

Synchronization of phase oscillators with chemical coupling: Removal of oscillators and external feedback control

Bruno M. Czajkowski^a, Carlos A.S. Batista^b, Ricardo L. Viana^{a,c,*}

^a Universidade Federal do Paraná, Setor de Ciências Exatas, Departamento de Física, 81531-990, Curitiba, Paraná, Brazil

^b Universidade Federal do Paraná, Campus Pontal do Paraná, Centro de Estudos do Mar, 83255-976, Pontal do Paraná, Paraná, Brazil

^c Universidade de São Paulo, Instituto de Física, Departamento de Física Aplicada, 05315-970, São Paulo, Brazil

ARTICLE INFO

Article history:

Received 27 October 2022

Received in revised form 16 December 2022

Available online 20 December 2022

Keywords:

Phase oscillators

Chemical coupling

Non-local couplings

Phase synchronization

Frequency synchronization

Suppression of synchronization

ABSTRACT

In many applications of biological interest spatially localized phase oscillators present a non-local coupling characterized by the diffusion of a mediating chemical. The latter is both produced and absorbed by the phase oscillators according to their dynamical behavior. We investigate numerically phase and frequency synchronization in a model for chemical coupling of uniform phase oscillators with natural frequencies randomly distributed according to some statistical distribution. We study the effect of removal of oscillators, caused by lesions, on the synchronization properties of the network. The partial or complete suppression of synchronization is also investigated through the influence of a time-delayed feedback external signal.

© 2022 Elsevier B.V. All rights reserved.

1. Introduction

Phase oscillators appear in models for a variety of physical, chemical, and biological systems [1,2]. It is often found that, in biological applications, cellular phase oscillators are able to absorb and/or secrete some chemical which undergoes diffusion in the spatial medium embracing the cells. A well-known example is a neuron with chemical synapses, which releases a neurotransmitter which rapidly diffuses along the synaptic cleft and is absorbed by a post-synaptic neuron [3]. Since both the release and the absorption of this chemical are often dependent on the dynamics of the systems, it is effectively a coupling between them, mediated by the diffusion of a substance, or simply *chemical coupling*. This is a kind of non-local coupling, for it considers not only the nearest neighbors, but also all the systems, with an intensity that depends on their mutual distances.

Kuramoto [4] proposed a mathematical model for chemical coupling in which the coupling term acting on each oscillator is proportional to the local concentration of the mediating chemical. The latter satisfies a diffusion equation, and the limit of fast relaxation is taken, meaning that the characteristic diffusion time is supposed to be much smaller than the oscillator typical period. The geometrical setting and the boundary conditions are determined by the corresponding Green function in each case. Kuramoto and Nakao shown that such a coupling causes the appearance of power-law spatial correlations [5].

One of the collective effects that can be investigated in the context of these chemical coupling models is phase and frequency synchronization, which meets a number of examples of biological interest. One of them is the suprachiasmatic nucleus (SCN), which is a small region in the brain hypothalamus whose function is to control circadian rhythms [6]. The

* Corresponding author at: Universidade Federal do Paraná, Setor de Ciências Exatas, Departamento de Física, 81531-990, Curitiba, Paraná, Brazil.
E-mail address: viana@fisica.ufpr.br (R.L. Viana).

SCN contains *circa* 10^4 clock cells with a natural variety of individual frequencies. Their coupling is thought to be mediated by a neurotransmitter (GABA) which diffuses through the spatial medium in which the SCN cells are embedded [7]. Since the SCN acts as a pacemaker, in order to generate a collective single rhythm each clock cell must synchronize its own frequency, what can be achieved through their chemical coupling [8]. Moreover, in the neurological perspective, synchronization of coupled neurons has been linked with a variety of diseases like Parkinson, essential tremor and epilepsy [9].

In this paper we address the issue of phase and frequency synchronization of phase oscillators with chemical coupling, according to the Kuramoto model. We consider Green functions for the fast-relaxation limit of the coupling in one and two dimensions. We investigate numerically the onset of phase and frequency synchronization of an assembly of nonidentical phase oscillators whose natural frequencies are randomly distributed according to a normal distribution.

Networks of biological interest involving neurons or related clock cells should exhibit some degree of robustness against lesions that causes the progressive removal of oscillators or simply weaken their connections [10]. In this work we consider the effect of oscillator removal on the synchronization properties, using different protocols for changing the coupling strength according to the number of remaining oscillators.

Another theme to be considered in this manuscript is the effect of an external time-delayed feedback control signal, whose main purpose is to suppress totally or partially phase synchronization of the network. This procedure can be regarded as a mathematical model of deep-brain stimulation techniques now widely used in neurosurgical practice [11]. A low-intensity electrical signal can be applied to a number of carefully chosen neurons in order to take them out of a synchronized state related to some neurological disorder [12]. We show that there are intervals of control parameter values for which the control of synchronization is successful, even with small perturbation amplitudes.

The rest of the paper is organized as follows: in Section 2 we outline the Kuramoto model for chemical coupling, detailing the geometrical settings to be used in one and two spatial dimensions. Section 3 is devoted to the definition and numerical characterization of phase and frequency synchronization. The effect of removal of oscillators is treated in Section 4, whereas the issue of synchronization control through an external feedback signal is addressed in Section 5. The last Section contains our Conclusions.

2. Model for chemical coupling

Let us consider an assembly of N uniform phase oscillators characterized by their geometrical phases $0 \leq \theta_j < 2\pi$, where $j = 1, 2, \dots, N$, such that, in absence of coupling, they satisfy $\dot{\theta}_j = \omega_j$, where ω_j are constant frequencies. The latter are randomly chosen from a probability distribution $g(\omega)$, such that $g(\omega)d\omega$ is the probability of finding an oscillator with a frequency between ω and $\omega + d\omega$. This distribution is unimodal, normalized to the unity, and centered at $\omega = 0$.

