only sufficiently small a, such that the series $\log_{\mathfrak{C}}(D(a))$, $\log_{\mathfrak{C}}(D_\omega(a))$ converge. We have:

$$z_{22} = \log_{\mathfrak{C}}(-D(a)), \quad z_{32} = \log_{\mathfrak{C}}(-D_\omega(a)). \tag{3.10}$$

We denote the Siegel matrix of $M_t(a)$ corresponding to the basis (3.8), by $\mathfrak{S}(a)$. (3.8) and (3.10) imply that

$$\mathfrak{S}(a) = S(\mathfrak{s}(a)) = (\mathfrak{s}(a); \omega), \tag{3.11}$$

where

$$\mathfrak{s}(a) = \frac{\log_{\mathfrak{C}}(-D_\omega(a)) - \omega \cdot \log_{\mathfrak{C}}(-D(a))}{\pi_1}.$$

The function $a \mapsto \mathfrak{s}(a)$ is an additive power series in a having a nonzero radius of convergence. It is nonzero: its first term is

$$\frac{\mathfrak{d}_0(\omega - \bar{\omega})}{\pi_1} a$$

and $\mathfrak{d}_0 \neq 0$. Hence, $a \mapsto \mathfrak{s}(a)$ is a local isomorphism in a neighborhood of 0. This completes the proof of Theorem 3.2: for s_{11} near 0 we find a such that $\mathfrak{s}(a) = s_{11}$. For this a we have: $L(M_t(a)) = L(s_{11})$. □

Let us consider two applications.

3.1.

Let $s_{11} \in \mathbb{R}_{\infty,2}$. The lattice $L(s_{11})$ has no dual. So, earlier we could not guarantee that it is the lattice of a t-motive (except the trivial case $s_{11} = 0$). Theorem 3.2 tells us that for all sufficiently small s_{11} there exists a t-motive such that $L(s_{11})$ is its lattice.

3.2.

Let a sufficiently small $s_{11} \in \mathbb{C}_\infty - \mathbb{R}_{\infty,2}$. The lattice $L(s_{11})$ has dual. From one side, this means that $L(s_{11})$ is the lattice of the only one pure t-motive of rank 3, dimension 2 (= dual of a Drinfeld module of rank 3). From another side, Theorem 3.2 tells us that $L(s_{11})$ is the lattice of a non-pure t-motive $M_t(a)$, as above. We get once again the result that the lattice map is not injective on the set of non-pure t-motives.

4. Further Questions

Here we consider some possibilities of further research related to the subject of this paper.

Question 4.1. Conjecture 1.8 implies that the set of $M_{np}(A)$ has dimension 3 over $\mathbb{C}_\infty$. Clearly, the set of lattices of $M_{np}(A)$ has dimension 2 over $\mathbb{C}_\infty$, because it is defined by a 1×2 Siegel matrix. Hence, we conjecture that the fibers of the lattice map on the set of $M_{np}(A)$ have dimension 1. Conjecturally, they are analytic curves in the space $\langle a_{11}, a_{12}, a_{21}\rangle$. Is it really true? What are these curves?

Question 4.2. Does there exist a simply described two-dimensional subset $\mathfrak{S}_2$ of the set of A defined in (1.7) such that for any lattice $L \subset \mathbb{C}_\infty^2$ of rank 3 there exists $A \in \mathfrak{S}_2$ such that $L = L(M_{np}(A))$? That is, this $\mathfrak{S}_2$ is a representative of the set of the above curves.

Existence of this $\mathfrak{S}_2$ will help us to prove the conjecture that the lattice map from the set of $M_{np}(A)$ defined by (1.6), to the set of lattices of rank 3 in $\mathbb{C}_\infty^2$, is surjective.

Question 4.3. The set of $M(a)$ defined by (2.2) is $\mathrm{Ext}^1_{\mathbb{C}_\infty[T,\tau]}(\mathfrak{C}_2, \mathfrak{C}_1)$. We have: $\mathrm{Ext}^1_{\mathbb{C}_\infty[T,\tau]}(\mathfrak{C}_2, \mathfrak{C}_1)$ is a module over $Z(\mathbb{C}_\infty[T,\tau]) = \mathbb{F}_q[T]$. We showed in Proposition 2.2 that $\mathrm{Ext}^1_{\mathbb{C}_\infty[T,\tau]}(\mathfrak{C}_2, \mathfrak{C}_1)$ can be considered as having dimension 1 over $\mathbb{C}_\infty$. Hence, it is meaningful to consider a new invariant of a non-uniformizable t-motive M, namely, the dimension of $\mathrm{Ext}^1_{\mathbb{C}_\infty[T,\tau]}(M, Z_1)$ (here $Z_1 = \mathbb{C}_\infty\{T\}$ is from [4], (5.9.22)) over $\mathbb{C}_\infty$. Recall that M is uniformizable $\iff$ $\mathrm{Ext}^1_{\mathbb{C}_\infty[T,\tau]}(M, Z_1) = 0$. As a $\mathbb{F}_q[T]$-module, most likely, $\mathrm{Ext}^1_{\mathbb{C}_\infty[T,\tau]}(M, Z_1)$ is infinite-dimensional, but we can expect that over $\mathbb{C}_\infty$ the dimension is finite and gives us an invariant of M.