The model proposed by Kuramoto for chemical coupling assumes that: (i) the oscillators occupy fixed positions $\mathbf{r}_j$ in a d -dimensional Euclidean space; (ii) the coupling term for each oscillator depends on the local concentration of a chemical $A(\mathbf{r})$; (iii) the oscillators are able to produce the chemical at a rate dependent on their phases; (iv) the mediating chemical diffuses through the spatial region containing the phase oscillators [4,5]. The model is thus mathematically represented by the following coupled ordinary/partial differential equations

$$\dot{\theta}_j = \omega_j + KA(\mathbf{r}_j, t), \quad (j = 1, 2, \dots, N) \quad (1)$$

$$\varepsilon \frac{\partial A}{\partial t} = -\eta A + D \nabla^2 A + \sum_{k=1}^N h(\theta_k) \delta(\mathbf{r} - \mathbf{r}_k), \quad (2)$$

where $K > 0$ is the coupling strength, ε is a small constant, η is a coefficient of chemical degradation, and D is the diffusion coefficient. The source term for the diffusion equation depends on the oscillator phases by the function $h(\cdot)$. All the quantities here are assumed to be suitably normalized.

Additionally, we assume that the diffusion characteristic time is much smaller than any of the oscillator periods $2\pi/\omega_j$, such that we can set $\varepsilon \approx 0$ and consider that the chemical concentration undergoes a fast relaxation and converge very rapidly to its stationary limit. Recently we have obtained a similar model, but with arbitrary diffusion time, with respect to the oscillator periods [13]. The resulting (time-independent) form of Eq. (2) can be solved to yield the local equilibrium concentration of the mediating chemical, namely

$$A(\mathbf{r}) = \sum_{k=1}^N h(\theta_k) G(\mathbf{r}, \mathbf{r}_k), \quad (3)$$

where $G(\mathbf{r}, \mathbf{r}')$ is the corresponding Dirichlet–Green function. For simplicity we have considered free boundary conditions, such that $\lim_{|\mathbf{r}| \rightarrow \infty} G(\mathbf{r}, \mathbf{r}') = 0$. Substituting (3) into (1) results a general equation for chemical coupling

$$\dot{\theta}_j = \omega_j + K \sum_{k=1}^N h(\theta_k) G(\mathbf{r}_j, \mathbf{r}_k), \quad (j = 1, 2, \dots, N). \quad (4)$$

In the following we will consider a nonlinear form of the coupling given by

$$h(\theta_k) = \sin(\theta_k - \theta_j). \quad (5)$$

Considering an infinite one-dimensional chain of oscillators the corresponding Green function yields [14]

$$\dot{\theta}_j = \omega_j + KC_{1,j}(\gamma, N) \sum_{k=1}^N e^{-\gamma|x_k - x_j|} \sin(\theta_k - \theta_j), \quad (6)$$

where $C_{1,j}$ is a normalization factor and

$$\gamma = \sqrt{\frac{\eta}{D}}, \quad (7)$$

is a coupling length. In regular lattices the sites are separated by a normalized distance Δ , and we choose periodic boundary conditions such that

$$|x_k - x_j| = \Delta \times \min \left\{ \Psi_k^j, N - \Psi_k^j \right\}, \quad (8)$$

where Ψ_k^j is the remainder of the integer division of $|k - j|$ by N .

By imposing the following normalization condition for the Green functions

$$\int d^d r G(\mathbf{r}, \mathbf{r}_k) = 1,$$

there follows that

$$C_{1,j}(\gamma, N)^{-1} = \begin{cases} \sum_{k=1}^N e^{-\gamma|x_k - x_j|} - 1, & N \text{ even} \\ 2 \sum_{k=1}^{(N-1)/2} e^{-\Delta\gamma k}. & N \text{ odd} \end{cases} \quad (9)$$

For N odd Eq. (6) can thus be recast in a symmetrical form

$$\dot{\theta}_j = \omega_j + KC_{1,j}(\gamma, N) \times \sum_{k=1}^{(N-1)/2} e^{-\Delta\gamma k} \left\{ \sin(\theta_{j-k} - \theta_j) + \sin(\theta_{j+k} - \theta_j) \right\}. \quad (10)$$

In a two-dimensional region the Green function is proportional to the modified Bessel function of the second kind $\mathcal{K}_0(\gamma|\mathbf{r} - \mathbf{r}_k|)$, such that the equation for the chemical coupling reads in this case

$$\dot{\theta}_j = \omega_j + KC_{2,j}(\gamma, N) \sum_{k=1, k \neq j}^N \mathcal{K}_0(\gamma|\mathbf{r}_k - \mathbf{r}_j|) \sin(\theta_k - \theta_j), \quad (11)$$

where the corresponding normalization factor is

$$C_{2,j}(\gamma, N)^{-1} = \sum_{k=1, k \neq j}^N \mathcal{K}_0(\gamma|\mathbf{r}_k - \mathbf{r}_j|). \quad (12)$$

Considering a rectangular lattice with $n \times n$ sites with uniform spacing L_x in the x -axis direction and L_y in the y -axis direction, and assuming periodic boundary conditions, the distance between two oscillators located at $\mathbf{r}_k = (i_1 L_x, j_1 L_y)$ and $\mathbf{r}_j = (i_2 L_x, j_2 L_y)$ is

$$|\mathbf{r}_k - \mathbf{r}_j| = \sqrt{(\Delta x)_{k,j}^2 + (\Delta y)_{k,j}^2}, \quad (13)$$

$$(\Delta x)_{k,j} = L_x \min \left\{ \Psi_{i_1}^{i_2}, n - \Psi_{i_1}^{i_2} \right\}, \quad (14)$$

$$(\Delta y)_{k,j} = L_y \min \left\{ \Psi_{j_1}^{j_2}, n - \Psi_{j_1}^{j_2} \right\}. \quad (15)$$

3. Phase and frequency synchronization

In the numerical simulations to be shown in this paper we adopt a normal distribution for the uncoupled oscillator frequencies.

$$g(\omega) = (2/\pi) e^{-4\omega^2/\pi}. \quad (16)$$

We use initial profiles with randomly chosen phases in the interval $[0, 2\pi)$ with a uniform probability distribution. The coupled differential equations for coupled oscillator lattices in one and two dimensions have been numerically solved using a fourth-order Runge–Kutta method with variable stepsize and a tolerance of 10^{-6} . For simplicity we assume $\Delta = L_x = L_y = 1$. The number of oscillators in the one-dimensional and two-dimensional lattices will be chosen as $N = 3000$ and $55^2 = 3025$, respectively.

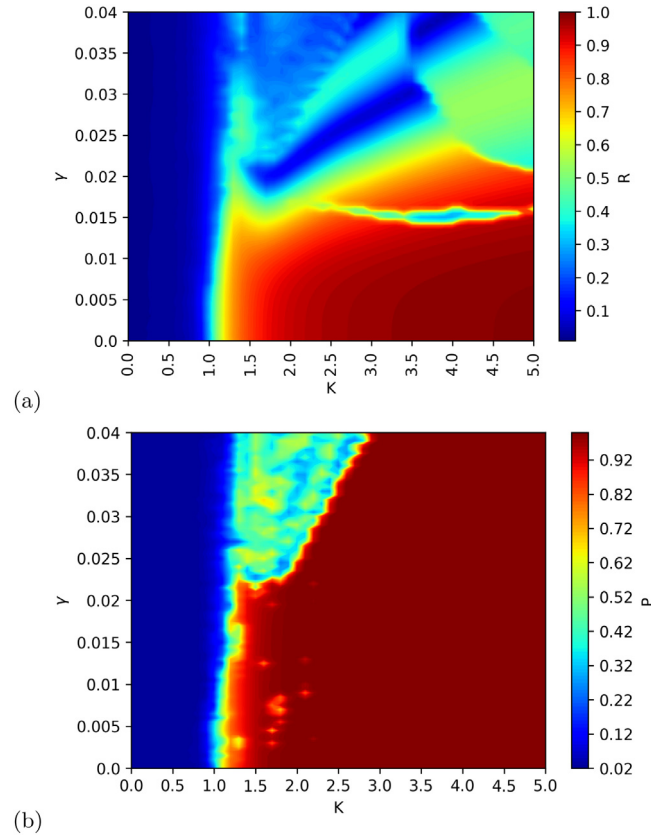


Fig. 1. (a) Mean order parameter magnitude and (b) frequency order parameter as a function of the coupling strength K and length γ for a one-dimensional chain of $N = 3000$ phase oscillators with randomly chosen natural frequencies.

As a numerical diagnostic of phase synchronization, for a fixed time we compute the order parameter magnitude of a one-dimensional chain as [15]

$$R(t) = \frac{1}{N} \left| \sum_{k=1}^N e^{i\theta_k(t)} \right|, \quad (17)$$

with an analogous definition for a two-dimensional lattice. After some transient $R(t)$ oscillates with time and we take its mean $\bar{R}$ over the interval $1500 < t < 2000$. If $\bar{R} \approx 1$ all the oscillators are phase-synchronized. Numerically we used $\bar{R} = 0.95$ as a threshold for complete phase synchronization, lower values characterizing partial synchronization.

The coupling parameters are the strength K and the effective length γ . For each pair of their values we computed the mean order parameter magnitude $\bar{R}$, the results being shown in Fig. 1(a). For small values of γ we have a transition from non-synchronized states to completely synchronized ones, for $K > K_c$. In particular, for $\gamma = 0$ we estimate $K_c \approx 1.2$. Indeed, taking the limit $\gamma \rightarrow 0$ in Eq. (10) we obtain essentially the classical Kuramoto model for global coupling, in which every oscillator interacts with all other neighbors with the same intensity, no matter how far or close they are. From (7) there follows that $\gamma \rightarrow 0$ means either large diffusion coefficient or vanishing chemical degradation of the coupling substance, both factors resulting in a global coupling. A mean-field analysis of the classical Kuramoto model in the thermodynamical limit ($N \rightarrow \infty$) reveals that $K_c = 2/(\pi g(0))$ [16]. From (16), it results that $K_c = 1$. On the other hand, for finite N this value turns to be close but slightly higher than the limit value, as we have observed here. The same trend can be observed for $\gamma \lesssim 0.015$, where a qualitatively different behavior shows up. A dramatic increase of K_c is found, with a narrow “valley” of non-synchronized behavior in between. We observed that this valley results from the emergence of spatially inhomogeneous oscillators coexistent with synchronized regions, the so-called chimera states [17]. The net result is that the order parameter magnitude experiences a decrease for those states.

For even larger values of the coupling length γ we found that phase synchronization cannot be achieved for values of the coupling strength in the interval $0 \leq K \leq 5.0$. This can be understood by taking the limit $\gamma \rightarrow \infty$ in Eq. (10), for which only the nearest neighbors of a given site are included in the coupling term. In this case we have the diffusive (or local) coupling limit, for which synchronization is not likely since the coupling strength is usually too small to overcome the