It is possible to ask the same for Coker $\exp_M \subset \mathrm{Ext}^1_{\mathbb{C}_\infty[T,\tau]}(M, Z_1)$.

Remark 4.4. It is not too difficult to find v_∞ of the coefficients of the power series $\mathfrak{s}(a)$, and hence its Newton polygon. Hence, we can find explicitly the size of the set of s_{11} that belong to the image of $\mathfrak{s}$. Maybe this set is even the whole $\mathbb{C}_\infty$.

Our purpose is to prove the conjecture that the lattice map for $r = 3, n = 2$ is surjective. Therefore, we should not restrict ourselves by the Siegel matrices of the form (s_{11}, ω), we should consider all matrices satisfying (3.2) and hence more general M than the ones defined by (3.3).

5. Lattices of t-Motives Defined by (1.6), (1.7)

We want to start to solve Problem 0.3. Here we apply the methods of [3, 8] to the non-pure t-motives M of rank 3, dimension 2 defined by (1.6), (1.7), in order to get an analog of formulas (3.8)–(3.10) of [8].

A basis of this M over $\mathbb{C}_\infty[T]$ is $\{f_*\} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ \tau e_1 \end{pmatrix}$. The action of τ on $\{f_*\}$ is defined by a matrix $Q \in \mathbb{C}_\infty[T]$:

$$\tau f_* = Q f_* \text{ where in our case } Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & T-\theta & -a_{21} \\ T-\theta & -a_{12}(T-\theta) & -d \end{pmatrix},$$

where $d = |A| = a_{11} - a_{12}a_{21}$. In order to find $L(M)$, we must solve a system

$$QX = X^{(1)}, \tag{5.1}$$

where $X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \in \mathbb{C}_\infty\{T\}^3$. Hence, (5.1) is

$$X_3 = X_1^{(1)}, \tag{5.2}$$

$$(T-\theta)X_2 - a_{21}X_3 = X_2^{(1)}, \tag{5.3}$$

$$(T-\theta)X_1 - a_{12}(T-\theta)X_2 - dX_3 = X_3^{(1)}. \tag{5.4}$$

Let us make eliminations. Equations (5.2)–(5.4) imply

$$(T-\theta^q)X_1^{(1)} - a_{12}^q(T-\theta^q)X_2^{(1)} - d^q X_1^{(2)} - X_1^{(3)} = 0, \tag{5.5}$$

$$X_1^{(1)} = \frac{T-\theta}{a_{21}}X_2 - \frac{1}{a_{21}}X_2^{(1)}, \tag{5.6}$$

$$X_1^{(2)} = \frac{T-\theta^q}{a_{21}^q}X_2^{(1)} - \frac{1}{a_{21}^q}X_2^{(2)}, \tag{5.7}$$

$$X_1^{(3)} = \frac{T-\theta^{q^2}}{a_{21}^{q^2}}X_2^{(2)} - \frac{1}{a_{21}^{q^2}}X_2^{(3)}. \tag{5.8}$$

Substituting (5.6)–(5.8) to (5.5), we get

$$\frac{1}{a_{21}^{q^2}}X_2^{(3)} + \left(-\frac{T}{a_{21}^{q^2}} + \frac{d^q}{a_{21}^q} + \frac{\theta^{q^2}}{a_{21}^{q^2}}\right)X_2^{(2)} - (T-\theta^q)\left(\frac{1}{a_{21}} + a_{12}^q + \frac{d^q}{a_{21}^q}\right)X_2^{(1)} + \frac{(T-\theta^q)(T-\theta)}{a_{21}}X_2 = 0. \tag{5.9}$$

In order to simplify further calculations, we introduce new variables

$$u = \frac{1}{a_{21}} + a_{12}^q + \frac{d^q}{a_{21}^q} = \frac{1}{a_{21}} + \frac{a_{11}^q}{a_{21}^q},$$

$$v = \frac{d^q}{a_{21}^q} + \frac{\theta^{q^2}}{a_{21}^{q^2}}.$$

If $a_{21} \neq 0$ then the set of (a_{21}, a_{11}, a_{12}) is in one-to-one correspondence with the set of (a_{21}, u, v). Equation (5.9) has the form

$$\frac{1}{a_{21}^{q^2}}X_2^{(3)} + \left(-\frac{T}{a_{21}^{q^2}} + v\right)X_2^{(2)} - (T-\theta^q)\cdot u \cdot X_2^{(1)} + \frac{(T-\theta^q)(T-\theta)}{a_{21}}X_2 = 0. \tag{5.10}$$

This is an analog of [8, (3.8)]. Continuing the calculations similar to the ones of [3, 8], we shall find h_1, h^1 of all $M_{np}(A)$.

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