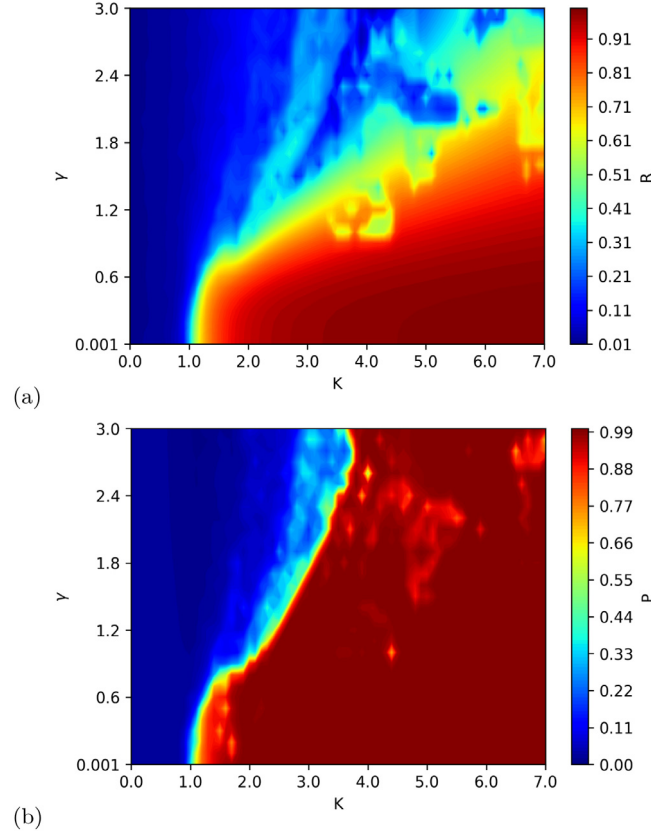


Fig. 2. (a) Mean order parameter magnitude and (b) frequency order parameter as a function of the coupling strength K and length γ for a two-dimensional lattice of 55^2 phase oscillators with randomly chosen natural frequencies.

natural disorder of the oscillator frequencies. This fact can also be inferred from Eq. (7), since $\gamma \rightarrow \infty$ means a vanishing diffusion coefficient or large degradation, both factors preventing a long-range coupling.

A weaker form of collective behavior of coupled phase oscillators is frequency synchronization, in which the oscillators have not necessarily the same phases but their frequencies [18]

$$\Omega_j = \lim_{t \rightarrow \infty} (1/t) \{ \theta_j(t+T) - \theta_j(T) \}, \quad (18)$$

are equal (up to a given tolerance). Numerically we found that the above limit always exists and is independent on T . Let us divide the frequency interval $[\min \Omega_j, \max \Omega_j]$ into sub-intervals of size δ and consider the fraction of oscillators belonging to the sub-interval with the largest number of oscillators. If the largest number of oscillators in a given interval is N_{max} , we define a frequency order parameter by $P = N_{max}/N$. If all the oscillators have the same frequency, then $P = 1$, in analogy with the former order parameter. In the numerical simulations we have chosen $\delta = \sigma/2001$, where $\sigma = \sqrt{\pi/8}$ is the standard deviation of the ω_j -distribution.

In Fig. 1(b) we plot the frequency order parameter P as a function of K and γ for the same parameter values as before. For $\gamma < 0.022$ we observe a similar behavior in comparison with the phase order parameter. As a matter of fact, the results should be similar, since phase synchronized oscillators are always frequency synchronized. The converse, however, is not always true, and hence the critical K for frequency synchronization should be slightly smaller than for phase synchronization. For $\gamma > 0.02$, however, the difference between phase and frequency synchronization is pronounced, since we find a large region of (K, γ) -values for which the frequency order parameter is large (> 0.8), whereas the phase order parameter is between 0.4 and 0.8. Another difference with the previous case is that the narrow valley observed in Fig. 1(a) at $\gamma \approx 0.015$ is absent in Fig. 1(b).

The phase and frequency order parameters are plotted in Fig. 2(a) and (b), respectively, for a two-dimensional rectangular lattice of 55^2 oscillators. We have here the same overall features already described for one-dimensional chains: for example, when γ is small we still have a transition to synchronized behavior for $K > K_c \approx 1.0$. Examples of non-synchronized and synchronized lattices are shown in the snapshots of the phases (in colorscale) of Figs. 3(a) and (b), respectively. A noteworthy difference with the one-dimensional case, however, lies in the range of the coupling length γ , which has a nearly ten-fold increase: a diffusive process in two dimensions involves a larger range for the same value of γ .

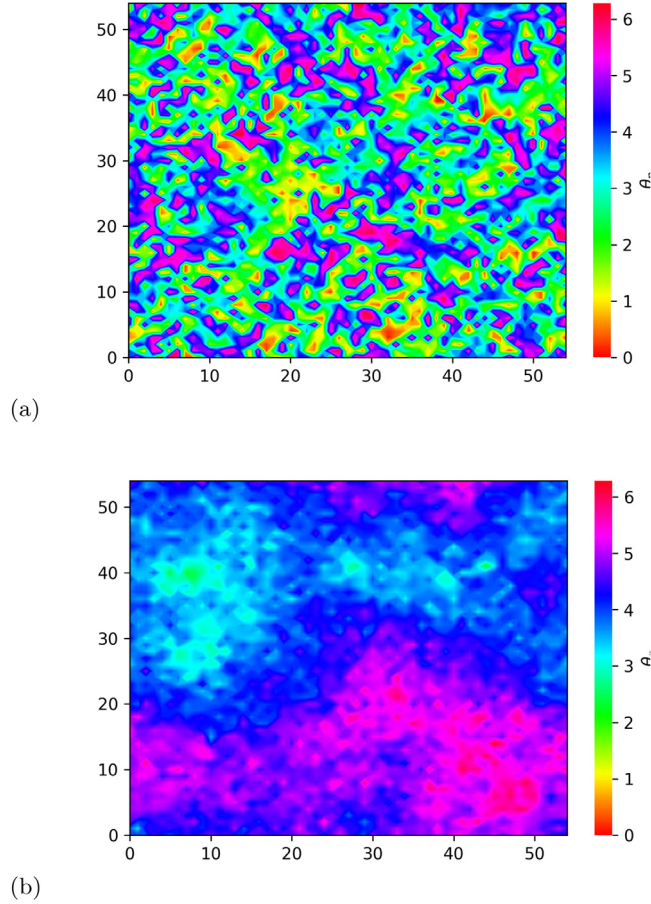


Fig. 3. Snapshots of the oscillator phases for a two-dimensional lattice of 55^2 phase oscillators with randomly chosen natural frequencies, $\gamma = 0.8$ and (a) $K = 0.9$ and (b) 2.5 .

4. Removal of oscillators

As we have commented in the Introduction, there is a biological interest in studying the effect of lesions in assemblies of coupled phase oscillators, specifically with respect to their synchronization properties. We consider lesion protocols whereby, given a network initially with N oscillators and a coupling strength K_i , a number N_d of them is removed over some time interval. The remaining $N_a = N - N_d$ ones will be called active oscillators. The network is supposed to adapt to these alterations, such that the coupling strength will vary with the number of removed oscillators in three possible ways [10]

1. enhanced coupling: $K(N_d) = K_i N / (N - N_d)$;
2. invariant coupling: $K(N_d) = K_i$;
3. reduced coupling: $K(N_d) = K_i N / (N + N_d)$

In our numerical experiments we start from picking up values of (K_i, γ) which lead to completely phase-synchronized states and solving the coupled oscillator equations for a long time, until the transients have died out. Then we remove a small quantity of sites and resume integration, such that the order parameter is recomputed. The process is repeated until the number of removed oscillators reaches a specified value of N_d .

The effect of this procedure on a one-dimensional chain with initially N_i oscillators is shown in Fig. 4(a), where we plot the order parameter as a function of time with invariant coupling, i.e. the coupling strength does not vary as we remove sites from the lattice. The dashed vertical bars indicate the times at which a small quantity of sites is removed until we get $N = N_d$. The oscillators are kept synchronized ($R > 0.95$) until, after a time $\sim 10^4$, they progressively lose phase synchronization and eventually become completely non-synchronized.

In Fig. 4(b) we show the average value of the order parameter (R) as a function of the number of removed oscillators N_d , considering the three protocols already mentioned. The average of R is taken over those time intervals for which the

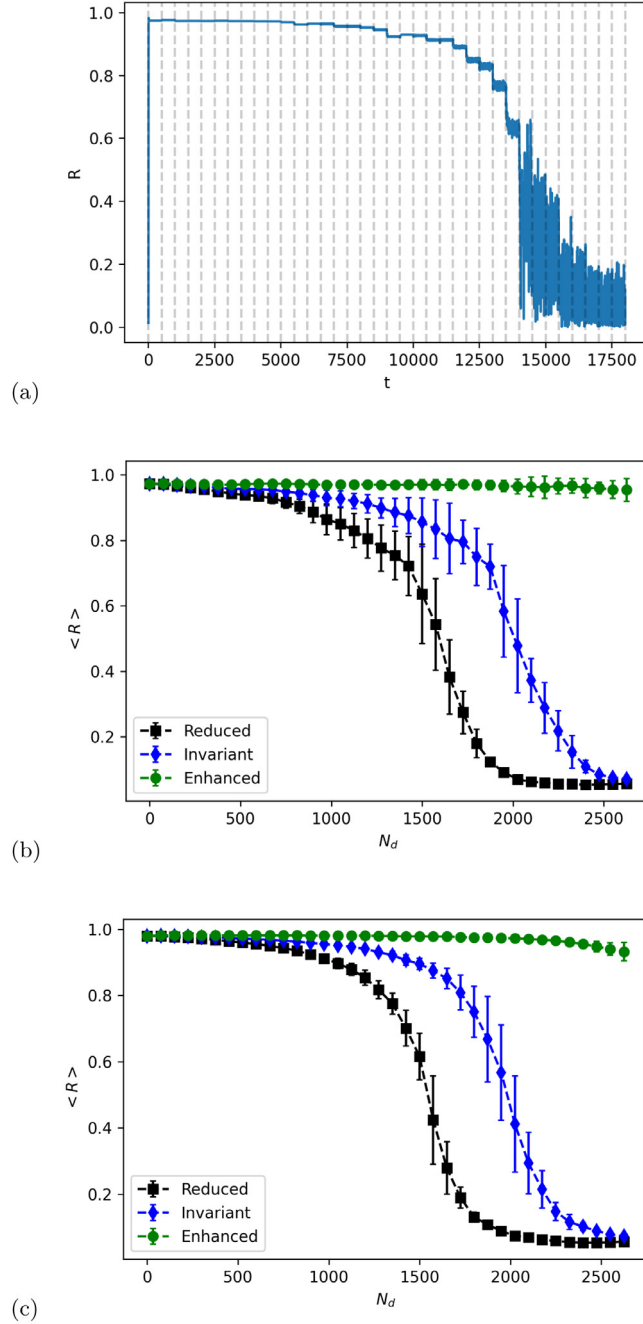


Fig. 4. (a) Order parameter magnitude as a function of time for a one-dimensional chain with $\gamma = 0.01$ and $K_i = 4$. (b) Average order parameter as a function of the number of removed oscillators for the reduced (black), invariant (blue), and enhanced (green) couplings, for a one-dimensional chain. (c) The same as (b), but for a two-dimensional lattice with $\gamma = 0.4$ and $K_i = 4.0$.

number of active oscillators $N_a = N - N_d$ is constant. To understand how relevant the way we remove the oscillators is for the results, we have made for each protocol eight simulations that differ only in the order of removal of the oscillators. So the points in this figure are actually the mean value of $\langle R \rangle$ in these eight simulations, and the error bars refer to one standard deviation from this mean. In this sense, small errors imply that the values of $\langle R \rangle$ are not greatly affected by how the oscillator removal occurs.

Using the invariant coupling (blue points) the chain goes progressively non-synchronized as the number of removed oscillators N_d is increased. In order to avoid this effect, we can use the enhanced coupling, for which K is increased according to the ratio N/N_a (green points). Indeed, using this protocol the chain is able to keep itself synchronized even

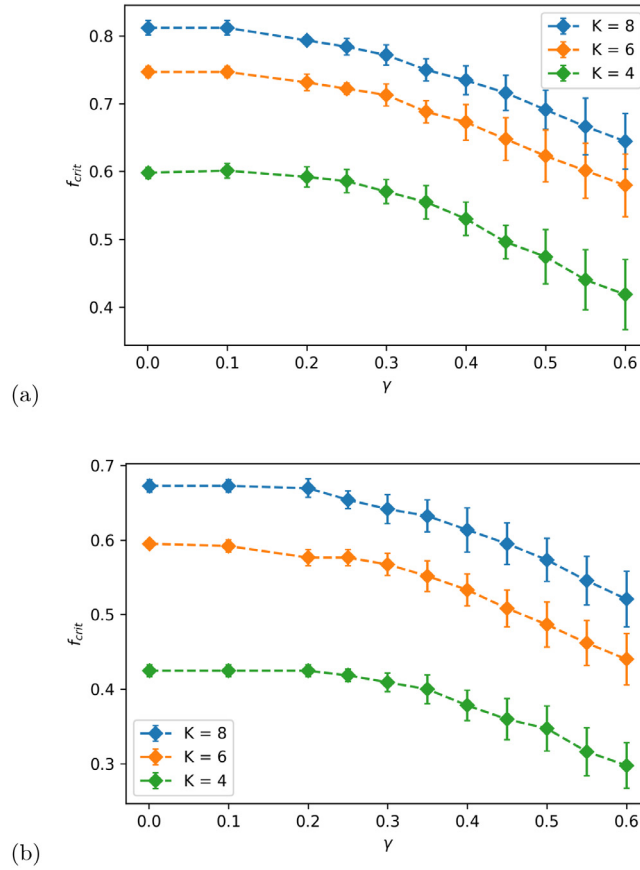


Fig. 5. Critical fraction of removed oscillators as a function of γ for different values of K_i with (a) invariant, (b) reduced coupling protocols.

if N_d is as large as 2500, out of a chain with initially 3000 sites. If, on the other hand, we use a reduced coupling (black points), the coupling strength decreases with N/N_d and, accordingly, the transition to non-synchronized behavior occurs even before the case for which K does not vary with N_d . These observations hold for two-dimensional lattices as well, as shown by Fig. 4(c), provided we choose appropriate values of the coupling length (namely, $\gamma = 0.4$).

We observed in our numerical experiments that the value of N_d for which the lattice start losing synchronization is practically not affected by the order by which each oscillator is removed, this can be seen by the small error bars at the beginning of the decline in synchronization in the Figs. 4(b) and 4(c). This suggests a critical value of N_d , denoted by N_{crit} , for given K and γ , such that, if $N_d \geq N_{crit}$, the lattice cannot synchronize. We estimate the critical fraction of removed oscillators, $f_{crit} = N_{crit}/N$, by taking the minimum value of N_d yielding $R(N_d) \leq 0.9R(N_d = 0)$. The values of f_{crit} as a function of the coupling length γ , for a two-dimensional lattice, are depicted in Figs. 5(a) and (b) for invariant and reduced couplings, respectively, and three different values of K_i . We did eight simulations for each pair (K_i, γ) that differed only in the order of removal of the oscillators. The points in the figure are the average values of f_{crit} in these 8 simulations, and the error bars represent one standard deviation from this average. We can clearly see that for lower γ values, f_{crit} is better defined by the model's parameters, as indicated by smaller error bars. While when we increase γ , f_{crit} has greater dependence on how the removal of oscillators takes place.

In all of these cases the value of f_{crit} decreases with γ , indicating that lattices with local coupling desynchronize earlier than globally coupled ones, as the number of removed oscillators increase. Moreover, for a fixed γ , lattices with smaller K will also desynchronize first. As for the different coupling protocols considered, the values of f_{crit} are smaller for the reduced coupling in comparison with the invariant coupling, as can be concluded by comparing Figs. 5(a) and (b).

5. Time-delayed feedback control

In biological applications it is sometimes desirable to suppress phase synchronization of a given network of coupled oscillators, either completely or in a partial way. One manner to achieve phase synchronization suppression is to apply an external control signal on a given number of oscillators [19]. A feedback control can be implemented by applying a time-delayed signal proportional to the coupling term at an early time [20].

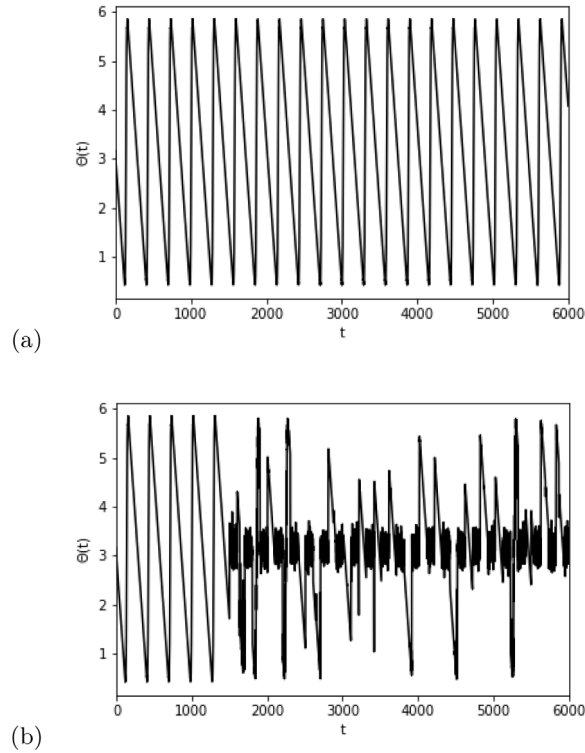


Fig. 6. Mean field of a two-dimensional lattice (a) without control and (b) with a delayed feedback control applied at $\alpha = 1500$, $\varepsilon = 4$, and $\tau = 100$.

Let us write the coupled oscillator Eqs. (6) and (11) in the following form

$$\dot{\theta}_j = \omega_j + KY_j(t), \quad (19)$$

where $Y_j(t)$ represents the coupling term corresponding to the respective Green function

$$Y_j(t) = \sum_{k=1}^N G(\mathbf{r}_j, \mathbf{r}_k) \sin(\theta_k(t) - \theta_j(t)). \quad (20)$$

We suppose that the coupling parameters K and γ are chosen so as to yield completely phase-synchronized behavior. The external time-delayed feedback control is applied in the following way

$$\dot{\theta}_j = \omega_j + KY_j(t) + \varepsilon Y_j(t - \tau)H(t - \alpha), \quad (21)$$

where τ is the time delay and ε is the control amplitude. $H(t - \alpha)$ is the Heaviside unit-step function, where α is the time after which the control is continuously applied, and it is chosen after transients have died out.

In order to measure to what extent the control is able to suppress phase synchronization, we use – instead of the order parameter – the mean field of the network at a given time

$$\Theta(t) = \frac{1}{N} \sum_{j=1}^N \theta_j(t), \quad (22)$$

and the corresponding variance $\text{Var}(\Theta)$. If there is complete phase synchronization, the mean field will have the same variation in time as any of the oscillator themselves, with a finite variance. However, if the lattice is completely non-synchronized, for large N we can assume that the phases are more or less uniformly distributed over the interval $[0, 2\pi)$, and the mean field have fluctuations of low amplitude, and a corresponding small variance.

We present numerical results for two-dimensional lattices only, since they are qualitatively similar to the one-dimensional chains. In Fig. 6(a) we plot the mean field $\Theta(t)$ of a free-running lattice, i.e. without control, showing regular oscillations characteristic of a completely synchronized behavior. The effect of a time-delayed feedback signal, applied after $\alpha = 1500$, is shown in Fig. 6(b). The evolution of the mean field, in this case, shows an irregular behavior dominated by low-amplitude fluctuations.

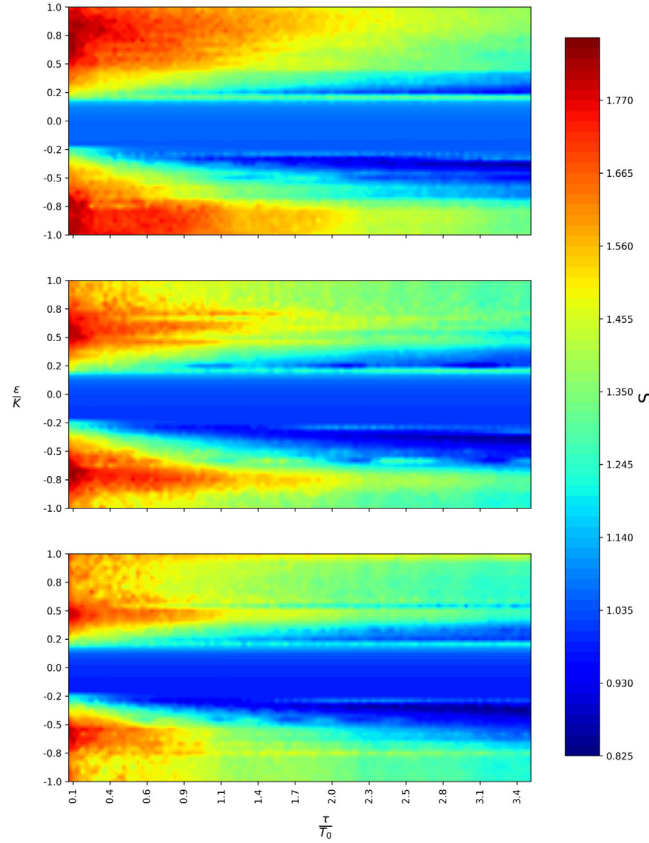


Fig. 7. Suppression coefficient for a two-dimensional lattice as a function of the control amplitude and time-delay, for $K = 6.0$ and (a) $\gamma = 0.02$, (b) 0.4 and (c) 0.6 .

To quantify the ability of the control signal to suppress phase synchronization, Pikowsky and Rosenblum defined the following coefficient [21]

$$S = \sqrt{\frac{\text{Var}(\Theta)}{\text{Var}(\Theta_f)}}, \quad (23)$$

where $\Theta_f(t)$ is the mean field after the control signal in (21) has been applied. Hence, if the control is able to suppress synchronization the denominator in (23) is small and S can be as large as the control is efficient in its goal. Conversely, if the control has no effect then $S = 1$. Values of $0 < S < 1$ would mean an enhanced synchronization, and hence of no interest in the present discussion.

We plot in Fig. 7 the suppression coefficient (in colorscale) as a function of the control amplitude ε (normalized by K) and the time delay τ (normalized by the period T_0 of the synchronized mean field oscillations). The control signal is switched on at $\alpha = 1500$ and the mean field is computed at $t = 15000$. We kept $K = 6$ and consider three values of γ . For $\gamma = 0.02$ [Fig. 7(a)] and small values of the time-delay the suppression becomes noticeable only for $\varepsilon > 0.2K$ and $\varepsilon < -0.2K$. There is an approximate symmetry $\varepsilon \rightarrow -\varepsilon$. As the time delay increases, this critical ε increases, showing that the suppression effect becomes progressively more difficult. If $\gamma = 0.4$ [Fig. 7(b)] there are two tongues of suppression centered at $\varepsilon \approx \pm 0.8K$. These tongues fade out, however, as γ increases further [Fig. 7(c)].

6. Conclusions

In the context of the model for a non-local coupling mediated by the fast diffusion of some chemical, we compared phase and frequency synchronization as we vary the parameters K and γ . In one-dimensional oscillator chains, for $\gamma \lesssim 0.02$ we observe a transition between non-synchronized and synchronized behavior as K is increased. The critical value of K_c for this transition is around 1.0, in accordance with the value predicted by the mean-field approach of the classical

Kuramoto model. If $\gamma > 0.02$ this transition still occurs but with important differences: frequency synchronization takes place before phase synchronization, and the value of K_c increases with γ . For $\gamma \approx 0.015$ there is a narrow region of non-synchronized behavior related to the emergence of chimera states. For two-dimensional lattices the picture is roughly the same, with quantitative differences.

Removing oscillators from the lattice causes the degradation of its synchronized behavior. Finite size effects in non-local chains have been long observed: in the classical Kuramoto model (global coupling), for large coupling strengths it has been numerically found that decreasing N deteriorates the synchronization properties of the lattice. Extending this observation to our model of chemical coupling there follows that, on diminishing N by oscillator removal, the synchronization worsens [18]. One way to circumvent this effect would be to adjust the coupling strength in order to cope with the deterioration of synchronized behavior, what we achieved by enhancing the parameter K according to the fraction of remaining oscillators. We also studied the critical fraction of removed oscillators, showing that it decreases with γ (for fixed K) and increases with K (for fixed γ).

It is worth mentioning that there is another class of lesions, characterized by impairing of connections rather than removing the oscillators themselves. This distinction has been investigated in the context of neuronal bursting oscillators, where we distinguish between type-I lesions, causing only deactivation of the synapses without taking neurons off the network; and type-II lesions, for which the neurons are actually removed from the network [10]. Notice that the transition to non-synchronized behavior resembles a first-order phase transition for a type-I lesion, and a second-order one for a type-II lesion.

Finally, we considered the issue of partial or total suppression of synchronization through the application of an external time-delayed feedback signal. This control procedure has been inspired by a surgical practice of deep-brain stimulation aiming to reduce pathological neurological rhythms. The control signal is applied according to the network mean field at an earlier time: the variance of the mean field is a diagnostic of synchronization as well. We observed good suppression effects as the control amplitude increases, this effect being weakened as the time delay increases.

This control procedure works well since it represents an external time-dependent forcing which is able to perturb a number of sites of the synchronized lattice. As these sites become non-synchronized, due to the perturbation, the remaining ones, while still synchronized, cause the global mean field to have smaller variances. While the control effect we are interested in should take the coupled system out of the synchronized state, our numerical results suggest that the external signal can also actually enhance synchronization of the network, depending on the coupling parameters. We found that this is particularly true for very small coupling strength. Hence, we can combine both approaches shown in this paper: the deteriorating synchronization caused by oscillator removal can be partially reversed by application of a small amplitude control signal, without changing the coupling strength, which is not always possible in real-world networks.

CRediT authorship contribution statement

Bruno M. Czajkowski: Data curation, Formal analysis, Investigation, Methodology, Validation, Visualization, Writing – review & editing. **Carlos A.S. Batista:** Formal analysis, Investigation, Methodology, Validation, Writing – review & editing. **Ricardo L. Viana:** Conceptualization, Formal analysis, Funding acquisition, Investigation, Methodology, Project administration, Resources, Software, Supervision, Validation, Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

Acknowledgments

R. L. V. received partial financial support from the following Brazilian government agencies: CNPq (403120/2021-7, 301019/2019-3), CAPES (88881.143103/2017-01), FAPESP (2022/04251-7).

References

- [1] A. Winfree, *The Geometry of the Biological Time*, Springer Verlag, New York, 1980.
- [2] S.H. Strogatz, *Nonlinear Dynamics and Chaos: with Applications to Physics, Biology, Chemistry, and Engineering*, Addison-Wesley Pub. Reading, MA, 1994.
- [3] E.R. Kandel, *Principles of Neural Science*, fifth ed., McGraw-Hill Education, 2012.
- [4] Y. Kuramoto, *Progr. Theoret. Phys.* 94 (1995) 321.
- [5] Y. Kuramoto, H. Nakao, *Physica D* 103 (1997) 294–313.
- [6] M.H. Hastings, E.J. Herzog, *J. Biol. Rhythms* 19 (2004) 400.
- [7] C. Liu, D.R. Weaver, S.H. Strogatz, S.M. Reppert, *Cell* 91 (1997) 855.

- [8] F.A. dos S. Silva, S.R. Lopes, R.L. Viana, *Commun. Nonlinear Sci. Numer. Simul.* 35 (2016) 37–52.
- [9] C.A.S. Batista, S.R. Lopes, R.L. Viana, A.M. Batista, *Neural Netw.* 23 (2010) 114–124.
- [10] F.A.S. Ferrari, R.L. Viana, *Pap. Phys.* 7 (2015) 070002.
- [11] E. Richter, A.M. Lozano, *Bioelectromagnetic Medicine*, CRC Press, Boca Raton, 2004, pp. 271–282.
- [12] C.A.S. Batista, A.M. Batista, J.A.C. de Pontes, R.L. Viana, S.R. Lopes, *Phys. Rev. E* 76 (2007) 016218.
- [13] R.P. Aristides, R.L. Viana, *Nonlinear Dynam.* 100 (2020) 3759–3770.
- [14] H. Nakao, *Chaos* 9 (1999) 902.
- [15] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence*, Springer-Verlag, New York, 1984.
- [16] J.A. Acebrón, L.L. Bonilla, C.J.P. Vicente, F. Ritort, R. Spigler, *Rev. Modern Phys.* 77 (2005) 137–185.
- [17] Y. Kuramoto, D. Battogtokh, *Nonlinear Phenom. Complex Syst.* 5 (2002) 380.
- [18] J.L. Rogers, L.T. Wille, *Phys. Rev. E* 54 (1996) R2193.
- [19] M.V. Ivanchenko, G.V. Osipov, V.D. Shalfeev, J. Kurths, *Phys. Rev. Lett.* 93 (2004) 134101.
- [20] E. Schöll, P. Hövel, V. Flunkert, M.A. Dahlem, Time-delayed feedback control: from simple models to lasers and neural systems, in: F.M. Atay (Ed.), *Complex Time-Delay Systems*, Springer, New York, 2010.
- [21] M. Rosenblum, A. Pikovsky, *Phys. Rev. E* 70 (2004) 041904.